Isomorphism for transitive groupoid C^*-algebras

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Abstract

We shall prove that the C^*-algebra of the locally compact second countable transitive groupoid G is $*$-isomorphic to the C^*-algebra of the groupoid $G^{(0)} \times H \times G^{(0)}$ endowed with the Haar system $\{\varepsilon_u \times \nu_H \times \mu\}$, where H is the isotropy group G_u^u at any unit $u \in G^{(0)}$, ε_u is the unit point mass at u, ν_H is a Haar measure on H, and μ is a quasi invariant Radon measure with respect to the Haar system of G. The C^* -algebra of $G^{(0)} \times H \times G^{(0)}$ is easily seen to be $*$-isomorphic to $C^*(H) \otimes \mathcal{K}\left(L^2(\mu)\right)$, where $C^*(H)$ denotes the group C^*-algebra of H, and $\mathcal{K}\left(L^2(\mu)\right)$ denotes the compact operators on $L^2(\mu)$. Therefore the C^*-algebra of G is $*$-isomorphic to $C^*(H) \otimes \mathcal{K}\left(L^2(\mu)\right)$. Thus we re-establish a result of P. Muhly, J. Renault and D. Williams (Theorem 3.1/p. 16 [7]).

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1 Introduction

The construction of the C^*-algebra of a groupoid extends the case of a group. The space of continuous functions with compact support on groupoid is made into a $*$-algebra and endowed with the smallest C^*-norm making its representations continuous. For this $*$-algebra the multiplication is convolution. For defining the convolution on a locally compact groupoid, one needs an analogue of Haar measure on locally compact groups. This analogue is a system of measures, called Haar system, subject to suitable invariance and smoothness conditions called respectively "left invariance" and "continuity". Unlike the case of locally compact group, Haar system on groupoid need not exists, and if it does, it will not usually be unique. However on locally compact second countable groupoids one can construct systems of measures satisfying "left invariance" condition. But the continuity assumption has topological consequences for groupoid. It entails that the range map (and hence the domain map) is open (Proposition I. 4 [13]). A. K. Seda has proved that the "continuity" condition is crucial in construction of the groupoid C^*-algebra ([12]). In this paper we shall only use locally compact second countable transitive groupoids. For this kind of groupoids the "continuity" condition is a consequence of the "left invariance"

condition (Theorem 4.4 [2] or Theorem 2.2 B/p. 8 [7] and Theorem 2/p. 430 [11]). Thus for this kind of groupoids always there are Haar systems. As in the general case, the Haar system need not to be unique. A result of P. Muhly, J. Renault and D. Williams (Theorem 2.8/p. 10 [7]) states that the C^*-algebras associate with two different Haar systems on a locally compact second countable groupoid (not necessarily transitive) are strongly Morita equivalent. It is not known if they are $*$-isomorphic. For a transitive groupoid G, P. Muhly, J. Renault and D. Williams have showed that then the C^*-algebra of G is isomorphic to $C^*(H) \otimes \mathcal{K}\left(L^2(\mu)\right)$, where H is the isotropy group G_u^u at any unit $u \in G^{(0)}$, μ is an essentially unique measure on $G^{(0)}$, $C^*(H)$ denotes the group C^*-algebra of H, and $\mathcal{K}\left(L^2(\mu)\right)$ denotes the compact operators on $L^2(\mu)$. For proving that result they firstly established that $C^*(G)$ and $C^*(H)$ are strong Morita equivalent via a $C^*(H)$ module X_1. As a consequence the C^*-algebra of G is the imprimitivity algebra of X_1. Then they needed another $C^*(H)$ module X_2 whose imprimitivity algebra is $C^*(H) \otimes \mathcal{K}\left(L^2(\mu)\right)$ for a suitable measure μ. By this result, it follows that the C^*-algebras associate with two Haar systems on a locally compact second countable transitive groupoid are $*$-isomorphic.

We shall obtain the isomorphism between the C^*-algebra of G and $C^*(H) \otimes \mathcal{K}\left(L^2(\mu)\right)$ more directly: we shall show that the C^*-algebra of G is isomorphic with the C^*-algebra of the groupoid $G^{(0)} \times H \times G^{(0)}$ endowed with the Haar system $\{\delta_u \times \nu_H \times \mu\}$,where ν_H is a Haar measure on H, and μ is quasi invariant probability measure with respect to the Haar system of G. In order to prove that we shall construct a groupoid Borel isomorphism ϕ between G and $G^{(0)} \times H \times G^{(0)}$ which carry the Haar system of G to a Haar system of $G^{(0)} \times H \times G^{(0)}$ of the form $\{\delta_u \times \nu_H \times \mu\}$.Then we shall use the fact that any compactly supported Borel bounded function can be viewed as an element of the C^*-algebra (Proposition 4/p. 82, Proposition 5/p. 86 [3]) and we shall prove that the $f \to \phi \circ f$ can be extended to a $*$-isomorphism of C^*-algebras.

2 Haar systems on G and Haar systems on the trivial groupoid $G^{(0)} \times G_e^e \times G^{(0)}$

Throughout this paper G will designate a second countable locally compact transitive groupoid and $\left\{\nu^u,\, u \in G^{(0)}\right\}$ a Haar system on G. We shall use the terminology and notation of [10].

According to Lemma 4.5/p. 277 [9] or Proposition I.3.8 [10] all quasi-invariant measures carried by an orbit $[u]$ are equivalent. Since the groupoid G is transitive, it follows that it has a single orbit, and therefore all quasi-invariant measures are equivalent.

In Section 3 [1], we have proved that we can choose a quasi-invariant probability measure μ such that there exists a family $\left\{\nu_{u,v},\, u, v \in G^{(0)}\right\}$ of σ-finite measures on G with the following properties:

1. $\nu_{u,v}$ is supported on G_v^u, and $\nu_{u,v} \neq 0$, for all $u, v \in G^{(0)}$.
2. For all $f \geq 0$ Borel on G,
$$(u, v) \mapsto \int f(y)\, d\nu_{u.v}(y) \left[: G^{(0)} \times G^{(0)} \to \overline{\mathbf{R}}\right]$$
is an extended real-valued Borel function.

3. For all $f \geq 0$ Borel on G,

$$\int f(xy)\, d\nu_{d(x),v}(y) = \int f(y)\, d\nu_{r(x),v}(y) \quad \text{for all } x \in G,\ v \in G^{(0)}$$

4. For all u in $G^{(0)}$, $\nu^u = \int \nu_{u,v} d\mu(v)$.

Moreover, if Δ is the modular function of μ, then Δ may be chosen to be a strict homomorphisms and

5. For all $f \geq 0$ Borel on G,

$$\Delta(x) \int f(yx)\, d\nu_{u,r(x)}(y) = \int f(y)\, d\nu_{u,d(x)}(y) \quad \text{for all } x \in G,\ u \in G^{(0)}$$

6. For all $f \geq 0$ Borel on G,

$$\int f(y)\, d\nu_{u,v}(y) = \int f\left(y^{-1}\right) \Delta\left(y^{-1}\right) d\nu_{v,u}(y) \ \text{ for all } u, v \in G^{(0)}$$

A similar decomposition for more general groupoids can be found in [4].

Let $\delta : G \to \mathbf{R}_+^*$ be a strict homomorphism and $\left\{\beta_{u,v},\ u, v \in G^{(0)}\right\}$a system of measures on G, satisfying:

(i) For all $f \geq 0$ Borel on G,

$$\int f(xy)\, d\beta_{d(x),v}(y) = \int f(y)\, d\beta_{r(x),v}(y)$$

for all $x \in G$, $v \in G^{(0)}$.

(ii) For all $f \geq 0$ Borel on G,

$$\delta(x) \int f(yx)\, d\beta_{u,r(x)}(y) = \int f(y)\, d\beta_{u,d(x)}(y)$$

for all $x \in G$, $u \in G^{(0)}$.

Using the uniqueness of the Haar measure on the locally compact group G_v^v and arguing as in Section 2 [1], it results that there exists a positive function $h : G^{(0)} \to \mathbf{R}_+^*$ such that

$$\begin{aligned} \nu_{u,v} &= h(v)\, \beta_{u,v} \text{ for all } u, v \in G^{(0)} \\ h(r(x))\, \Delta(x) &= \delta(x)\, h(d(x)) \text{ for all } x \in G. \end{aligned}$$

In [3] (p. 84). we have constructed $\delta : G \to \mathbf{R}_+^*$ and $\left\{\beta_{u,v},\ u, v \in G^{(0)}\right\}$ with the above properties and such that

$$\sup_{u,v} \beta_{u,v}(K) < \infty,\ \sup_{x \in K} \delta(x) < \infty \text{ for all compact set } K \subset G.$$

We sketch that construction. We need the following lemma :

Lemma 1 *Let X and Y be metric spaces and let $f : X \to Y$. Let A be a σ-compact subset of X, and $K_1, K_2, ..., K_n, ...$ be a sequence of compact sets whose union is A. If $f|_{K_n}$ is continuous for each n, then there is a Borel function $g : f(A) \to A$ such that $g(f(K_n)) \subset K_n$ for each n and $f(g(y)) = y$ for all $y \in f(A)$.*

Lemma 1 is a slight reformulation (according to [8, Lemma 4.12. p. 99]), based on a result of Federer and Morse [5], of a lemma proved by Mackey (Lemma 1.1 [6]). We shall call the function g in the Lemma 1 a regular Borel cross section of f. let e be a unit in $G^{(0)}$. Applying Lemma 1 to the continuous surjection $d : G^e \to G^{(0)}$, it follows that there is a regular Borel cross section $\sigma : G^{(0)} \to G^e$. By Theorem 2.2 B/p. 8 [7] or Corollary 5.7 [2], if G is a transitive locally compact second countable groupoid, then the application $d : G^u \to G^{(0)}$ is open. Hence for any compact subset K of $G^{(0)}$ there is a compact subset L in G^u such that K is contained in $d(L)$. Thus the closure of $\sigma(K)$ is a compact set for all compact sets K.

Let U_0 be a closed symmetric d-relatively compact neighborhood of $G^{(0)}$, U be an open symmetric d-relatively compact neighborhood of $G^{(0)}$ such that $U_0 \subset U$, and $f_0 : G \to [0,\ 1]$ be a continuous function with $supp(f) \subset U$ and such that $f(x) = 1$ for all $x \in U_0$. Choose μ_e a (left) Haar measure on the locally compact group G_e^e such that

$$\int f_0(y)\, d\mu_e(y) = 1.$$

Define $\beta_{u,v}$ and δ by

$$\begin{aligned} \int f(y)\, \beta_{u,v}(y) &= \int f\left(\sigma(u)^{-1} y \sigma(v)\right) d\mu_e(y), \text{ for all } f \geq 0 \text{ Borel} \\ \delta(y) &= \Delta_e\left(\sigma(r(y))\, y \sigma(d(y))^{-1}\right), \text{ for all } y \in G \end{aligned}$$

where Δ_e is the modular function on the locally compact group G_e^e. Using the same argument as in [3] (p. 85), we can prove that the function h (the connection between $\left\{\beta_{u,v},\ u, v \in G^{(0)}\right\}$ and $\left\{v_{u,v},\ u, v \in G^{(0)}\right\}$) is μ-integrable on the compact subsets of $G^{(0)}$, i.e. for all compact subsets K of $G^{(0)}$

$$\int 1_K(v)\, h(v)\, d\mu(v) < \infty$$

Remark 2 *Let e be a unit. Let us consider the trivial groupoid $G^{(0)} \times G_e^e \times G^{(0)}$. The topology of $G^{(0)} \times G_e^e \times G^{(0)}$ is the product topology, and the operations are*

$$\begin{aligned} (u, x, v)(v, y, w) &= (u, xy, w) \\ (u, x, v)^{-1} &= (v, x, u) \end{aligned}$$

Under this structure $G^{(0)} \times G_e^e \times G^{(0)}$ is a locally compact second countable groupoid. The system of measures $\left\{\varepsilon_u \times \mu_e \times h \cdot \mu,\ u \in G^{(0)}\right\}$ is Haar system on this groupoid.

It is not hard to prove that the C^*-algebra of $G^{(0)} \times G_e^e \times G^{(0)}$ is isomorphic to $C^*(G_e^e) \otimes \mathcal{K}\left(L^2(\mu)\right)$.

Proposition 3 *Let G be a locally compact second countable transitive groupoid. Let e be a unit and $\sigma : G^{(0)} \to G^e$ be a regular Borel cross section of $d : G^e \to G^{(0)}$. Then $\phi : G \to G^{(0)} \times G_e^e \times G^{(0)}$ defined by*

$$\phi(x) = \left(r(x), \sigma(r(x))\, x \sigma(d(x))^{-1}, d(x)\right)$$

is a Borel isomorphism which transport the Haar system of G into a Haar system of $G^{(0)} \times G_e^e \times G^{(0)}$ of the form $\left\{\varepsilon_u \times \mu_e \times \lambda,\, u \in G^{(0)}\right\}$, where ε_u is the unit point mass at $u \in G^{(0)}$, μ_e is a Haar measure on G_e^e, and λ is a suitable Radon measure on $G^{(0)}$.

Proof. Straightforward calculation. The Haar system $\left\{\nu^u,\, u \in G^{(0)}\right\}$ is carried to $\left\{\varepsilon_u \times \mu_e \times h \cdot \mu,\, u \in G^{(0)}\right\}$. ■

3 Isomorphism for transitive groupoid C^*-algebras

The following proposition was proved in [3](Proposition 4/p. 82, Proposition 5/p. 86)

Proposition 4 *With the notation of preceding section, let $\nu_0 = \Delta^{-\frac{1}{2}}\mu$ and $f \in L^1(G, \nu_0)$ such that*

$$\int\int\left(\int |f(x)|\, \Delta(x)^{-\frac{1}{2}}\, d\nu_{u,v}(x)\right)^2 d\mu(v)\, d\mu(u) < \infty.$$

Then there is a sequence $(f_n)_n$, in $C_c(G)$ such that

$$\lim_n \|f - f_n\|_{II} = 0.$$

where for any $g \in L^1(G, \nu_0)$, $\|g\|_{II}$ is defined by

$$\|g\|_{II} = \sup\left\{\int |g(x)\, j(d(x))\, k(r(x))|\, d\nu_0(x),\ \int |j|^2\, d\mu = \int |k|^2\, d\mu = 1\right\}.$$

Proposition 5 *If $f \in L^1(G, \nu_0)$ and*

$$\int\int\left(\int |f(x)|\, \Delta(x)^{-\frac{1}{2}}\, d\nu_{u,v}(x)\right)^2 d\mu(v)\, d\mu(u) < \infty.$$

then f can be viewed as an element in $C^*(G)$.

Remark 6 *In particular, any function in $\mathcal{B}_c(G)$, the space of compactly supported Borel bounded function on G, can be viewed as an element in $C^*(G)$.*

Theorem 7 *Let G be a locally compact second countable transitive groupoid. Let $\left\{\nu^u, u \in G^{(0)}\right\}$ be a Haar system on G. Let e be a unit and let us endow the trivial groupoid $G^{(0)} \times G_e^e \times G^{(0)}$ with the Haar system $\left\{\varepsilon_u \times \mu_e \times h\mu,\, u \in G^{(0)}\right\}$, where μ_e and h are chosen as in Section 2. Then the C^*-algebra of G associated with the Haar system $\left\{\nu^u, u \in G^{(0)}\right\}$ and the C^*-algebra of $G^{(0)} \times G_e^e \times G^{(0)}$ associated with the Haar system $\left\{\varepsilon_u \times \mu_e \times h\mu,\, u \in G^{(0)}\right\}$ are $*$-isomorphic.*

Proof. Let ϕ be the groupoid isomorphism defined in Proposition 3. Any nondegenerate representation of $C_c(G)$ is equivalent with the integrated form of a representation of G (Theorem 1.21 /pg.65 [10], or Theorem 3.29 /pg.74 [8]). The same thing is true for $G^{(0)} \times G_e^e \times G^{(0)}$. Since $\phi : G \to G^{(0)} \times G_e^e \times G^{(0)}$ is a Borel groupoid isomorphism, $L \to L \circ \phi$ is an one to one correspondence between the representation of the two groupoids Also ϕ carry the Haar system$\left\{\nu^u, u \in G^{(0)}\right\}$ into the Haar system $\left\{\varepsilon_u \times \mu_e \times h\mu,\ u \in G^{(0)}\right\}$.Therefore $\Phi : B_c\left(G^{(0)} \times G_e^e \times G^{(0)}\right) \to \mathcal{B}_c(G)$ defined by

$$\Phi(f) = f \circ \phi$$

is a *-isomorphism which can be extended to the $C^*\left(G^{(0)} \times G_e^e \times G^{(0)}\right)$. ■

References

[1] M. Buneci, *Consequences of Hahn structure theorem for the Haar measure*, to appear in Mathematical Reports, **4** (54) nr. 4, 2002.

[2] M. Buneci, *Haar systems and homomorphisms on groupoids*, to appear in Proceedings of the 4th Operator Algebras and Mathematical Physics Conference, Constanţa, 2001, Theta Foundation.

[3] M. Buneci, *C^*-algebras associated to the transitive groupoids*, An. Univ. Craiova Ser. Mat. Inform, Vol. XXVIII, 2001, 79-92.

[4] M. Buneci, *The structure of the Haar systems on locally compact groupoids*, Math. Phys. Electron. J, **8**(4), 2002.

[5] H. Federer and A. Morse, *Some properties of measurable functions*, Bull. Amer. Math. Soc, **49**(1943), 270-277.

[6] G. Mackey, *Induced representations of locally compact groups. I*, Ann. of Math., **55**(1952), 101-139.

[7] P. Muhly, J. Renault and D. Williams, *Equivalence and isomorphism for groupoid C*-algebras*, J. Operator Theory **17**(1987), 3-22.

[8] P. Muhly, *Coordinates in operator algebra*, (Book in preparation).

[9] A. Ramsay, *Virtual groups and groups actions*, Adv. in Math. **6**(1971), 253-322.

[10] J. Renault, *A groupoid approach to C^*- algebras*, Lecture Notes in Math., Springer-Verlag, **793**, 1980.

[11] A. K. Seda, *A continuity property of Haar systems of measures*, Ann. Soc. Sci. Bruxelles Ser I **89**(1975).

[12] A.K. Seda, *On the continuity of Haar measure on topological groupoids,* Proc. Amer. Math. Soc., **96** (1986), 115-120.

[13] J. Westman, *Nontransitive groupoid algebras*, Univ. of California at Irvine, 1967.

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