Spectral properties for bounded operators on locally convex spaces*

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Abstract
We introduce the spectral radius \( r_P(T) \) for a quotient bounded operator on a locally convex space \( X \). Similarly to the case of bounded operators on a Banach space we prove that the Neumann series \( \sum_{n=0}^{\infty} \frac{T^n}{\lambda+n} \) converges to \( R(\lambda, T) \), whenever \( \lambda > r_P(T) \), and \( |\sigma(Q_P, T)| = r_P(T) \). Also we study the universally bounded operators on locally convex spaces.

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1 Introduction
It is known that if \( T \) is a linear operator on a Banach space \( X \) then the spectral radius of \( T \) is defined by the Gelfand formula \( r(T) = \lim_{n \to \infty} \sqrt[|\lambda|]{\|T^n\|} \), \( |\sigma(Q, T)| = r(T) \) and the resolvent \( R(\lambda, T) \) is given by the Newmann series \( \sum_{n=0}^{\infty} \frac{T^n}{\lambda+n} \), whenever \( \lambda > r(T) \).

If we want to generalize this theory on locally convex space \( X \) one major difficulty is that is not clear which class of operators we can use, because there are several non-equivalent ways of defining bounded operators on \( X \). The concept of bounded element of a locally convex algebra was introduced by Allan [1]. An element is said to be bounded if some scalar multiple of it generates a bounded semi group.

Definition 1 Let \( X \) be a locally convex algebra. The radius of boundness of an element \( x \in X \) is the number

\[
\beta(x) = \inf \left\{ \alpha > 0 \mid \text{the set } \left\{ (\alpha^{-1}x)^n \right\}_{n \geq 1} \text{ is bounded} \right\}
\]

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Through this paper all locally convex spaces will be assumed Hausdorff, over complex field $\mathbb{C}$, and all operators will be linear. If $X$ and $Y$ are topological vector spaces we denote by $\mathcal{L}(X, Y)$ (or $L(X, Y)$) the algebra of linear operators (continuous operators) from $X$ to $Y$.

Any family $\mathcal{P}$ of seminorms who generate the topology of locally convex space $X$ (in the sense that the topology of $X$ is the coarsest with respect to which all seminorms of $\mathcal{P}$ are continuous) will be called a calibration on $X$. A calibration is characterized by the property that for every seminorms $p \in \mathcal{P}$ and every constant $\epsilon > 0$ the sets

$$S(p, \epsilon) = \{x \in X | p(x) < \epsilon\},$$

constitute a neighborhood sub-base at 0. A calibration on $X$ will be principal if it is directed. The set of calibrations for $X$ is denoted by $\mathcal{C}(X)$ and the set of principal calibration is denoted by $\mathcal{C}_0(X)$.

A family of seminorms on a linear space is partially ordered by relation $\leq$, where

$$p \leq q \iff p(x) \leq q(x), \forall x \in X.$$

A family of seminorms is preordered by the relation $\prec$, where

$$p \prec q \iff \text{there exists some } r > 0 \text{ such that } p(x) \leq rq(x), \forall x \in X.$$

If $p \prec q$ and $q \prec p$, we write $p \approx q$.

**Definition 2** Two families $\mathcal{P}_1$ and $\mathcal{P}_2$ of seminorms on a linear space are called $Q$-equivalent (denoted $\mathcal{P}_1 \approx \mathcal{P}_2$) provided:

1. for each $p_1 \in \mathcal{P}_1$ there exists $p_2 \in \mathcal{P}_2$ such that $p_1 \approx p_2$;
2. For each $p_2 \in \mathcal{P}_2$ there exists $p_1 \in \mathcal{P}_1$ such that $p_2 \approx p_1$.

It is obviously that two $Q$-equivalent and separating families of seminorms on a linear space generate the same locally convex topology.

Similarly to the norm of a linear operator on a normed space $V$, Troitsky [19] define the mixed operator seminorm of an operator between locally convex spaces. If $(X, \mathcal{P}), (Y, \mathcal{Q})$ are locally convex spaces, then for all seminorms $p, q \in \mathcal{P}$ the application $m_{pq} : L(X, Y) \to \mathbb{R} \cup \{\infty\}$, defined by

$$m_{pq}(T) = \sup_{p(x) \neq 0} \frac{q(Tx)}{p(x)},$$

is called the mixed operator seminorm of $T$ associated with $p$ and $q$. When $X = Y$ and $p = q$ we use notation $\hat{p} = m_{pp}$.

**Lemma 3** ([19]) If $(X, \mathcal{P}), (Y, \mathcal{Q})$ are locally convex spaces and $T \in L(X, Y)$, then:

1. $m_{pq}(T) = \sup_{p(x) = 1} q(Tx) = \sup_{p(x) \leq 1} q(Tx), \forall p \in \mathcal{P}, \forall q \in \mathcal{Q};$
2. \( q(Tx) \leq m_{pq}(T)p(x), (\forall) x \in X, \) whenever \( m_{pq}(T) < \infty \)

**Corollary 4** If \((X, \mathcal{P}), (Y, \mathcal{Q})\) are locally convex spaces and \(T \in L(X, Y)\), then

\[
m_{pq}(T) = \inf \{ M > 0 \mid q(Tx) \leq Mp(x), (\forall) x \in X \},
\]

whenever \( m_{pq}(T) < \infty \).

**Proof.** If \( p, q \in \mathcal{P} \) then from the previous lemma we have

\[
q(Tx) \leq m_{pq}(T)p(x), (\forall) x \in X.
\]

If \( M > 0 \) such that

\[
q(Tx) \leq Mp(x), (\forall) x \in X,
\]

then using (1) we obtain

\[
m_{pq}(T) = \sup_{p(x) = 1} q(Tx) \leq M.
\]

**Definition 5** An operator \( T \) on a locally convex space \( X \) is quotient bounded with respect to a calibration \( \mathcal{P} \in \mathcal{C}(X) \) if for every seminorm \( p \in \mathcal{P} \) there exists some \( c_p > 0 \) such that

\[
p(Tx) \leq c_p p(x), (\forall) x \in X.
\]

The class of quotient bounded operators with respect to a calibration \( \mathcal{P} \in \mathcal{C}(X) \) is denoted by \( Q_\mathcal{P}(X) \).

**Lemma 6** ([10]) If \( X \) is a locally convex space and \( \mathcal{P} \in \mathcal{C}(X) \), then for every \( p \in \mathcal{P} \) the application \( \hat{p} : Q_\mathcal{P}(X) \to \mathbb{R} \) defined by

\[
\hat{p}(T) = \inf \{ r > 0 \mid p(Tx) \leq rp(x), (\forall) x \in X \},
\]

is a submultiplicative seminorm on \( Q_\mathcal{P}(X) \), satisfying the relation \( \hat{p}(I) = 1 \).

We denote by \( \hat{\mathcal{P}} \) the family \( \{ \hat{p} \mid p \in \mathcal{P} \} \).

**Proposition 7** ([10]) Let \( X \) be a locally convex space and \( \mathcal{P} \in \mathcal{C}(X) \).

1. \( Q_\mathcal{P}(X) \) is a unital subalgebra of the algebra of continuous linear operators on \( X \);
2. \( Q_\mathcal{P}(X) \) is a unital local multiplicative convex algebra with respect to the topology determined by \( \hat{\mathcal{P}} \);
3. If \( \mathcal{P}' \in \mathcal{C}(X) \) such that \( \mathcal{P} \approx \mathcal{P}' \), then \( Q_{\mathcal{P}'}(X) = Q_\mathcal{P}(X) \) and \( \hat{\mathcal{P}} = \hat{\mathcal{P}}' \);
4. The topology generated by \( \hat{\mathcal{P}} \) on \( Q_\mathcal{P}(X) \) is finer than the topology of uniform convergence on bounded subsets of \( X \).

**Lemma 8** If \( X \) is a sequentially complete locally convex space, then \( Q_\mathcal{P}(X) \) is a sequentially complete \( m \)-convex algebra for all \( \mathcal{P} \in \mathcal{C}(X) \).

**Proof.** Let \( \mathcal{P} \in \mathcal{C}(X) \) and \( (T_n)_n \subseteq Q_\mathcal{P}(X) \) be a Cauchy sequence. Then, for each \( \epsilon > 0 \) and each \( \hat{p} \in \mathcal{P} \) there exists some index \( n_{p,\epsilon} \in \mathbb{N} \) such that

\[
| \hat{p}(T_n) - \hat{p}(T_m) | \leq \hat{p}(T_n - T_m) < \epsilon, \quad (\forall) \ n, m \geq n_{p,\epsilon} \tag{1}
\]

so, it follows that \( (\hat{p}(T_n))_n \) is convergent sequence of real numbers, for each \( \hat{p} \in \mathcal{P} \). If \( x \in X \), then

\[
p(T_n(x) - T_m(x)) \leq \hat{p}(T_n - T_m)p(x), \quad (\forall) \ p \in \mathcal{P}, \tag{2}
\]

so \( (T_n(x))_n \subseteq X \) is a Cauchy sequence. But, since \( X \) is sequentially complete and separated, there exists an unique element \( y \in X \) such that

\[
\lim_{n \to \infty} T_n x = y.
\]

Therefore, the operator \( T : X \to X \) defined by

\[
T(x) = \lim_{n \to \infty} T_n x, \quad (\forall) \ x \in X,
\]

is well defined. It is obvious that \( T \) is a linear operator. Using the continuity of seminorms \( \hat{p} \in \mathcal{P} \) we have

\[
p(Tx) = p\left( \lim_{n \to \infty} T_n x \right) = \lim_{n \to \infty} p(T_n x) \leq \lim_{n \to \infty} \hat{p}(T_n)p(x) = c_\mathcal{P}p(x),
\]

for all \( x \in X \) and for each \( p \in \mathcal{P} \) (where \( c_\mathcal{P} = \lim_{n \to \infty} \hat{p}(T_n) \)). This implies that \( T \in Q_\mathcal{P}(X) \). Now we prove that \( T_n \to T \) in \( Q_\mathcal{P}(X) \). From relations (1) and (2) it follows that for each \( \epsilon > 0 \) and each \( p \in \mathcal{P} \) there exists \( n_{p,\epsilon} \in \mathbb{N} \) such that

\[
p(T_n(x) - T_m(x)) \leq c_p(x), \quad (\forall) \ n, m \geq n_{p,\epsilon}.
\]

so

\[
p(T_n(x) - T(x)) \leq c_p(x), \quad (\forall) \ n \geq n_{p,\epsilon}.
\]

This implies that

\[
\hat{p}(T_n - T) < \epsilon, \quad (\forall) \ n \geq n_{p,\epsilon}.
\]

which prove that \( T_n \to T \) in \( Q_\mathcal{P}(X) \) and \( Q_\mathcal{P}(X) \) is a sequentially complete \( m \)-convex algebra.

Given \((X, \mathcal{P})\), for each \( p \in \mathcal{P} \) denote by \( N^p \) the null space and by \( X_p \) the quotient space \( X/N^p \). For each \( p \in \mathcal{P} \) consider the natural mapping

\[
x \to x_p \equiv x + N^p
\]
It is obvious that $X_p$ is a normed space, for each $p \in \mathcal{P}$, with norm $\| \cdot \|_p$ defined by

$$\| x_p \|_p = p(x), \forall x \in X.$$ 

Consider the algebra homomorphism $T \rightarrow T^p$ of $Q_{\mathcal{P}}(X)$ into $\mathcal{L}(X_p)$ defined by

$$T^p(x_p) = (Tx)_p, \forall x \in X.$$ 

This operators are well defined because $T(N^p) \subset N^p$. Moreover, for each $p \in \mathcal{P}$, $\mathcal{L}(X_p)$ is a unital normed algebra and we have

$$\| T^p \|_p = \sup \{ \| T^p x_p \|_p \mid \| x_p \|_p \leq 1 \text{ for } x_p \in X_p \}$$

= $\sup \{ p(Tx) \mid p(x) \leq 1 \text{ for } x \in X \}$

For $p \in \mathcal{P}$ consider the normed space $(\hat{X}_p, \| \cdot \|_p)$ the completion of $(X_p, \| \cdot \|_p)$. If $T \in Q_{\mathcal{P}}(X)$, then the operator $T^p$ has a unique continuous linear extension $\hat{T}^p$ on $(\hat{X}_p, \| \cdot \|_p)$.

**Definition 9** Let $(X, \mathcal{P})$ a locally convex space and $T \in Q_{\mathcal{P}}(X)$. We say that $\lambda \in \rho(Q_{\mathcal{P}}, T)$ if the inverse of $\lambda I - T$ exists and $(\lambda I - T)^{-1} \in Q_{\mathcal{P}}(X)$. Spectral sets $\sigma(Q_{\mathcal{P}}, T)$ are defined to be complements of resolvent sets $\rho(Q_{\mathcal{P}}, T)$.

For each $p \in \mathcal{P}$ we denote by $\sigma(X_p, T^p)$ (respectively the spectral set of $\hat{T}^p$ in $\mathcal{L}(\hat{X}_p)$) the spectral set of the operator $T^p$ in $\mathcal{L}(X_p)$ (respectively the spectral set of $\hat{T}^p$ in $\mathcal{L}(\hat{X}_p)$). The resolvent set of the operator $T^p$ in $\mathcal{L}(X_p)$ (respectively the spectral set of $\hat{T}^p$ in $\mathcal{L}(\hat{X}_p)$) is denoted by $\rho(X_p, T^p)(\rho(\hat{X}_p, \hat{T}^p))$.

**Lemma 10 ([9])** Let $(X, \mathcal{P})$ be a sequentially complete locally convex space and $T \in Q_{\mathcal{P}}(X)$. Then, $T$ is invertible in $Q_{\mathcal{P}}(X)$ if and only if $T^p$ is invertible in $\mathcal{L}(X_p)$ for all $p \in \mathcal{P}$.

**Corollary 11 ([9])** If $(X, \mathcal{P})$ is a sequentially complete convex space and $T \in Q_{\mathcal{P}}(X)$, then

$$\sigma(Q_{\mathcal{P}}, T) = \bigcup \{ \sigma(X_p, T^p) \mid p \in \mathcal{P} \} = \bigcup \{ \sigma(\hat{X}_p, \hat{T}^p) \mid p \in \mathcal{P} \}.$$ 

## 2 Spectral radius of quotient bounded operators

Let $(X, \mathcal{P})$ be a locally convex space and $T \in Q_{\mathcal{P}}(X)$. We said that $T$ is **bounded element** of the algebra $Q_{\mathcal{P}}(X)$ if it is a bounded element of $Q_{\mathcal{P}}(X)$ in the sense of G.R.Allan [1]. The class of bounded elements of $Q_{\mathcal{P}}(X)$ is denoted by $(Q_{\mathcal{P}}(X))_0$. 


Definition 12 If $(X, \mathcal{P})$ is a locally convex space and $T \in Q_{\mathcal{P}}(X)$ we denote by $r_{\mathcal{P}}(T)$ the radius of boundedness of operator $T$ in $Q_{\mathcal{P}}(X)$, i.e.

\[ r_{\mathcal{P}}(T) = \inf \{ \alpha > 0 \mid \alpha^{-1}T \text{ generates a bounded semigroup in } Q_{\mathcal{P}}(X) \}. \]

We said that $r_{\mathcal{P}}(T)$ is the $\mathcal{P}$-spectral radius of the operator $T$.

Remark 13 Proposition 1.7 (3) implies that for each $\mathcal{P}_0 \in \mathcal{C}(X)$, $\mathcal{P} \approx \mathcal{P}_0$, we have $Q_{\mathcal{P}}(X) = Q_{\mathcal{P}_0}(X)$, so if $H$ is a $Q$-equivalence class in $\mathcal{C}(X)$, then

\[ r_{\mathcal{P}}(T) = r_{\mathcal{P}_0}(T), \forall \mathcal{P}, \mathcal{P}_0 \in H. \]

Proposition 14 ([1]) If $X$ is a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$, then for each $T \in Q_{\mathcal{P}}(X)$ we have

\[ r_{\mathcal{P}}(T) = \sup \{ \limsup_{n \to \infty} (p(T^n))^{1/n} \mid p \in \mathcal{P} \}. \]

From real analysis we have the following lemma.

Lemma 15 If $(t_n)_n$ is a sequence in $R^+ \cup \{ \infty \}$ then

\[ \limsup_{n \to \infty} \sqrt[n]{t_n} = \inf \left\{ \nu > 0 \mid \lim_{n \to \infty} \frac{t_n}{\nu^n} = 0 \right\}. \]

Proposition 16 If $X$ is a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$, then for each $T \in Q_{\mathcal{P}}(X)$ we have:

1. $r_{\mathcal{P}}(T) \geq 0$ and $r_{\mathcal{P}}(\lambda T) = | \lambda | \cdot r_{\mathcal{P}}(T), (\forall) \lambda \in C$, where by convention $0^\infty = \infty$;
2. $r_{\mathcal{P}}(T) < +\infty$ if and only if $T \in (Q_{\mathcal{P}}(X))_0$;
3. $r_{\mathcal{P}}(T) = \inf \{ \lambda > 0 \mid \lim_{n \to \infty} \frac{T^n}{\lambda^n} = 0 \}$.

Proof. 1) From proposition 13 results that for each $\lambda \in N^*$, $\lambda \neq 0$, we have

\[ r_{\mathcal{P}}(\lambda T) = \sup \{ \limsup_{n \to \infty} (\lambda t^n)^{1/n} \mid p \in \mathcal{P} \} = \sup \{ \limsup_{n \to \infty} (|\lambda| t^n)^{1/n} \mid p \in \mathcal{P} \} = |\lambda| \sup \{ \limsup_{n \to \infty} (t^n)^{1/n} \mid p \in \mathcal{P} \} = |\lambda| \cdot r_{\mathcal{P}}(T). \]

The case $\lambda = 0$ is obvious.

2) This equivalence results from definition of $\mathcal{P}$-spectral radius of the operator $T$.

3) The equality results directly from proposition 14 and lemma 15.

\[ \blacksquare \]
Lemma 15 implies that for a bounded operator on Banach space we have

\[ r(T) = \lim_{n \to \infty} \sqrt[n]{\|T^n\|} = \inf \left\{ \nu > 0 \mid \left( \frac{T^n}{\nu^n} \right)_n \text{ converges to zero in operator norm topology} \right\}. \]

If we consider this relation as an alternative definition of the spectral radius, then proposition 16 implies that \( \mathcal{P} \)-spectral radius of a quotient bounded operator can be considered to be natural generalization of the spectral radius of a bounded operator on a Banach space.

**Proposition 17** If \( X \) is a locally convex space and \( \mathcal{P} \in \mathcal{C}(X) \), then for each \( T \in Q_{\mathcal{P}}(X) \) and each \( p \in \mathcal{P} \) the sequence \( (\hat{p}_n)_n \) is convergent and

\[ \lim_{n \to \infty} (\hat{p}(T^n))^{1/n} = \inf_{n \geq 1} (\hat{p}(T^n))^{1/n}. \]

**Proof.** If there exists some \( n_0 \in \mathbb{N}^* \) such that \( \hat{p}(T^{n_0}) = 0 \), then

\[ \hat{p}(T^{n+n_0}) \leq \hat{p}(T^n) \hat{p}(T^{n_0}) = 0, \forall n > 0. \]

Therefore,

\[ \lim_{n \to \infty} (\hat{p}(T^n))^{1/n} = \inf_{n \geq 1} (\hat{p}(T^n))^{1/n}. \]

Assume that \( \hat{p}(T^n) > 0 \) for each \( n \in \mathbb{N}^* \) and let \( m \in \mathbb{N}^* \) be arbitrary fixed. For each \( n \in \mathbb{N}^* \) we consider the relations \( n = m \cdot q(n) + r(n) \), where \( 0 \leq r(n) < m \). Using this notations we have

\[ (\hat{p}(T^n))^{1/n} = \left( \hat{p} \left( T^{mq(n)+r(n)} \right) \right)^{1/n} \leq \left[ \hat{p} \left( T^{mq(n)} \right) \hat{p} \left( T^r(n) \right) \right]^{1/n} \leq \hat{p}(T^m)^{q(n)/n} \hat{p}(T)^{r(n)/n} \]

so

\[ \limsup_{n \to \infty} (\hat{p}(T^n))^{1/n} \leq (\hat{p}(T^m))^{1/m}. \]

Since \( m \in \mathbb{N}^* \) is arbitrary fixed, from previous inequality results that

\[ \limsup_{n \to \infty} (\hat{p}(T^n))^{1/n} \leq \inf_{m \geq 1} (\hat{p}(T^m))^{1/m} \leq \liminf_{n \to \infty} (\hat{p}(T^n))^{1/n}. \]

Therefore, the sequence \( (\hat{p}(T_n)_n \) is convergent and

\[ \lim_{n \to \infty} (\hat{p}(T^n))^{1/n} = \inf_{n \geq 1} (\hat{p}(T^n))^{1/n}. \]

**Corollary 18** If \( X \) is a locally convex space and \( \mathcal{P} \in \mathcal{C}(X) \), then for each \( T \in Q_{\mathcal{P}}(X) \) we have:
1. \( r_P(T) = \sup \{ \limsup_{n \to \infty} (p(T^n))^{1/n} \mid p \in P \} = \sup \{ \limsup_{n \to \infty} (\hat{p}(T^n))^{1/n} \mid p \in P \} = \sup \{ \inf_{n \geq 1} (\hat{p}(T^n))^{1/n} \mid p \in P \}; \)

2. \( r_P(T) \leq \hat{p}(T), (\forall) p \in P. \)

**Lemma 19** Let \((X, P)\) be a locally convex space and \(T, S \in Q_P(X)\). If \(TS = ST\), then for each \(p \in P\) we have

\[
\limsup_{n \to \infty} (\hat{p}((TS)^n))^{1/n} \leq \limsup_{n \to \infty} (\hat{p}(T^n))^{1/n} \limsup_{n \to \infty} (\hat{p}(S^n))^{1/n}.
\]

**Proof.** From inequality

\[
\hat{p}(TS) \leq \hat{p}(T) \hat{p}(S),
\]

we have

\[
\hat{p}((TS)^n) \leq \hat{p}(T^n) \hat{p}(S^n), (\forall) n \in N^*.
\]

Therefore,

\[
\limsup_{n \to \infty} (\hat{p}((TS)^n))^{1/n} \leq \limsup_{n \to \infty} (\hat{p}(T^n))^{1/n} \limsup_{n \to \infty} (\hat{p}(S^n))^{1/n}.
\]

**Corollary 20** Let \(X\) be a locally convex space and \(P \in C(X)\). If \(T, S \in Q_P(X)\) such that \(TS = ST\), then

\[
r_P(TS) \leq r_P(T)r_P(S).
\]

**Lemma 21** Let \((X, P)\) be a locally convex space and \(T, S \in Q_P(X)\). If \(TS = ST\), then we have

\[
\limsup_{n \to \infty} (\hat{p}((T + S)^n))^{1/n} \leq \limsup_{n \to \infty} (\hat{p}(T^n))^{1/n} + \limsup_{n \to \infty} (\hat{p}(S^n))^{1/n}.
\]

**Proof.** For each \(\varepsilon > 0\) there exists some seminorm index \(n_\varepsilon \in \mathbb{N}\) such that

\[
(\hat{p}(T^n)^{1/m} < \limsup_{n \to \infty} (\hat{p}(T^n))^{1/n} + \varepsilon,
\]

\[
(\hat{p}(S^n)^{1/m} < \limsup_{n \to \infty} (\hat{p}(S^n))^{1/n} + \varepsilon,
\]

for every \(m \geq n_\varepsilon\). Therefore, there exists \(M \geq 1\) such that

\[
\hat{p}(T) < M \left( \limsup_{n \to \infty} (\hat{p}(T^n))^{1/n} + \varepsilon \right)^m,
\]

\[
\hat{p}(S^n) < M \left( \limsup_{n \to \infty} (\hat{p}(S^n))^{1/n} + \varepsilon \right)^m,
\]
for every $m \in \mathbb{N}$. Moreover, $p$ is a submultiplicative seminorm, so we have

$$
\hat{p}((T + S)^m) \leq \sum_{k=1}^{m} C_k^m \hat{p}(T^k) \hat{p}(S^{m-k}) \\
\leq M^2 \sum_{k=1}^{m} C_k^m \left( \limsup_{n \to \infty} (\hat{p}(T^n))^{1/n} + \varepsilon \right)^k \left( \limsup_{n \to \infty} (\hat{p}(S^n))^{1/n} + \varepsilon \right)^{m-k} = \\
= M^2 \left( \limsup_{n \to \infty} (\hat{p}(T^n))^{1/n} + \limsup_{n \to \infty} (\hat{p}(S^n))^{1/n} + 2\varepsilon \right)^m.
$$

for every $n \geq n_\varepsilon$. Therefore

$$
\limsup_{n \to \infty} (\hat{p}((T + S)^n))^{1/n} \leq \limsup_{n \to \infty} (\hat{p}(T^n))^{1/n} + \limsup_{n \to \infty} (\hat{p}(S^n))^{1/n} + 2\varepsilon.
$$

Since $\varepsilon > 0$ is arbitrary chosen from previous relation results that

$$
\limsup_{n \to \infty} (\hat{p}((T + S)^n))^{1/n} \leq \limsup_{n \to \infty} (\hat{p}(T^n))^{1/n} + \limsup_{n \to \infty} (\hat{p}(S^n))^{1/n}.
$$

Corollary 22 Let $X$ be a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$. If $T, S \in Q_\mathcal{P}(X)$ such that $TS = ST$, then

$$
r_\mathcal{P}(T + S) \leq r_\mathcal{P}(T) + r_\mathcal{P}(S).
$$

From the definition of the $\mathcal{P}$-spectral radius of a quotient bounded operator and the properties we proved above result the following proposition.

Proposition 23 Let $X$ be a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$.

1. If $T \in (Q_\mathcal{P}(X))_0$, then

$$
\lim_{n \to \infty} \frac{T^n}{\lambda^n} = 0, (\forall) | \lambda | > r_\mathcal{P}(T);
$$

2. If $T \in (Q_\mathcal{P}(X))_0$ and $0 < | \lambda | < r_\mathcal{P}(T)$, then the set $\{ \frac{T^n}{\lambda^n} \}_{n \geq 1}$ is unbounded.

3. For each $T \in Q_\mathcal{P}(X)$ and every $n > 0$ we have $r_\mathcal{P}(T^n) = r_\mathcal{P}(T)^n$.

Proposition 24 Let $X$ be a sequentially complete locally convex space and $\mathcal{P} \in \mathcal{C}(X)$. If $T \in (Q_\mathcal{P}(X))_0$ and $| \lambda | > r_\mathcal{P}(T)$, then the Neumann series $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^n}$ converges to $R(\lambda, T)$ (in $Q_\mathcal{P}(X)$) and $R(\lambda, T) \in Q_\mathcal{P}(X)$. 

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Proof. If $|\lambda| > r_{P}(T)$ then there exists $\beta \in \mathbb{C}$, such that $0 < |\beta| < 1$ and $r_{P}(T) < \beta \lambda$. From proposition 23(1) we obtain that for each $\epsilon > 0$ and every $p \in \mathcal{P}$, there exists some index $n_{p,\epsilon} \in \mathbb{N}$ with the property

$$\hat{p}\left(\frac{T^{n}}{(\beta\lambda)^{n}}\right) < \epsilon, (\forall) n \geq n_{p,\epsilon}.$$ 

therefore, using corollary 4 we obtain

$$p\left(\frac{T^{n}}{(\beta\lambda)^{n}}x\right) \leq \hat{p}\left(\frac{T^{n}}{(\beta\lambda)^{n}}\right) p(x) < \epsilon p(x), (\forall) n \geq n_{p,\epsilon}, (\forall) x \in X.$$ 

Since $0 < |\beta| < 1$, there exists $n_{0} \in \mathbb{N}$, such that

$$\sum_{k=n}^{m} |\beta|^{k} < 1, (\forall) m > n \geq n_{0}.$$ 

From previous relations result that for each $\epsilon > 0$ and every $p \in \mathcal{P}$ there exists an index $m_{p,\epsilon} = \max\{n_{p,\epsilon}, n_{0}\} \in \mathbb{N}$, for which we have

$$p\left(\sum_{k=n}^{m} \frac{T^{k}}{\lambda^{k}}x\right) \leq \epsilon \left(\sum_{k=n}^{m} |\beta|^{k}\right) p(x) < \epsilon p(x), \quad (3)$$

for every $m > n \geq m_{p,\epsilon}$ and every $x \in X$. Therefore, $\left(\sum_{k=0}^{m} \frac{T^{k}}{\lambda^{k+1}}x\right)_{m \geq 0}$ is a Cauchy sequence, for each $x \in X$. But $X$ is sequentially complete, so for every $x \in X$ there exists an unique element $y \in X$ such that

$$y = \lim_{m \to \infty} \sum_{k=0}^{m} \frac{T^{k}}{\lambda^{k+1}}x.$$ 

We consider the operator $S: X \to X$ given by

$$S(x) = \lim_{m \to \infty} \sum_{k=0}^{m} \frac{T^{k}}{\lambda^{k+1}}x, (\forall) x \in X.$$ 

It is obvious that $S$ is linear operator. Moreover, from equality

$$\sum_{k=0}^{m} \frac{T^{k}}{\lambda^{k+1}}(\lambda x - Tx) = x - \frac{T^{m+1}}{\lambda^{m+1}}x, (\forall) x \in X,$$

result that if $m \to \infty$ then

$$S(\lambda x - Tx) = x, (\forall) x \in X,$$

so $S(\lambda I - T) = I$. From continuity of the operator $T$ result that

$$STx = \lim_{m \to \infty} \sum_{k=0}^{m} \frac{T^{k}}{\lambda^{k+1}}Tx = \lim_{m \to \infty} T \left(\sum_{k=0}^{m} \frac{T^{k}}{\lambda^{k+1}}x\right) = \text{10}.$$
\[
\lim_{m \to \infty} \sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}} x = TSx,
\]
for all \( x \in X \), therefore \( S(\lambda I - T) = (\lambda I - T)S = I \).

The definition of \( \mathcal{P} \)-spectral radius implies that the family \( \left( \frac{T^n}{(\beta \lambda)^n} \right) \) is bounded in \( Q_\mathcal{P}(X) \), therefore for every \( p \in \mathcal{P} \) there exists a constant \( \epsilon_p > 0 \) such that
\[
\hat{\rho} \left( \frac{T^n}{(\beta \lambda)^n} \right) < \epsilon_p, \forall n \geq 1.
\]

Using again the corollary 4 we have
\[
p \left( \frac{T^n}{\lambda^n} \right) < \epsilon_p \left( \sum_{k=0}^{m} \left( \frac{\beta}{\lambda} \right)^k \right) p(x) < \epsilon_p \frac{1}{\lambda(1 - |\beta|)} p(x),
\]
and if \( m \to \infty \) then
\[
p(Sx) < \epsilon_p \frac{1}{\lambda(1 - |\beta|)} p(x),
\]
for every \( m \geq 1 \) and every \( x \in X \), which implies that \( S = R(\lambda, T) \in Q_\mathcal{P}(X) \).

If we write the relation (3) under the form
\[
p \left( \sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}} x - \sum_{k=0}^{n} \frac{T^k}{\lambda^{k+1}} x \right) < \epsilon \frac{1}{\lambda} p(x),
\]
then for \( m \to \infty \) result that for every \( \epsilon > 0 \) and every \( p \in \mathcal{P} \) there exists some index \( n_{p, \epsilon} \in \mathbb{N} \), such that
\[
p \left( Sx - \sum_{k=0}^{n} \frac{T^k}{\lambda^{k+1}} x \right) \leq \epsilon \frac{1}{\lambda} p(x), \forall n \geq n_{p, \epsilon}, \forall x \in X
\]

Corollary 4 implies that for each \( p \in \mathcal{P} \) there exists some index \( n_{p, \epsilon} \in \mathbb{N} \), such that
\[
\hat{\rho} \left( S - \sum_{k=0}^{n} \frac{T^k}{\lambda^{k+1}} \right) \leq \epsilon \frac{1}{\lambda}, \forall n \geq n_{p, \epsilon}, \forall x \in X,
\]
which prove that the Neumann series \( \sum_{n=0}^{\infty} \frac{T^n}{\lambda^n} \) converges to \( R(\lambda, T) \) in \( Q_\mathcal{P}(X) \).

\[\blacksquare\]

**Proposition 25** Let \( X \) be a sequentially complete locally convex space and \( \mathcal{P} \in \mathcal{C}(X) \). If \( T \in Q_\mathcal{P}(X) \), then \( |\sigma(Q_\mathcal{P}, T)| = r_\mathcal{P}(T) \).
Proof. Inequality $|\sigma(Q, T)| \leq r_{\mathcal{P}}(T)$ is implied by previous proposition. We prove now the reverse inequality. From corollary 11 we have

$$\sigma(Q, T) = \cup \{\sigma(X, T) \mid p \in \mathcal{P}\} = \cup \{\sigma(\tilde{X}, T) \mid p \in \mathcal{P}\}.$$ 

so, if $|\lambda| > |\sigma(Q, T)|$, then

$$|\lambda| > |\sigma(\tilde{X}, T)|, (\forall) p \in \mathcal{P}.$$ 

But, $\tilde{X}$ is Banach space for each $p \in \mathcal{P}$, therefore

$$|\sigma(\tilde{X}, T)| = r(\tilde{X}, T),$$

where $r(\tilde{X}, T)$ is spectral radius of bounded operator $\tilde{T}$ in $\tilde{X}$.

This observation implies that for each $p \in \mathcal{P}$ we have

$$(\forall) p \in \mathcal{P},$$

which means that for every $p \in \mathcal{P}$ and every $\epsilon > 0$ there exists $n_{\epsilon,p} \in \mathbb{N}$, such that

$$\hat{p} \left( \frac{T^n}{\lambda^n} \right) = \left\| \frac{T^n}{\lambda^n} \right\|_{\mathcal{P}} < \epsilon, (\forall) n \geq n_{\epsilon,p}.$$ 

Using proposition 16(3) and previous relation we have $r_{\mathcal{P}}(T) \leq |\lambda|$. But $|\lambda| > |\sigma(Q, T)|$ is arbitrary chosen, so $r_{\mathcal{P}}(T) \leq |\sigma(Q, T)|$. 

Definition 26 If $X$ is a locally convex space and $T \in Q_{\mathcal{P}}(X)$, we denote by $
abla(X)$ the set

$$\cap \{\sigma(Q, T) \mid p \in \mathcal{P} \in C(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\}.$$ 

Lemma 27 If $X$ is a locally convex space and $T \in Q_{\mathcal{P}}(X)$ then

$$|\sigma(Q, T)| \leq \inf \{r_{\mathcal{P}}(T) \mid p \in \mathcal{C}(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\}.$$ 

Proof. This is a direct consequence of the proposition 25. 

Definition 28 An operator $T$ is quotient bounded operator on a locally convex space $X$ if there exists some calibration $\mathcal{P}$ on $X$ such that $T \in Q_{\mathcal{P}}(X)$.

Remark 29 An operator $T$ is quotient bounded on a locally convex space $X$ if and only if there exists some calibration $\mathcal{P} \in C(X)$ such that $\hat{p}(T)$ is finit for each $p \in \mathcal{P}$.

Lemma 30 If $T$ is a quotient bounded operator on a locally convex space $X$, then there exists some principal calibration $\mathcal{P}' \in C_0(X)$ such that $T \in Q_{\mathcal{P}'}(X)$.

Proof. Let $\mathcal{P}$ be a calibration on $X$ such that $T \in Q_{\mathcal{P}}(X)$ and denote by $\mathcal{P}'$ the set of all seminorms given by the relations

$$\hat{p}'(x) = \max_{i=1,n} p_i(x), (\forall) x \in X,$$
where \( p_i \in \mathcal{P}', i = \overline{1,n} \), and \( n \in \mathbb{N} \).

Let \( p' \in \mathcal{P}' \) be arbitrary chosen. Since \( T \in Q_{\mathcal{P}}(X) \), from previous remark and lemma 3(2) results that

\[
p_i(Tx) \leq \hat{p}_i(T)(x), (\forall) x \in X, i = \overline{1,n},
\]

If \( c_{p'} = \max_{i=\overline{1,n}} \hat{p}_i(T) \), then

\[
p_i(Tx) \leq c_{p'} p_i(x) \leq c_{p'} p'(x), (\forall) x \in X, i = \overline{1,n},
\]

so

\[
p'(Tx) \leq c_{p'} p'(x), (\forall) x \in X,
\]

Therefore, \( T \in Q_{\mathcal{P}'}(X) \).

\[\Box\]

**Lemma 31** If \( X \) is a locally convex space and \( T \in Q_{\mathcal{P}}(X) \) then

\[
\inf\{r_{\mathcal{P}}(T) \mid \mathcal{P} \in C_0(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\} =
\]

\[
= \inf\{r_{\mathcal{P}}(T) \mid \mathcal{P} \in C(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\}.
\]

**Proof.** Assume that \( \mathcal{P} \in C(X) \) such that \( T \in Q_{\mathcal{P}}(X) \).

If \( |\lambda| > r_{\mathcal{P}}(T) \), then the family \( \left( \frac{T^n}{\lambda^n} \right)_{n \geq 0} \) is bounded in \( Q_{\mathcal{P}}(X) \), i.e. for every \( p \in \mathcal{P} \) there exists \( \epsilon_p > 0 \) such that

\[
\hat{p} \left( \frac{T^n}{\lambda^n} \right) \leq \epsilon_p, (\forall) n \geq 0.
\]

Let \( \mathcal{P}' \) be the principal calibration associated with the calibration \( \mathcal{P} \), i.e. for each \( p' \in \mathcal{P}' \) there exists \( p_1, p_2, \ldots, p_n \in \mathcal{P} \) such that \( p' = \max\{p_1, p_2, \ldots, p_n\} \).

If \( \epsilon_{p'} = \max\{\epsilon_{p_1}, \ldots, \epsilon_{p_n}\} \), then

\[
\hat{p}' \left( \frac{T^n}{\lambda^n} \right) \leq \epsilon_{p'}, (\forall) n \geq 0.
\]

so \( |\lambda| > r_{\mathcal{P}'}(T) \). Since \( \lambda \) is arbitrary chosen results \( r_{\mathcal{P}'}(T) \leq r_{\mathcal{P}}(T) \).

Therefore,

\[
\inf\{r_{\mathcal{P}}(T) \mid \mathcal{P} \in C_0(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\} \leq
\]

\[
\leq \inf\{r_{\mathcal{P}}(T) \mid \mathcal{P} \in C(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\}.
\]

The reverse inequality is obvious.

\[\Box\]

**Lemma 32** If \( X \) is a locally convex space and \( T \in Q_{\mathcal{P}}(X) \) then

\[
\sigma(Q, T) = \cap\{\sigma(Q_{\mathcal{P}}, T) \mid \mathcal{P} \in C_0(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\}.
\]
Proof. From definition of the set $\sigma(Q, T)$ results the inclusion

$$\sigma(Q, T) \subset \cap \{ \sigma(Q_{\mathcal{P}}, T) \mid \mathcal{P} \in \mathcal{C}_0(X) \text{ such that } T \in Q_{\mathcal{P}}(X) \}.$$  

If $\lambda \notin \sigma(Q, T)$, then there exists some calibration $\mathcal{P} \in \mathcal{C}(X)$ such that $\lambda \in \rho(Q_{\mathcal{P}}, T)$, so for every $p \in \mathcal{P}$ we have $\hat{p}(R(\lambda, T)) < \infty$.

Denote by $\mathcal{P}'$ the principal calibration of all seminorms

$$p' = \max_{i=1,n} p_i(x), \forall x \in X,$$

where $p_i \in \mathcal{P}'$, $i = 1n$, and $n \in \mathbb{N}$.

Let $p' \in \mathcal{P}'$ be such seminorm. Since $R(\lambda, T) \in Q_{\mathcal{P}}(X)$, the lemma 3(2) implies that

$$p_i(R(\lambda, T) x) \leq \hat{p}_i(R(\lambda, T)) p_i(x), \forall x \in X, i = 1n,$$

If $c_{p'} = \max_{i=1,n} \hat{p}_i(R(\lambda, T))$, then

$$p_i(R(\lambda, T) x) \leq c_{p'} p_i(x) \leq c_{p'} p'(x), \forall x \in X, i = 1n,$$

so we have

$$p'(R(\lambda, T) x) \leq c_{p'} p'(x), \forall x \in X,$$

Therefore, $R(\lambda, T) \in Q_{\mathcal{P}'}(X)$ and $\lambda \notin \sigma(Q_{\mathcal{P}'}, T)$, which implies that

$$\cap \{ \sigma(Q_{\mathcal{P}}, T) \mid \mathcal{P} \in \mathcal{C}_0(X) \text{ such that } T \in Q_{\mathcal{P}}(X) \} \subset \sigma(Q, T).$$

\section{Universally bounded operators}

\textbf{Definition 33} Let $\mathcal{P}$ be a calibration on a locally convex space $X$. An operator $T : X \rightarrow X$ is universally bounded with respect to the calibration $\mathcal{P}$ if there exists $c_0 > 0$ such that

$$p(Tx) \leq c_0 p(x), \forall x \in X, \forall p \in \mathcal{P}.$$  

We denote by $B_{\mathcal{P}}(X)$ the class of the universally bounded operators with respect to the calibration $\mathcal{P} \in \mathcal{C}(X)$.

\textbf{Remark 34} For each calibration $\mathcal{P} \in \mathcal{C}(X)$ we have $B_{\mathcal{P}}(X) \subset Q_{\mathcal{P}}(X) \subset \mathcal{L}(X)$.

\textbf{Definition 35} Let $X$ be a locally convex space. For each $\mathcal{P} \in \mathcal{C}(X)$ we define the application $\| \bullet \|_{\mathcal{P}} : B_{\mathcal{P}}(X) \rightarrow \mathbb{R}$ by the formula

$$\| T \|_{\mathcal{P}} = \inf \{ c > 0 \mid p(Tx) \leq cp(x), \forall x \in X, \forall p \in \mathcal{P} \}.$$  

\textbf{Proposition 36} ([10]) Let $X$ be a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$. Then:
1. $B_\mathcal{P}(X)$ is a subalgebra of $\mathcal{L}(X)$;

2. $(B_\mathcal{P}(X), \|\bullet\|_\mathcal{P})$ is unitary normed algebra;

3. for each $\mathcal{P}' \in \mathcal{C}(X)$ with the property $\mathcal{P} \approx \mathcal{P}'$, we have
   
   $$B_\mathcal{P}(X) = B_\mathcal{P}'(X) \text{ and } \|\bullet\|_\mathcal{P} = \|\bullet\|_{\mathcal{P}'}.$$  

Proposition 37 ([4]) Let $X$ be a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$. Then:

1. the topology given by the norm $\|\bullet\|_\mathcal{P}$ on the algebra $B_\mathcal{P}(X)$ is finer than the topology of uniform convergence;

2. if $(T_n)_n$ is a Cauchy sequences in $(B_\mathcal{P}(X), \|\bullet\|_\mathcal{P})$ which converges to an operator $T$, we have $T \in B_\mathcal{P}(X)$;

3. the algebra $(B_\mathcal{P}(X), \|\bullet\|_\mathcal{P})$ is complete if $X$ is sequentially complete.

Proposition 38 ([10]) Let $(X, \mathcal{P})$ be a locally convex space. An operator $T \in Q_\mathcal{P}(X)$ is bounded in the algebra $Q_\mathcal{P}(X)$ if and only if there exists some calibration $\mathcal{P}' \in \mathcal{C}(X)$ such that $\mathcal{P} \approx \mathcal{P}'$ and $T \in B_\mathcal{P'}(X)$.

Definition 39 Let $(X, \mathcal{P})$ be a locally convex space and $T \in B_\mathcal{P}(X)$. We said that $\alpha \in \mathbb{C}$ is in resolvent set $\rho(B_\mathcal{P}, T)$ if there exists $(\alpha I - T)^{-1} \in B_\mathcal{P}(X)$. The spectral set $\sigma(B_\mathcal{P}, T)$ will be the complementary set of $\rho(B_\mathcal{P}, T)$.

Remark 40 It is obvious that we have the following inclusions

$$\sigma(T) \subset \sigma(Q_\mathcal{P}, T) \subset \sigma(B_\mathcal{P}, T).$$

Proposition 41 ([4]) If $(X, \mathcal{P})$ is a locally convex space and $T \in B_\mathcal{P}(X)$, then the set $\sigma(B_\mathcal{P}, T)$ is compact.

Lemma 42 If $\mathcal{P}$ is a calibration on a locally convex space $X$, then

$$\|T\|_\mathcal{P} = \sup\{\hat{p}(T) \mid p \in \mathcal{P}, \forall T \in B_\mathcal{P}(X)\}.$$  

Proof. Let be $T \in B_\mathcal{P}(X)$ and $\alpha = \sup\{\hat{p}(T) \mid p \in \mathcal{P}\}$. From the definition of the norm $\|\bullet\|_\mathcal{P}$ and of the operatorial seminorm $\hat{p}$, $p \in \mathcal{P}$, results that $\|T\|_\mathcal{P} \leq \alpha$. If $\|T\|_\mathcal{P} < \alpha$, then there exists some seminorm $p \in \mathcal{P}$ such that

$$\|T\|_\mathcal{P} \leq \hat{p}(T) \leq \alpha.$$  

which implies that there exists some element $x_0 \in X$ for which we have

$$p(Tx_0) > \|T\|_\mathcal{P} p(x_0).$$

Since, this relation contradicts the definition of the norm $\|T\|_\mathcal{P}$, results that $\alpha \leq \|T\|_\mathcal{P}$. 

\[\blacksquare\]
Corollary 43 If $X$ is a locally convex algebra $\mathcal{P} \in \mathcal{C}(X)$, then for each $T \in B_P(X)$ the inequality $r_P(T) \leq \|T\|_P$, holds for each calibration $\mathcal{P}' \in \mathcal{C}(X)$ such that $\mathcal{P} \approx \mathcal{P}'$ and $T \in B_P(X)$.

**Proof.** If $\mathcal{P}' \in \mathcal{C}(X)$ such that $\mathcal{P} \approx \mathcal{P}'$ and $T \in B_P(X)$, then from remark 13 results that the equality $r_P(T) = r_{\mathcal{P}'}(T)$ hold. From corollary 18(1) and the previous lemma results that

$$r_P(T) = r_{\mathcal{P}'}(T) = \sup \left\{ \lim_{n \to \infty} (\hat{p}(T^n))^{1/n} \mid p \in \mathcal{P}' \right\} \leq \sup \{ \hat{p}(T) \mid p \in \mathcal{P}' \} = \|T\|_{\mathcal{P}'}.$$


Corollary 44 If $(X, \mathcal{P})$ is a locally convex space, then for each operator $T \in (Q_{\mathcal{P}}(X))_0$ the spectral set $\sigma(Q_{\mathcal{P}}, T)$ is bounded.

**Proof.** If $T \in (Q_{\mathcal{P}}(X))_0$ then by proposition 38 there exists a calibration $\mathcal{P}' \in \mathcal{C}(X)$ such that $\mathcal{P} \approx \mathcal{P}'$ and $T \in B_{\mathcal{P}'}(X)$. Therefore, the proposition 24 the previous corollary implies that

$$|\sigma(Q_{\mathcal{P}}, T)| \leq r_P(T) \leq \|T\|_{\mathcal{P}'}$$

so, the set $\sigma(Q_{\mathcal{P}}, T)$ is bounded.


Definition 45 Let $(X, \mathcal{P})$ be a locally convex space and $T \in B_P(X)$. We denote by $r(B_P, T)$ the spectral radius of the operator $T$ with respect to the algebra $B_P(X)$ given by the relation

$$r(B_P, T) = \limsup_{n \to \infty} \|T^n\|^{1/n}_P.$$


Proposition 46 Let $(X, \mathcal{P})$ be a sequentially complete convex space and $T \in B_P(X)$. If $\lambda > \|T\|_P$, then the Neumann series $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ converges to $R(\lambda, T)(\text{in } B_P(X))$. Moreover, $|\sigma(B_P, T)| \leq \|T\|_P$.

**Proof.** Since $\|T\|_P \leq 1$ and $B_P(X)$ is a Banach space, the Neumann series $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ is convergent in $B_P(X)$. Let denote $S = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$. From the equalities

$$(\lambda I - T)S = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^n} - \sum_{n=0}^{\infty} \frac{T^{n+1}}{\lambda^{n+1}} = I$$

$$S(\lambda I - T) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^n} - \sum_{n=0}^{\infty} \frac{T^{n+1}}{\lambda^{n+1}} = I$$

results that $S = R(\lambda, T)$. The inequality $|\sigma(B_P, T)| \leq \|T\|_P$ results from the
proof of corollary 44 and proposition 36(3).

**Corollary 47** Let $X$ be a sequentially complete locally convex space and $P \in \mathcal{C}(X)$. Then, for each $T \in B_P(X)$ we have

$$|\sigma(B_P, T)| \leq \liminf_{n \to \infty} \|T^n\|^{1/n}_{B_P} \leq r(B_P, T)$$

**Proof.** From equalities

$$(\lambda^n I - T^n) = (\lambda I - T)(\lambda^{n-1} I + \lambda^{n-2} T + \ldots + \lambda T^{n-2} + T^{n-1}) =$$

$$= (\lambda^{n-1} I + \lambda^{n-2} T + \ldots + \lambda T^{n-2} + T^{n-1})(\lambda I - T),$$

results that if the operator $\lambda^n I - T^n \in B_P(X)$ is invertible in $B_P(X)$, then the operator $(\lambda I - T)$ has the same property.

Therefore, if $\lambda \in \sigma(B_P, T)$, then $\lambda^n \in \sigma(B_P, T^n)$. From the previous proposition we have

$$|\lambda|^n = |\lambda^n| \leq \|T^n\|_{B_P}, (\forall) n \geq 1,$$

which is equivalently with the inequality

$$|\lambda| \leq \|T^n\|^{1/n}_{B_P}, (\forall) n \geq 1,$$

Since $\lambda \in \sigma(B_P, T)$ is arbitrarily chosen results that

$$|\sigma(B_P, T)| \leq \|T^n\|^{1/n}_{B_P}, (\forall) n \geq 1,$$

so

$$|\sigma(B_P, T)| \leq \liminf_{n \to \infty} \|T^n\|^{1/n}_{B_P} \leq \limsup_{n \to \infty} \|T^n\|^{1/n}_{B_P} = r(B_P, T).$$

**References**


