Spectral Sets For Locally Bounded Operators

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Abstract

For a quotient bounded operator $T$ on a locally convex space we define the set $\sigma(Q,T)$. If $T$ is a locally bounded operators on a sequentially complete locally convex space then the equalities

$$r_{lb}(T) = |\sigma(T)| = |\sigma_{lb}(T)| = |\sigma(Q,T)|$$

holds and the spectral set $\sigma(T)$ is compact.

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1 Introduction

Let $X$ be a locally convex Hausdorff space and $L(X)$ ($\mathcal{L}(X)$) denote the algebra of linear operators (continuous operators) from $X$ into itself.

A family $\mathcal{P}$ of seminorms who generate the topology of a locally convex space $X$ (in the sense that the topology of $X$ is the coarsest with respect to which all seminorms of $\mathcal{P}$ are continuous) will be called a calibration on $X$. A calibration on $X$ will be principal if it is directed. The set of calibrations for $X$ is denoted by $\mathcal{C}(X)$ and the set of principal calibration is denoted by $\mathcal{C}_0(X)$.

Any family of seminorms on a linear space is partially ordered by relation $\leq$, where

$$p \leq q \iff p(x) \leq q(x), \forall x \in X.$$ 

A family of seminorms is preordered by the relation $<$, where

$$p < q \iff \text{there exists some } r > 0 \text{ such that } p(x) \leq rq(x), \forall x \in X.$$ 

If $p < q$ and $q < p$, we write $p \approx q$. 

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Definition 1.1 Two families $P_1$ and $P_2$ of seminorms on a linear space are called $Q$-equivalent (denoted $P_1 \approx P_2$) provided:

1. for each $p_1 \in P_1$ there exists $p_2 \in P_2$ such that $p_1 \approx p_2$;
2. For each $p_2 \in P_2$ there exists $p_1 \in P_1$ such that $p_2 \approx p_1$.

It is obviously that two $Q$-equivalent and separating families of seminorms on a linear space generate the same locally convex topology.

If $(X, \mathcal{P})$, $(Y, \mathcal{Q})$ are locally convex spaces, then for all seminorms $p, q \in \mathcal{P}$ the application $m_{pq} : L(X, Y) \to \mathbb{R} \cup \{\infty\}$, defined by

$$m_{pq}(T) = \sup_{p(x) \neq 0} \frac{q(Tx)}{p(x)},$$

is called the mixed operator seminorm of $T$ associated with $p$ and $q$. When $X = Y$ and $p = q$ we use notation $\hat{p} = m_{pp}$.

Lemma 1.1 ([21]) If $(X, \mathcal{P})$ is a locally convex spaces and $T \in L(X)$, then

1. $m_{pq}(T) = \sup_{p(x) = 1} q(Tx) = \sup_{p(x) \leq 1} q(Tx), (\forall) p \in \mathcal{P}, (\forall) q \in \mathcal{Q}$;
2. $q(Tx) \leq m_{pq}(T) p(x), (\forall) x \in X$, whenever $m_{pq}(T) < \infty$.
3. $m_{pq}(T) = \inf \{ M > 0 \mid q(Tx) \leq Mp(x), (\forall) x \in X \}$, whenever $m_{pq}(T) < \infty$.

Definition 1.2 Let $X$ be a locally convex space. An operator $T \in L(X)$ is:

1. quotient bounded operator with respect to a calibration $\mathcal{P} \in \mathcal{C}(X)$ if for every seminorm $p \in \mathcal{P}$ there exists $c_p > 0$ such that

   $$p(Tx) \leq c_p p(x), (\forall) x \in X.$$

2. universally bounded with respect to the calibration $\mathcal{P} \in \mathcal{C}(X)$ if there exists $c_0 > 0$ such that

   $$p(Tx) \leq c_0 p(x), (\forall) x \in X, (\forall) p \in \mathcal{P}.$$

3. locally bounded if maps some zero neighborhood in a bounded set.

Definition 1.3 If $X$ is a locally convex space, then we denote by:
1. $Q_\mathcal{P}(X)$ the class of quotient bounded operators with respect to some calibration $\mathcal{P} \in \mathcal{C}(X)$;

2. $B_\mathcal{P}(X)$ the class of universally bounded operators with respect to some calibration $\mathcal{P} \in \mathcal{C}(X)$,

3. $\mathcal{LB}(X)$ the class of the locally bounded operators on $X$:

4. $\mathcal{LB}_0(X)$ the subalgebra of $\mathcal{L}(X)$ generated by $\mathcal{LB}(X)$ and the identity operator $I$.

**Remark 1.1**

1. Let $X$ be a locally convex space. If $T \in \mathcal{LB}(X)$, then there exists some calibration $\mathcal{P} \in \mathcal{C}(X)$ such that $T \in B_\mathcal{P}(X) \subset Q_\mathcal{P}(X) \subset \mathcal{L}(X)$.

2. If $X$ is a seminormed space, then $Q_\mathcal{P}(X) = B_\mathcal{P}(X) = \mathcal{LB}(X)$.

The class of locally bounded operators is an algebra and it will be usually equipped with the topology of uniform convergence on a zero neighborhood. We say that a sequence $(S_n)_{n \in \mathbb{N}}$ converges uniformly to zero on a zero neighborhood $U$ if for each zero neighborhood $V$ there exists a positive index $n_0 \in \mathbb{N}$ such that $S_n(U) \subset V, \forall n \geq n_0$.

In terms of operator seminorms we have the following definition:

**Definition 1.4** Let $X$ be a locally convex space. A sequence $(S_n)_{n \in \mathbb{N}} \subset \mathcal{LB}(X)$ converges uniformly to zero on some zero-neighborhood if for each principal calibration $\mathcal{P} \in \mathcal{C}(X)$ there exists some seminorm $p \in \mathcal{P}$ such that for every $q \in \mathcal{P}$ and every $\epsilon > 0$ there exists an index $n_\epsilon \in \mathbb{N}$, with the property

$$m_{pq}(S_n) < \epsilon, \forall n \geq n_\epsilon.$$

A family $G \subset \mathcal{LB}(X)$ is uniformly bounded on some zero-neighborhood if there exists some seminorm $p \in \mathcal{P}$ such that for every $q \in \mathcal{P}$ there exists $\epsilon_q > 0$ with the property

$$m_{pq}(S) < \epsilon_q, \forall S \in G.$$

If $X$ is a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$, then for every $p \in \mathcal{P}$ the application $\hat{p} : Q_\mathcal{P}(X) \to \mathbb{R}$ defined by

$$\hat{p}(T) = \inf \{ r > 0 \mid p(Tx) \leq r \, p(x), \forall x \in X \},$$

is a submultiplicative seminorm on $Q_\mathcal{P}(X)$, satisfying $\hat{p}(I) = 1$. We denote by $\mathcal{P}$ the family $\{ \mathcal{P} \mid p \in \mathcal{P} \}$. 

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Proposition 1.1 ([10]) Let $X$ be a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$.

1. $Q_\mathcal{P}(X)$ is a unital subalgebra of the algebra of continuous linear operators on $X$.

2. $Q_\mathcal{P}(X)$ is a unitary l.m.c.-algebra with respect to the topology determined by $\hat{\mathcal{P}}$.

3. If $\mathcal{P}' \in \mathcal{C}(X)$ such that $\mathcal{P} \approx \mathcal{P}'$, then $Q_{\mathcal{P}'}(X) = Q_\mathcal{P}(X)$ and $\hat{\mathcal{P}} \approx \hat{\mathcal{P}}'$.

4. The topology generated by $\hat{\mathcal{P}}$ on $Q_\mathcal{P}(X)$ is finer than the topology of uniform convergence on bounded subsets of $X$.

Definition 1.5 If $(X, \mathcal{P})$ is a locally convex space and $T \in Q_\mathcal{P}(X)$ we denote by $r_\mathcal{P}(T)$ the radius of boundness of operator $T$ in $Q_\mathcal{P}(X)$, i.e.

$$r_\mathcal{P}(T) = \inf \{ \alpha > 0 \mid \alpha^{-1}T \text{ generates a bounded semigroup in } Q_\mathcal{P}(X) \}.$$ 

We have said that $r_\mathcal{P}(T)$ is the $\mathcal{P}$-spectral radius of the operator $T$. Proposition (1.1) implies that for each $\mathcal{P}' \in \mathcal{C}(X)$, $\mathcal{P} \approx \mathcal{P}'$, we have $Q_{\mathcal{P}'}(X) = Q_\mathcal{P}(X)$, so if $H$ is a $Q$-equivalence class in $\mathcal{C}(X)$, then

$$r_\mathcal{P}(T) = r_\mathcal{P}'(T), \forall \mathcal{P}, \mathcal{P}' \in H.$$ 

Proposition 1.2 ([20]) If $X$ is a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$, then for each $T \in Q_\mathcal{P}(X)$ we have:

$$r_\mathcal{P}(T) = \inf \left\{ \lambda > 0 \mid \lim_{n \to \infty} \frac{T^n}{\lambda^n} = 0 \right\}.$$ 

Proposition 1.3 ([20]) Let $X$ be a sequentially complete locally convex space and $\mathcal{P} \in \mathcal{C}(X)$. If $T \in Q_\mathcal{P}(X)$, then $\sigma(Q_\mathcal{P}, T) \subseteq r_\mathcal{P}(T)$.

Definition 1.6 Given a linear operator $T$ on a topological vector space $X$, we consider

$$r_{lb}(T) = \inf \left\{ \nu > 0 \mid \left| \frac{T^n}{\nu^n} \right| \to 0 \text{ uniformly on some zero neighborhood} \right\}.$$ 

Lemma 1.2 If $\mathcal{P}$ a calibration on $X$, then $B_\mathcal{P}(X)$ is a unital normed algebra with respect to the norm $\| \cdot \|_\mathcal{P}$ defined by

$$\|T\|_\mathcal{P} = \inf \{ M > 0 \mid p(Tx) \leq Mp(x), \forall x \in X, \forall p \in \mathcal{P} \}.$$
Corollary 1.1  If $\mathcal{P} \in \mathcal{C}(X)$, then for each $T \in B_\mathcal{P}(X)$ we have

$$\|T\|_\mathcal{P} = \sup\{\hat{p}(T) \mid p \in \mathcal{P}\}, (\forall) T \in B_\mathcal{P}(X).$$

Proof. Let be $T \in B_\mathcal{P}(X)$ and $\alpha = \sup\{\hat{p}(T) \mid p \in \mathcal{P}\}$. From the
definition of the norm $\|\cdot\|_\mathcal{P}$ and of the operatorial seminorm $\hat{p}$, $p \in \mathcal{P}$,
results that $\|T\|_\mathcal{P} \leq \alpha$.

If $\|T\|_\mathcal{P} < \alpha$, then there exists some seminorm $p \in \mathcal{P}$ such that

$$\|T\|_\mathcal{P} < \hat{p}(T) \leq \alpha.$$ 

which implies that there exists some element $x_0 \in X$ for which we have

$$p(Tx_0) > \|T\|_\mathcal{P} p(x_0).$$

Since, this relation contradicts the definition of the norm $\|T\|_\mathcal{P}$, results
that $\alpha \leq \|T\|_\mathcal{P}$.

Proposition 1.4 ([10])  Let $X$ be a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$. Then:

1. $B_\mathcal{P}(X)$ is a subalgebra of $\mathcal{L}(X)$;
2. $(B_\mathcal{P}(X), \|\cdot\|_\mathcal{P})$ is unitary normed algebra;
3. for each $\mathcal{P}' \in \mathcal{C}(X)$, with the property $\mathcal{P} \approx \mathcal{P}'$, we have

$$B_\mathcal{P}(X) = B_{\mathcal{P}'}(X) \text{ and } \|\cdot\|_\mathcal{P} = \|\cdot\|_{\mathcal{P}'}.$$ 

Proposition 1.5 ([4])  Let $X$ be a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$. Then:

1. the topology given by the norm $\|\cdot\|_\mathcal{P}$ on the algebra $B_\mathcal{P}(X)$ is finer
   than the topology of uniform convergence;
2. if $(T_n)_n$ is a Cauchy sequences in $(B_\mathcal{P}(X), \|\cdot\|_\mathcal{P})$ which converges to
   an operator $T$, we have $T \in B_\mathcal{P}(X)$;
3. the algebra $(B_\mathcal{P}(X), \|\cdot\|_\mathcal{P})$ is complete if $X$ is sequentially complete.

Proposition 1.6 ([10])  Let $(X, \mathcal{P})$ be a locally convex space. An operator
$T \in Q_\mathcal{P}(X)$ is bounded in the algebra $Q_\mathcal{P}(X)$ if and only if there exists some
calibration $\mathcal{P}' \in \mathcal{C}(X)$ such that $\mathcal{P} \approx \mathcal{P}'$ and $T \in B_{\mathcal{P}'}(X)$. 

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Definition 1.7 Let \((X, \mathcal{P})\) be a locally convex space.

1. If \(T \in Q_\mathcal{P}(X) (T \in B_\mathcal{P}(X))\) we said that \(\alpha \in \mathbb{C}\) is in the resolvent set \(\rho(Q_\mathcal{P}, T) (\rho(B_\mathcal{P}, T))\) if there exists \((\alpha I - T)^{-1} \in Q_\mathcal{P}(X) ((\alpha I - T)^{-1} \in B_\mathcal{P}(X))\). The spectral set \(\sigma(Q_\mathcal{P}, T) (\sigma(B_\mathcal{P}, T))\) will be the complementary set of \(\rho(Q_\mathcal{P}, T) (\rho(B_\mathcal{P}, T))\).

2. Let \(T\) be a locally bounded operator on locally convex space \(X\). We say that \(\lambda \in \rho_{\text{lb}}(T)\) if there exists a scalar \(\alpha\) and a locally bounded operator \(S\) on \(X\) such that \((\lambda I - T)^{-1} = \alpha I + S\).

Remark 1.2

1. The set \(\rho_{\text{lb}}(T)\) will be the spectrum of \(T\) in the algebra \(\mathcal{L}\mathcal{B}_0(X)\). The spectral set \(\sigma_{\text{lb}}(T)\) is defined to be the complement of the resolvent set \(\rho_{\text{lb}}(T)\).

2. It is obvious that if \(T \in B_\mathcal{P}(X)\), then we have the following inclusions

\[\sigma(T) \subset \sigma(Q_\mathcal{P}, T) \subset \sigma(B_\mathcal{P}, T)\]

Proposition 1.7 ([4]) Proposition If \((X, \mathcal{P})\) is a locally convex space and \(T \in B_\mathcal{P}(X)\), then the set \(\sigma(B_\mathcal{P}, T)\) is compact.

Corollary 1.2 If \((X, \mathcal{P})\) is a locally convex space and \(T \in (Q_\mathcal{P}(X))_0\) then there exist \(c_T > 0\) such that

\[\rho(T) \leq c_T, (\forall) p \in \mathcal{P}\]

Given \((X, \mathcal{P})\), for each \(p \in \mathcal{P}\) denote by \(N^p\) the null space and by \(X_p\) the quotient space \(X/N^p\). For each \(p \in \mathcal{P}\) consider the natural mapping

\[x \rightarrow x_p \equiv x + N^p\]

(from \(X\) to \(X_p\)). It is obvious that \(X_p\) is a normed space, for each \(p \in \mathcal{P}\), with norm \(\| \cdot \|_p\) defined by

\[\|x_p\|_p = p(x), (\forall) x \in X\]

Consider the algebra homomorphism \(T \rightarrow T^p\) of \(Q_\mathcal{P}(X)\) into \(\mathcal{L}(X_p)\) defined by

\[T^p(x_p) = (Tx)_p, (\forall) x \in X\].
This operators are well defined because $T(N^p) \subset N^p$. Moreover, for each $p \in \mathcal{P}$, $\mathcal{L}(X_p)$ is a unital normed algebra and we have

$$\|T_p\|_p = \sup \left\{ \|T_p x_p\|_p \mid \|x_p\|_p \leq 1 \text{ for } x_p \in X_p \right\}$$

$$= \sup \left\{ p(Tx) \mid p(x) \leq 1 \text{ for } x \in X \right\}$$

For $p \in \mathcal{P}$ consider that the normed space $(\tilde{X}_p, \|\cdot\|_p)$ is the completion of $(X_p, \|\cdot\|_p)$. If $T \in Q_{\mathcal{P}}(X)$, then the operator $T^p$ has a unique continuous linear extension $\tilde{T}^p$ on $(\tilde{X}_p, \|\cdot\|_p)$.

**Definition 1.8** If $X$ is a locally convex space and $T \in Q_{\mathcal{P}}(X)$, we denote by $\sigma(Q, T)$ the set

$$\cap \{ \sigma(Q_{\mathcal{P}}, T) \mid \mathcal{P} \in \mathcal{C}(X) \text{ such that } T \in Q_{\mathcal{P}}(X) \}.$$  

**Lemma 1.3** If $X$ is a locally convex space and $T \in Q_{\mathcal{P}}(X)$ then

$$|\sigma(Q, T)| \leq \inf\{ r_{\mathcal{P}}(T) \mid \mathcal{P} \in \mathcal{C}(X) \text{ such that } T \in Q_{\mathcal{P}}(X) \}.$$  

**Proof.** This is a direct consequence of the proposition (1.3).

**Definition 1.9** An operator $T$ is quotient bounded operator on a locally convex space $X$ if there exists some calibration $\mathcal{P}$ on $X$ such that $T \in Q_{\mathcal{P}}(X)$.

**Remark 1.3** An operator $T$ is quotient bounded on a locally convex space $X$ if and only if there exists some calibration $\mathcal{P} \in \mathcal{C}(X)$ such that $\hat{p}(T)$ is finit for each $p \in \mathcal{P}$.

**Lemma 1.4** If $T$ is a quotient bounded operator on a locally convex space $X$, then there exists some principal calibration $\mathcal{P}' \in \mathcal{C}_0(X)$ such that $T \in Q_{\mathcal{P}'}(X)$.

**Proof.** Let $\mathcal{P}$ be a calibration on $X$ such that $T \in Q_{\mathcal{P}}(X)$ and denote by $\mathcal{P}'$ the set of all seminorms given by the relations

$$p'(x) = \max_{i=1,n} p_i(x), \forall x \in X,$$

where $p_i \in \mathcal{P}', i = 1, n$, and $n \in \mathbb{N}$.

Let $p' \in \mathcal{P}'$ be arbitrary chosen. Since $T \in Q_{\mathcal{P}}(X)$, from previous remark and lemma (1.1) results that

$$p_i(Tx) \leq \hat{p}_i(T)p_i(x), \forall x \in X, i = 1, n.$$
If \( c_{p'} = \max_{i=1}^{n} \hat{p}_i(T) \), then
\[
p_i(Tx) \leq c_{p'} p_i(x) \leq c_{p'} p_i'(x), \forall x \in X, \ i = 1, n,
\]
so
\[
p_i'(Tx) \leq c_{p'} p_i'(x), \forall x \in X,
\]
Therefore, \( T \in Q_{p'}(X) \).

**Lemma 1.5** If \( X \) is a locally convex space and \( T \in Q_{p}(X) \) then
\[
\inf \{r_{P}(T) \mid P \in C_0(X) \text{ such that } T \in Q_{P}(X)\} =
\]
\[
= \inf \{r_{P}(T) \mid P \in C(X) \text{ such that } T \in Q_{P}(X)\}.
\]

**Proof.** Assume that \( P \in C(X) \) such that \( T \in Q_{P}(X) \).
If \( |\lambda| > r_{P}(T) \), then the family \( \left( \frac{T_n}{\lambda} \right)_{n \geq 0} \) is bounded in \( Q_{P}(X) \), i.e. for every \( p \in P \) there exists \( \epsilon_{p} > 0 \) such that
\[
\hat{p} \left( \frac{T_n}{\lambda} \right) \leq \epsilon_{p}, (\forall) n \geq 0.
\]
Let \( P' \) be the principal calibration associated with the calibration \( P \), i.e. for each \( p' \in P' \) there exists \( p_1, \ldots, p_n \in P \) such that \( p' = \max \{p_1, \ldots, p_n\} \).
If \( \epsilon_{p'} = \max \{\epsilon_{p_1}, \ldots, \epsilon_{p_n}\} \), then
\[
\hat{p}' \left( \frac{T_n}{\lambda} \right) \leq \epsilon_{p'}, (\forall) n \geq 0.
\]
so \( |\lambda| > r_{P'}(T) \) (proposition (1.2)). Since \( \lambda \) is arbitrary chosen results \( r_{P'}(T) \leq r_{P}(T) \).
Therefore,
\[
\inf \{r_{P}(T) \mid P \in C_0(X) \text{ such that } T \in Q_{P}(X)\} \leq
\]
\[
\leq \inf \{r_{P}(T) \mid P \in C(X) \text{ such that } T \in Q_{P}(X)\}.
\]
The reverse inequality is obvious.

**Lemma 1.6** If \( X \) is a locally convex space and \( T \in Q_{P}(X) \) then
\[
\sigma(Q, T) = \cap \{\sigma(Q_{P}, T) \mid P \in C_0(X) \text{ such that } T \in Q_{P}(X)\}.
\]
Proof. From definition of the set $\sigma(Q,T)$ results the inclusion

$$\sigma(Q,T) \subset \cap \{ \sigma(Q_P,T) \mid P \in C_0(X) \text{ such that } T \in Q_P(X) \}.$$

If $\lambda \notin \sigma(Q,T)$, then there exists some calibration $P \in C(X)$ such that $\lambda \in \rho(Q_P,T)$, so for every $p \in P$ we have $\tilde{p}(R(\lambda,T)) < \infty$.

Denote by $P'$ the principal calibration of all seminorms

$$p'(x) = \max_{i=1,n} p_i(x), (\forall) x \in X,$$

where $p_i \in P', i = \overline{1,n}$, and $n \in \mathbb{N}$.

Let $p' \in P'$ be such seminorm. Since $R(\lambda,T) \in Q_{P}(X)$, the lemma (1.1) implies that

$$p_i(R(\lambda,T)x) \leq \hat{p}_i(R(\lambda,T))p_i(x), (\forall) x \in X, i = \overline{1,n},$$

If $c_{p'} = \max_{i=1,n} \hat{p}_i(R(\lambda,T))$, then

$$p_i(R(\lambda,T)x) \leq c_{p'}p_i(x) \leq c_{p'}p'(x), (\forall) x \in X, i = \overline{1,n},$$

so we have

$$p'(R(\lambda,T)x) \leq c_{p'}p'(x), (\forall) x \in X,$$

Therefore, $R(\lambda,T) \in Q_{P}(X)$ and $\lambda \notin \sigma(Q_{P'},T)$, which implies that

$$\cap \{ \sigma(Q_P,T) \mid P \in C_0(X) \text{ such that } T \in Q_P(X) \} \subset \sigma(Q,T).$$

2 Locally bounded operators

Lemma 2.1 ([12]) If $T_1$ and $T_2$ are locally bounded operators on $X$, then there exists a calibration $P'$ on $X$ such that $T_1, T_2 \in B_{P'}(X)$.

Proposition 2.1 ([3]) Let $T$ be a locally bounded operator on a sequentially complete locally convex space $X$ and $P \in C(X)$ such that $T \in Q_P(X)$. If $p \in P$ such that

$$m_{pq}(T) < \infty, (\forall) q \in P,$$

then $\rho(X_p,T^p) = \rho(\tilde{X}_p, \tilde{T}^p)$.

Proposition 2.2 ([21]) Let $X$ be a sequentially complete locally convex space and $P \in LB(X)$. If $| \lambda | > r_P(T)$, then the Neumann series $\sum_{n=0}^{\infty} \frac{T^n}{n!}$ converges to $R(\lambda,T)$ on a zero neighborhood. Moreover, $| \sigma_{lb}(T) | \leq r_{lb}(T)$. 9
Lemma 2.2 If $X$ is a locally convex space and $T \in \mathcal{LB}(X)$, then

$$r_{lb}(T) = \inf \left\{ \nu > 0 \mid \left( \frac{T^n}{\nu^n} \right)_n \text{ is bounded on a zero neighborhood} \right\}.$$ 

Proof. If $P \in C_0(X)$ we consider

$$r'(T) = \inf \left\{ \nu > 0 \mid \left( \frac{T^n}{\nu^n} \right)_n \text{ is uniformly bounded on a zero neighborhood} \right\}.$$ 

Assume that $\nu > r_{lb}(T)$, i.e. $(\frac{T^n}{\nu^n})_n$ is converges uniformly to zero on a zero neighborhood. Then there exists $p_1 \in P$ such that for each $q \in P$ and every $\epsilon > 0$ there exists $n_{q,\epsilon} \in \mathbb{N}$, with the property

$$m_{p_1 q} \left( \frac{T^n}{\nu^n} \right) < \epsilon, (\forall) n \geq n_{q,\epsilon}.$$ 

Since the operator $T$ is locally bounded there exists $p_2 \in P$ such that for every $q \in P$ we have $m_{p_2 q}(T) < \infty$. But the calibration $P$ is principal, so there exists $p_0 \in P$ such that $p_1 \leq p_0$ and $p_2 \leq p_0$.

Let $\epsilon > 0$ be arbitrary fixed. For each $q \in P$ we have

$$m_{p_0 q} \left( \frac{T^n}{\nu^n} \right) = \sup \left\{ q \left( \frac{T^n}{\nu^n} x \right) \mid p_0(x) \leq 1 \right\} \leq$$

$$\leq \sup \left\{ q \left( \frac{T^n}{\nu^n} x \right) \mid p_1(x) \leq 1 \right\} = m_{p_1 q} \left( \frac{T^n}{\nu^n} \right), (\forall) n \geq n_{q,\epsilon}.$$ 

$$m_{p_0 q} \left( \frac{T}{\nu} \right) = \sup \left\{ q \left( \frac{T}{\nu} x \right) \mid p_0(x) \leq 1 \right\} \leq$$

$$\leq \sup \left\{ q \left( \frac{T}{\nu} x \right) \mid p_2(x) \leq 1 \right\} = m_{p_2 q} \left( \frac{T}{\nu} \right).$$

Moreover, from lemma (1.1) we have

$$q \left( \frac{T^k}{\nu^k} x \right) \leq m_{p_2 q} \left( \frac{T}{\nu} \right) p_2 \left( \frac{T^{k-1}}{\nu^{k-1}} x \right) \leq \cdots \leq m_{p_2 q} \left( \frac{T}{\nu} \right) m_{p_2 p_2} \left( \frac{T}{\nu} \right)^{k-1} p_2(x),$$

for every, $x \in X$ where $k = 1, n_{q,\epsilon}$, so lemma (1.1) implies that

$$m_{p_2 q} \left( \frac{T^k}{\nu^k} \right) \leq m_{p_2 q} \left( \frac{T}{\nu} \right) m_{p_2 p_2} \left( \frac{T}{\nu} \right)^{k-1}, k = 1, n_{q,\epsilon}.$$
So, if $n_{q,\epsilon} \geq 1$ and
\[
\alpha_q = \max \left\{ \epsilon, m_{pq} \left( \frac{T}{\nu} \right), \ldots, m_{pq} \left( \frac{T}{\nu} \right) m_{pq} \left( \frac{T}{\nu} \right)^{n_{q,\epsilon}-1} \right\},
\]
then
\[
m_{pq} \left( \frac{T^n}{\nu^n} \right) \leq \alpha_q, (\forall) n \in \mathbb{N},
\]
This means that the family $\left( \frac{T^n}{\nu^n} \right)_n$ is uniformly bounded on a zero neighborhood, so $r'(T) \leq \nu$. Since $\nu > r_{lb}(T)$ is arbitrary chosen results that $r'(T) \leq r_{lb}(T)$.

Now we prove the opposite inequality. If $\alpha > r'(T)$, then there exists $\beta \in (r'(T), \alpha)$, such that $\left( \frac{T^n}{\nu^n} \right)_n$ is uniformly bounded on a zero neighborhood, i.e. there exists $p_0 \in \mathcal{P}$ such that for every $q \in \mathcal{P}$ there exists $\beta_q > 0$ with he property
\[
m_{pq} \left( \frac{T^n}{\beta^n} \right) < \beta_q, (\forall) n \in \mathbb{N}.
\]
Therefore,
\[
m_{pq} \left( \frac{T^n}{\alpha^n} \right) = \left( \frac{\beta}{\alpha} \right)^n m_{pq} \left( \frac{T^n}{\beta^n} \right) < \left( \frac{\beta}{\alpha} \right)^n \beta_q, (\forall) n \in \mathbb{N}.
\]
Since $\frac{\beta}{\alpha} < 1$ results that for each $q \in \mathcal{P}$ and every $\epsilon > 0$ there exists $n_{q,\epsilon} \in \mathbb{N}$, with the property
\[
\left( \frac{\beta}{\alpha} \right)^n \beta_q < \epsilon, (\forall) n \geq n_{q,\epsilon},
\]
Therefore
\[
m_{pq} \left( \frac{T^n}{\alpha^n} \right) < \epsilon, (\forall) n \geq n_{q,\epsilon}
\]
and $\left( \frac{T^n}{\alpha^n} \right)_n$ is uniformly bounded on a zero neighborhood and $r_{lb}(T) \leq \alpha$. But $\alpha > r'(T)$ is arbitrary chosen, so $r_{lb}(T) \leq r'(T)$.

Proposition 2.3 If $X$ is a locally convex space and $T \in \mathcal{LB}(X)$, then
\[
\sigma_{lb}(T) = \sigma(Q, T) = \sigma(T).
\]
**Proof.** The inclusion $\sigma(T) \subset \sigma(Q,T)$ is obvious. If $\lambda \notin \sigma_{lb}(T)$, then there exists $\alpha \in \mathbb{C}$ and $S \in \mathcal{L}(X)$ such that $(\lambda I - T)^{-1} = \alpha I + S$. From lemma (2.1) there exists $P \in \mathcal{C}(X)$ such that $T, S \in B_P(X)$ and $(\lambda I - T)^{-1} \in B_P(X) \subset Q_P(X)$. Therefore, $\lambda \notin \sigma(Q,P,T)$ and $\sigma(Q,T) \subset \sigma_{lb}(T)$.

Now we prove that $\sigma_{lb}(T) \subset \sigma(T)$. If $X$ is a locally bounded space, then the $X$ is seminormed space, so the remark (1.1) implies that $\sigma_{lb}(T) = \sigma(T)$.

Assume that $X$ is not locally bounded. If $\lambda \notin \sigma(T), \lambda \neq 0$, then $R(\lambda, T)$ is continuous. From equality $R(\lambda, T)(\lambda I - T) = I$ results

$$R(\lambda, T) = \frac{1}{\lambda} R(\lambda, T)T + \frac{1}{\lambda} I$$

which implies that the operator $R(\lambda, T)T$ is locally bounded, so $\lambda \notin \sigma_{lb}(T)$.

If $\lambda = 0 \notin \sigma(T)$, then the operator $R(0,T) = (-T)^{-1}$ is continuous. Since $I = (-T)^{-1}(-T)$, the identity operator is locally bounded, which is not true because $X$ is not a locally bounded space. Therefore, $\lambda = 0$ belongs to each sets $(\sigma(T), \sigma(Q,T)$, respectively $\sigma_{lb}(T))$ in the case when $X$ is not a locally bounded space. Moreover $\sigma_{lb}(T) \subset \sigma(T)$.

**Proposition 2.4** If $T$ is a locally bounded operator on a locally convex space $X$, then

$$r_{lb}(T) = \inf \{ r_P(T) \mid P \in \mathcal{C}(X) \text{ such that } T \in Q_P(X) \}.$$

**Proof.** Denote

$$r'(T) = \inf \{ r_P(T) \mid P \in \mathcal{C}(X) \text{ such that } T \in Q_P(X) \}.$$

If $\lambda > r_{lb}(T)$ and $P \in \mathcal{C}_0(X)$, such that $T \in Q_P(X)$, then there exists $\mu \in (r_{lb}(T), \lambda)$ such that the sequences $\left(\frac{T_n}{\mu^n}\right)_{n \geq 1}$ converges to zero on a zero neighborhood, i.e. there exists $p \in P$ such that for each $q \in P$ and every $\epsilon > 0$ there exists $n_{q,\epsilon} \in \mathbb{N}$, with the property

$$m_{pq} \left(\frac{T_n}{\mu^n}\right) < \epsilon, \ (\forall) \ n \geq n_{q,\epsilon}.$$

If we consider the family $Q = \{ q_m \mid m \geq 1, q \in \mathcal{P} \}$, where

$$q_m(x) = \max \{ mp(x), q(x) \}, (\forall) \ x \in X,$$

then $Q \in \mathcal{C}(X)$ and $T$ is quotient bounded with respect to the calibration $Q$. If $q_m \in Q$, then

$$q_m \left(\frac{T_n}{\mu^n}x\right) = \max \left\{ mp \left(\frac{T_n}{\mu^n}x\right), q \left(\frac{T_n}{\mu^n}x\right) \right\} \leq \frac{T_n}{\mu^n}.$$
\[
\max \left\{ \tilde{m} p \left( \frac{T^n}{\mu^n} \right) p(x), m_{pq} \left( \frac{T^n}{\mu^n} \right) p(x) \right\} \leq \max \left\{ \varepsilon p(x), \varepsilon p(x) \right\} = m\varepsilon p(x) \leq \varepsilon m(x),
\]
for every
\[
n \geq n_{q_{m,\varepsilon}} = \max \{n_{p,\varepsilon}, n_{q,\varepsilon}\}.
\]
Therefore
\[
\hat{q}_m \left( \frac{T^n}{\mu^n} \right) \leq \varepsilon, (\forall) n \geq n_{q_{m,\varepsilon}},
\]
and since \( \frac{\mu}{\lambda} < 1 \), results
\[
\hat{q}_m \left( \frac{T^n}{\mu^n} \right) = \left( \frac{\mu}{\lambda} \right)^n \hat{q}_m \left( \frac{T^n}{\mu^n} \right) < \varepsilon, (\forall) n \geq n_{q_{m,\varepsilon}}.
\]

Then \( r_{Q}(T) \leq \lambda \) (proposition (1.2)). This means that \( r'(T) \leq \lambda \), for every \( |\lambda| > r_{lb}(T) \), so \( r'(T) \leq r_{lb}(T) \).

Now we will prove the reverse inequality. If \( \lambda > r'(T) \), then lemma (1.5) and proposition (1.2) implies that there exists \( \mu \in [r'(T), \lambda) \) and \( P \in C_\mathcal{Q}(X) \) such that the sequence \( \left( \frac{T^n}{\mu^n} \right)_{n \geq 1} \) converges to zero in \( Q_{P}(X) \), i.e. for every \( \varepsilon > 0 \) and each \( p \in P \) there exists \( n_{p,\varepsilon} \in N \), with the property
\[
\hat{p} \left( \frac{T^n}{\mu^n} \right) < \varepsilon, (\forall) n \geq n_{p,\varepsilon}.
\]
But \( T \) is locally bounded, so there exists \( q \in P \) with the property that for each \( p \in P \) there exists \( c_p > 0 \) such that \( m_{qp} \left( \frac{T}{\mu} \right) < c_p \). Therefore, for every \( \varepsilon > 0 \) and each \( p \in P \) there exists \( n_{p,\varepsilon} \in N \), such that
\[
p \left( \frac{T^{n+1}}{\lambda^{n+1}} x \right) = \left( \frac{\mu}{\lambda} \right)^{n+1} p \left( \frac{T}{\mu} \left( \frac{T^n}{\mu^n} x \right) \right) \leq \left( \frac{\mu}{\lambda} \right)^{n+1} m_{qp} \left( \frac{T}{\mu} \right) q \left( \frac{T^n}{\mu^n x} \right) \leq
\]
\[
< \left( \frac{\mu}{\lambda} \right)^{n+1} c_p eq(x) < eq(x), (\forall) x \in X,
\]
for every \( n \geq n_{p,\varepsilon} \). Lemma (1.1) implies that for every \( \varepsilon > 0 \) and each \( p \in P \) there exists \( n_{p,\varepsilon} \in N \), such that
\[
m_{qp} \left( \frac{T^n}{\lambda^n} \right) < \varepsilon, (\forall) n \geq n_{p,\varepsilon},
\]
so the sequence \( \left( \frac{T^n}{\lambda^n} \right)_{n \geq 1} \) converges to zero on a zero neighborhood and \( r_{lb}(T) \leq \lambda \). Moreover, \( r_{lb}(T) \leq r'(T) \).
Lemma 2.3 Let $T$ be a locally bounded operator on a locally convex space $X$ and $P \in C(X)$, such that $T \in Q_P(X)$. If $p \in P$ such that

$$m_{pq}(T) < \infty, \forall q \in P,$$

and $\lambda \in \rho(T)$ has the property $\hat{p}(R(\lambda, T)) < \infty$, then $\lambda \in \rho(X_p, T^p)$ and

$$R(\lambda, T^p)(x + N_p) = R(\lambda, T)(x) + N_p, (\forall) x \in X.$$

**Proof.** Denote by $I$ the identity operator on the space $X$. From the definition of $T^p$ results that $\lambda \in \rho(X_p, T^p)$ and the $S: X^p \to X^p$, given by

$$S(x + N_p) = R(\lambda, T)(x) + N_p, (\forall) x \in X,$$

is a continuous linear operator on $X_p$. Moreover,

$$S(\lambda^p - T^p)(x + N_p) = S((\lambda I - T)(x + N_p)) = S((\lambda I - T)(x) + N_p) =
$$

$$= R(\lambda, T)(\lambda I - T)(x) + N_p = x + N_p.$$

$$(\lambda^p - T^p)S(x + N_p) = (\lambda^p - T^p)S(x + N_p) =
$$

$$= (\lambda^p - T^p)(R(\lambda, T)(x) + N_p) =
$$

$$(\lambda I - T)^p(R(\lambda, T)(x) + N_p) = (\lambda I - T)R(\lambda, T)(x) + N_p = x + N_p,$$

for every $x \in X$, so $S = R(\lambda, T^p)$.

Proposition 2.5 If $X$ is a sequentially complete locally convex space and $T \in LB(X)$, then

$$r_{lb}(T) = |\sigma(T)| = |\sigma_{lb}(T)| = |\sigma(Q, T)|.$$

**Proof.** Let remind to us that $\sigma_{lb}(T) = \sigma(Q, T) = \sigma(T)$ and

$$|\sigma(T)| = |\sigma_{lb}(T)| = |\sigma(Q, T)| \leq r_{lb}(T).$$

Let $P \in C_0(X)$ be arbitrary chosen such that $T \in Q_P(X)$. Since $T$ is locally bounded there exists $p \in P$ such that

$$m_{pq}(T) < c_q < \infty, (\forall) q \in P.$$
We prove that $\sigma(X_p, T^p) \subset \sigma_{lb}(T)$ and $r_{lb}(T) \leq r(X_p, \tilde{T}^p)$, where $r(\tilde{X}_p, \tilde{T}^p)$ is the spectral radius of $\tilde{T}^p$ in algebra $L(\tilde{X}_p)$.

If $p \leq p_1$, then

$$m_{p,q}(T) \leq m_{pq}(T) < c_q \leq \infty, (\forall) q \in \mathcal{P}.$$  

First we prove that $\sigma(\tilde{X}_p, \tilde{T}^p) \subset \sigma_{lb}(T)$. If $\lambda < r_{lb}(T)$, then $\lambda \in \rho(T)$ (proposition (2.3)). Assume that $r_{lb}(T) < 1$.

**Case I.** If $\rho_{lb}(T) < | \lambda |$, then the series $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{k+1}}$ converges uniformly on a zero neighborhood to $R(\lambda, T)$ (proposition (2.2)), i.e. there exists some seminorm $q_0 \in \mathcal{P}$ such that for each $q \in \mathcal{P}$ and every $\epsilon > 0$ there exists $n_{q, \epsilon} \in \mathbb{N}$, with the property

$$m_{q_0 q} \left( R(\lambda, T) - \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}} \right) < \epsilon, (\forall) n \geq n_{q, \epsilon}. $$

From lemma (2.2) results that $(\frac{T^n}{\lambda^n})_n$ is uniformly bounded on a zero neighborhood, i.e. there exists some seminorm $q_1 \in \mathcal{P}$ such that for each $q \in \mathcal{P}$ there exists $\beta_q > 0$ with the property

$$m_{q_1 q} \left( \frac{T^n}{\lambda^n} \right) < \beta_q, (\forall) n \in \mathbb{N}. $$

Since $\mathcal{P}$ is a principal calibration there exists $q_2 \in \mathcal{P}$ such that $q_1 \leq q_2$ and $q_0 \leq q_2$. Therefore, for each $q \in \mathcal{P}$ we have

$$q(R(\lambda, T) x) \leq q \left( R(\lambda, T) - \sum_{k=0}^{n_{q, \epsilon}} \frac{T^k}{\lambda^{k+1}} \right) x + q \left( \sum_{k=0}^{n_{q, \epsilon}} \frac{T^k}{\lambda^{k+1}} x \right) \leq$$

$$\leq m_{q_0 q} \left( R(\lambda, T) - \sum_{k=0}^{n_{q, \epsilon}} \frac{T^k}{\lambda^{k+1}} \right) q_0(x) + \sum_{k=0}^{n_{q, \epsilon}} q \left( \frac{T^k}{\lambda^{k+1}} x \right) \leq$$

$$\leq m_{q_0 q} \left( R(\lambda, T) - \sum_{k=0}^{n_{q, \epsilon}} \frac{T^k}{\lambda^{k+1}} \right) q_0(x) + \left( \lambda^{-1} + \sum_{k=1}^{n_{q, \epsilon}} m_{q_1 q} \left( \frac{T^k}{\lambda^{k+1}} \right) q_1(x) \right) \leq$$

$$\leq (\epsilon + \lambda^{-1} n_{q, \epsilon} \beta_q + \lambda^{-1}) q_2(x), (\forall) x \in X. $$

so, for each $q \in \mathcal{P}$ there exists

$$\gamma_q = \epsilon + \lambda^{-1} n_{q, \epsilon} \beta_q + \lambda^{-1} > 0$$

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such that
\[ q(R(\lambda, T) x) \leq \gamma_q q_2(x), \forall x \in X. \]

Let \( p_1 \in \mathcal{P} \) such that \( q_2 \leq p_1 \) and \( p \leq p_1 \). Then
\[ q(R(\lambda, T) x) \leq \gamma_q p_1(x), \forall x \in X, \]
and
\[ m_{p_1q}(R(\lambda, T)) < \infty, \forall q \in \mathcal{P}, \tag{1} \]

Moreover, \( \tilde{\rho}(R(\lambda, T)) < \infty \). From the remark we made at the beginning of the proof results that
\[ m_{p_1q}(T) \leq m_{pq}(T) < c_q < \infty. \]

Without the lost of generality of the proof we can replace the seminorm \( p \) initially used with \( p_1 \). Therefore, the condition of lemma (2.3) are fulfilled, so \( \lambda \in \rho(X_p, T^p) \) and \( \lambda \in \rho(\tilde{X}_p, \tilde{T}^p) \) (proposition (2.1)).

If \( |\lambda| \leq r_\delta(T) \), then \( r_\delta(T) < 1 < |\lambda^{-1}| \), so
\[ \lambda^{-1} \in \rho(X_p, T^p) = \rho(\tilde{X}_p, \tilde{T}^p). \]

Since \( \lambda, \lambda^{-1} \in \rho(T) \), from equalities
\[ (\lambda^{-1} - \lambda) R(\lambda, T) (\lambda^{-1} I - T) \left[ (\lambda^{-1} - \lambda)^{-1} I - (\lambda^{-1} I - T)^{-1} \right] = \]
\[ = R(\lambda, T) \left[ (\lambda^{-1} I - T) - (\lambda^{-1} - \lambda) I \right] = R(\lambda, T) (\lambda I - T) = I \]
results that
\[ \left[ (\lambda^{-1} - \lambda)^{-1} I - (\lambda^{-1} I - T)^{-1} \right]^{-1} = (\lambda^{-1} - \lambda) R(\lambda, T) \left( \lambda^{-1} I - T \right). \tag{2} \]

Therefore, \( (\lambda - \lambda^{-1})^{-1} \in \rho(R(\lambda^{-1}, T)) \). From relation 1 results that
\[ m_{pq}(R(\lambda^{-1}, T)) < \infty, \forall q \in \mathcal{P}, \]
so from lemma (2.3) and proposition (2.1) results that the operator
\[ (\lambda^{-1} - \lambda)^{-1} \tilde{T}^p - \left( \lambda^{-1} \tilde{T}^p - \tilde{T}^p \right)^{-1} \]
is invertible and continuous on the Banach space \( \tilde{X}_p \). Therefore
\[ \left[ (\lambda^{-1} - \lambda)^{-1} \tilde{T}^p - \left( \lambda^{-1} \tilde{T}^p - \tilde{T}^p \right)^{-1} \right]^{-1} \in L(\tilde{X}_p). \]
If we put the relation (2) under the form
\[
R(\lambda, T) = (\lambda^{-1} - \lambda)^{-1} \left[ (\lambda^{-1} - \lambda)^{-1} I - (\lambda^{-1} I - T)^{-1} \right]^{-1} (\lambda^{-1} I - T)^{-1}
\]
will result that \( \lambda \in \rho(\hat{X}_p, \hat{T}^p) \).

**Case II.** Assume that \( 1 \leq r_{lb}(T) < c < \infty \) (since \( T \) is locally bounded). Then, the operator \( T_1 = c^{-1} T \) is locally bounded and has the property requested at the beginning of the proof with respect to the seminorm \( p \) and \( r_{lb}(T_1) < 1 \).

Therefore, we are in the conditions of the case I, so
\[
c^{-1} \rho_{lb}(T) = \rho_{lb}(T_1) \subset \rho(\hat{X}_p, \hat{T}_1^p) = c^{-1} \rho(\hat{X}_p, \hat{T}^p),
\]
This will show that
\[
\sigma(\hat{X}_p, \hat{T}^p) \subset \sigma_{lb}(T).
\]

Now we prove that \( r_{lb}(T) \leq r(\hat{X}_p, \hat{T}^p) \). If \( \lambda > r(\hat{X}_p, \hat{T}^p) \), then there exists \( \mu \in (r(\hat{X}_p, \hat{T}^p), \lambda) \) such that the sequence \( \left( \frac{(T^p)^n}{\mu^n} \right)_{n \geq 1} \) converges to zero in \( L(\hat{X}_p) \), i.e. for every \( \epsilon > 0 \) there exists some index \( n_{p,\epsilon} \in \mathbb{N} \) such that
\[
\hat{p} \left( \frac{T^n}{\mu^n} \right) = \left\| \frac{(T^p)^n}{\mu^n} \right\|_p < \epsilon, (\forall) n \geq n_{p,\epsilon}.
\]

Therefore, for each \( q \in \mathcal{P} \), every \( \epsilon > 0 \) and every \( x \in X \) we have
\[
q \left( \frac{T^{n+1}}{\lambda^{n+1}} x \right) = \left( \frac{\mu}{\lambda} \right)^{n+1} q \left( \frac{T^{n+1}}{\mu^{n+1}} x \right) \leq \left( \frac{\mu}{\lambda} \right)^{n+1} m_{pq}(T) p \left( \frac{T^n}{\mu^n} x \right) \leq \lambda^{-1} \left( \frac{\mu}{\lambda} \right)^n m_{pq}(T) \frac{T^n}{\lambda^n} p(x) \leq \lambda^{-1} c_q \left( \frac{\mu}{\lambda} \right)^n \epsilon p(x),
\]
But \( \frac{\mu}{\lambda} < 1 \), so there exists some index \( n_1 \in \mathbb{N} \) such that
\[
\lambda^{-1} c_q \left( \frac{\mu}{\lambda} \right)^n < 1, (\forall) n \geq n_1,
\]
which implies that
\[
q \left( \frac{T^{n+1}}{\lambda^{n+1}} x \right) < \epsilon p(x).
\]
for all natural number \( n \) (sufficiently large) and every \( x \in X \). This shows that for each \( q \in \mathcal{P} \) and every \( \epsilon > 0 \) there exists some index \( n_{q, \epsilon} \in \mathbb{N} \) such that

\[
m_{pq} \left( \frac{T_n^q}{x^n} \right) < \epsilon, \quad (\forall) \ n \geq n_{q, \epsilon},
\]

i.e. the sequences \( \left( \frac{T_n^q}{x^n} \right)_n \) converges uniformly to zero on a zero neighborhood, so \( \lambda > r_{lb}(T) \). Since \( \lambda > r(\tilde{X}_p, \tilde{T}^p) \) is arbitrary chosen results \( r_{lb}(T) \leq r(\tilde{X}_p, \tilde{T}^p) \). Therefore, we have

\[
| \sigma(\tilde{X}_p, \tilde{T}^p) | \leq | \sigma(Q, T) | = | \sigma_{lb}(T) | \leq r_{lb}(T) \leq r(\tilde{X}_p, \tilde{T}^p),
\]

Since \( \tilde{T}^p \) is a bounded operator on the Banach space \( \tilde{X}_p \), so

\[
| \sigma(\tilde{X}_p, \tilde{T}^p) | = r(\tilde{X}_p, \tilde{T}^p)
\]

and

\[
r_{lb}(T) = | \sigma(T) | = | \sigma_{lb}(T) | = | \sigma(Q, T) |.
\]

\[\blacktriangleleft\]

Remark 2.1 Maeda [13] prove that if \( X \) is a quasi-complete locally convex space and \( T \in \mathcal{L}B(X) \), then the Waelbroeck spectrum in \( \mathcal{L}B_0(X) \) and classical spectrum of \( T \) are equal and \( \sigma(T) \) is compact.

Proposition 2.6 If \( X \) is a sequentially complete locally convex space and \( T \in \mathcal{L}B(X) \), then \( \sigma(T) \) is compact.

Proof. Assume that \( \mathcal{P} \in \mathcal{C}(X), \) satisfies the condition \((\lambda I - T)^{-1}, T \in B_{\mathcal{P}}(X)\), and \( \mu \in \mathbb{C} \).

Let remind to us that \( \sigma(T) = \sigma_{lb}(T) \). If \( \lambda \in \rho_{lb}(T) \), then from definition of \( \rho_{lb} \) result that there exists some scalar \( \alpha \in \mathbb{C} \) and some locally bounded operator \( S \) such that \( (\lambda I - T)^{-1} = \alpha I + S \). The lemma (2.1) implies that there exists some calibration \( \mathcal{P} \in \mathcal{C}(X) \) such that \( T, S \in B_{\mathcal{P}}(X) \). Moreover, we have \((\lambda I - T)^{-1} \in B_{\mathcal{P}}(X) \). Let \( \mu \in \mathbb{C} \) such that \( |\mu| \leq \left\| (\lambda I - T)^{-1} \right\|_{\mathcal{P}}^{-1} \).

We will show that \( \lambda + \mu \in \rho(T) \). Since

\[
\left\| \mu (\lambda I - T)^{-1} \right\|_{\mathcal{P}} < 1,
\]

the operatorial series

\[
\sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-(k+1)}
\]

converges in operatorial norm of the space \( B_{\mathcal{P}}(X) \). By proposition (1.5) the algebra \( B_{\mathcal{P}}(X) \) is complete, so there exists an operator \( R(\mu) \in B_{\mathcal{P}}(X) \) such that

\[
\sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-(k+1)} = R(\mu).
\]

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Using the equalities

\[
((\lambda + \mu) I - T) R(\mu) = (\lambda I - T) R(\mu) + \mu R(\mu) =
\]

\[
= (\lambda I - T) \left( \sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-(k+1)} \right) + \mu \left( \sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-(k+1)} \right) =
\]

\[
= \sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-k} - \sum_{k=0}^{\infty} (-\mu)^{k+1} (\lambda I - T)^{-(k+1)} =
\]

\[
= IR(\mu) ((\lambda + \mu) I - T) = \mu R(\mu) + R(\mu) (\lambda I - T) =
\]

\[
= \mu \left( \sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-(k+1)} \right) + \left( \sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-(k+1)} \right) + \lambda I - T =
\]

\[
= - \sum_{k=0}^{\infty} (-\mu)^{k+1} (\lambda I - T)^{-(k+1)} + \sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-k} = I
\]

we prove that \( \lambda + \mu \in \rho(B_P, T) \subset \rho(T) \). Therefore

\[
\{ \beta \mid \beta - \lambda < \| (\lambda I - T)^{-1} \|^{-1}_{\mathcal{P}} \} \subset \rho(T),
\]

Since \( \lambda \in \rho_{lb}(T) \) is an arbitrary chosen result that the set \( \rho(T) \) is open, so the spectral set \( \sigma(T) \) is closed. The spectral set \( \sigma(B_P, T) \) is compact, so from the inclusion \( \sigma(T) \subset \sigma(B_P, T) \) results that the set \( \sigma(T) \) is compact.

**Remark 2.2** The function \( \mu \to R(\mu) = ((\lambda + \mu) I - T)^{-1} \) is analytic at the point \( \mu = 0 \).

**Corollary 2.1** Let \( X \) be a sequentially complete locally convex space and \( T \in \mathcal{LB}(X) \). If \( \lambda \in \rho(T) \) and \( d(\lambda) \) is the distance from \( \lambda \) to the set \( \sigma(T) \), then

\[
\| (\lambda I - T)^{-1} \|_{\mathcal{P}} \geq \frac{1}{d(\lambda)},
\]

whenever \( \mathcal{P} \in \mathcal{C}(X) \), such that \( (\lambda I - T)^{-1}, T \in B_P(X) \).

**Proof.** Assume that \( \mathcal{P} \in \mathcal{C}(X) \), satisfies the condition \( (\lambda I - T)^{-1}, T \in B_P(X) \), and \( \mu \in \mathcal{C} \) such that \( |\mu| < \| (\lambda I - T)^{-1} \|^{-1}_{\mathcal{P}} \). Then from the proof of previous proposition results that

\[
\lambda + \mu \in \rho(B_P, T) \subset \rho(T),
\]

so \( \| (\lambda I - T)^{-1} \|^{-1}_{\mathcal{P}} \leq d(\lambda) \).
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References


