Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume **3** (2008), 195 – 209

A SURVEY ON DILATIONS OF PROJECTIVE ISOMETRIC REPRESENTATIONS

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Abstract. In this paper we present Laca-Raeburn's dilation theory of projective isometric representations of a semigroup to projective isometric representations of a group [4] and Murphy's proof of a dilation theorem more general than that proved by Laca and Raeburn. Murphy applied the theory which involves positive definite kernels and their Kolmogorov decompositions to obtain the Laca-Raeburn dilation theorem [6].

We also present Heo's dilation theorems for projective representations, which generalize Stinespring dilation theorem for covariant completely positive maps and generalize to Hilbert C^*-modules the Naimark-Sz-Nagy characterization of positive definite functions on groups [2].

In the last part of the paper it is given the dilation theory obtained in [6] in the case of unitary operator-valued multipliers [3].

1 Introduction

Throughout this paper the term *semigroup* will signify a semigroup with unit. A *subsemigroup* of a semigroup signifies a subset closed under the operation and containing the unit. We shall usually write the operation multiplicatively and denote the unit by e.

An involution on a semigroup S is a function $s \longmapsto s^*$ from S to itself having the properties $(st)^* = t^*s^*$ and $(s^*)^* = s$, for all $s, t \in S$. We call a pair consisting of a semigroup together with an involution a $*$-*semigroup*. If for all $x \in G$, there are $s, t \in S$ such that $x = s^{-1}t$, then we say that S *generates* G.

A subsemigroup S of a group G is *normal* if $xSx^{-1} \subseteq S$ for all $x \in G$.

A *von Neumann algebra* $\mathcal{M}$ is a *-algebra of bounded operators on a Hilbert space H that is closed in the weak operator topology and contains the identity operator.

Definition 1. *([3]) Let S be a semigroup with the unit e and let $\mathcal{M}$ be a von*

2000 Mathematics Subject Classification: 20C25; 43A35; 43A65; 46L45; 47A20.
Keywords: multiplier; isometric projective representation; positive definite kernel; Kolmogorov decomposition; dilation.

Neumann algebra on a Hilbert space H*.* The $\mathcal{U}(\mathcal{M})$-multiplier *on* S *is a* $\mathcal{U}(Z(\mathcal{M}))$*-valued map defined on* $S \times S$ *satisfying :*

(i) $\omega(e,s) = \omega(s,e) = 1$*;*

(ii) $\omega(s,t)\omega(st,u) = \omega(s,tu)\omega(t,u)$*, for all* $s,t,u \in S$*.*

Remark 2. *([3]) If* $\mathcal{M}$ *is a factor, i.e.* $Z(\mathcal{M}) = \mathbb{C}I$*, then the* $\mathcal{U}(\mathcal{M})$*-multiplier coincides with the unit circle* $\mathbf{T}$*-valued multiplier that we shall use in Section 2.*

Definition 3. *([3]) Let* S *be a semigroup with unit, let* $\mathcal{M}$ *be a be a von Neumann algebra on a Hilbert space* H *and let* ω *be a* $\mathcal{U}(\mathcal{M})$*-multiplier on* S*. A* projective isometric ω-representation *of* S *is a map* $\rho\colon S \to \mathcal{M}$ *having the following properties for all* $s,t \in S$ *:*

(i) $\rho(s)$ *is an isometry and* $\rho(e) = 1$*;*

(ii) $\rho(st) = \omega(s,t)\rho(s)\rho(t)$*.*

If $\rho(s)$ is unitary for $s \in S$, we say that ρ is a *projective unitary* ω*-representation*. If ρ is a projective isometric ω-representation of a group G, then ρ is automatically a projective unitary ω-representation, in fact $\rho(s)^* = \omega(s^{-1},s)\rho(s^{-1})$ for all $s \in G$.

Remark 4. *In particular, if* $\mathcal{M} = B(H)$*, we obtain the definition of the projective isometric* ω*-representation that we shall use in Section 2.*

Definition 5. *([6]) Let* X *be a non-empty set, let* H *be a Hilbert space and let* $B(H)$ *be the Banach algebra of all bounded operators on* H*. A map* k *from* $X \times X$ *to* $B(H)$ *is a* positive definite kernel *if for every positive integer* n *and* $x_1,\ldots,x_n \in X$*, the operator matrix* $(k(x_i,x_j))_{ij}$ *in the* C^**-algebra* $M_n(B(H))$ *is positive, i.e.* $\sum_{i,j} \langle k(x_i,x_j)h_j, h_i\rangle \geq 0$ *for all* $h_1,\ldots,h_n \in H$ *and* $x_1,\ldots,x_n \in X$*.*

Definition 6. *([6]) If* k *can be written in the form* $k(x,y) = V(x)^*V(y)$*, where* $V\colon X \to B(H,H_V)$*, for some Hilbert space* H_V*, then* k *is automatically positive definite. Such a map* V *is said to be a* Kolmogorov decomposition of k*. Moreover, if, in addition,* H_V *is the closed linear span of the set* $\bigcup_x V(x)H$*, then* V *is said to be* minimal*.*

Definition 7. *([3]) Let* G *be a group, let* $\mathcal{M}$ *be a von Neumann algebra on a Hilbert space* H *and let* ω *be a* $\mathcal{U}(\mathcal{M})$*-multiplier on* G*. We say that a map* $\varphi\colon G \to \mathcal{M}$ *is* ω-positive definite *if the map* k *on* $G \times G$ *defined by*

$$k(x,y) = \omega(x^{-1},x)\omega(x^{-1},y)^*\varphi(x^{-1}y)$$

is positive definite. We define a (minimal) Kolmogorov decomposition *for* φ *to be a (minimal) Kolmogorov decomposition for* k*.*

Remark 8. *In particular, if* $\mathcal{M} = B(H)$*, we obtain the definition of the* ω*-positive definite map that we shall use in Section 2.*

2 Dilation theory in the case of projective isometric representations on Hilbert spaces with T-valued multipliers

The following theorem shows that an isometric ω-representation of S is always the restriction of a ω-representation of S by unitary operators to an invariant subspace.

Theorem 9. *([4]) Suppose ω is a multiplier on a normal generating subsemigroup S of the group G and let ρ be an isometric ω-representation of S on a Hilbert space H. Then there is a unitary ω-representation ρ' of S on a Hilbert space H' containing a copy of H such that*

(i) $\rho'(s)$ leaves H invariant and $\rho'(s)|_H = \rho(s)$;

(ii) $\bigcup_{s\in S} \rho'(s)^* H$ is dense in H'.

Proof. Let H_0 be the set of functions $f\colon S \to H$ for which there is $s \in S$ such that

$$f(y) = \omega(ys^{-1}, s)\rho(ys^{-1})(f(s)) \tag{2.1}$$

for $y \in Ss$.

Such s will be called *admissible* for f. Note that if s is admissible for f and $r \in Ss$, then r is also admissible for f, for then $Sr \subset Ss$ and for all $y \in Sr$,

$$f(y) = \omega(ys^{-1}, s)\rho(ys^{-1})f(s) =$$
$$\omega(ys^{-1}, s)\omega(yr^{-1}, rs^{-1})\rho(yr^{-1})\rho(rs^{-1})f(s) =$$
$$\omega(ys^{-1}, s)\omega(yr^{-1}, rs^{-1})\overline{\omega(rs^{-1}, s)}\rho(yr^{-1})f(r) =$$
$$\omega(yr^{-1}, r)\rho(yr^{-1})f(r),$$

by Definition 1.

Suppose now f and g are in H_0 and s is admissible for both f and g (since S is normal, the product of an admissible value for f and one for g will do). If $y \in Ss$, then

$$\langle f(y), g(y)\rangle = \left\langle \omega(ys^{-1}, s)\rho(ys^{-1})f(s), \omega(ys^{-1}, s)\rho(ys^{-1})g(s)\right\rangle =$$
$$= \langle f(s), g(s)\rangle\,,$$

because $\rho(ys^{-1})$ is an isometry and ω takes values in the unit circle. Thus $\langle f(s), g(s)\rangle$ is constant on the set of values of s which are admissible for both functions and we can define a positive semidefinite sesquilinear functional on H_0 by $\langle f, g\rangle = \langle f(s), g(s)\rangle$, where s is any value admissible for both f and g.

Let H' be the Hilbert space completion of H_0 under the corresponding seminorm and notice that this identifies functions which coincide on an admissible set of the form Ss. To embed the original Hilbert space H, define for each $\xi \in H$, the function $\widehat{\xi}$ by $\widehat{\xi}(s) = \rho(s)\xi$ for $s \in S$. Since ρ is an isometric ω-representation, $\widehat{\xi}$ satisfies (2.1) for any $s \in S$, hence $\widehat{\xi} \in H_0$ and every $s \in S$ is admissible for $\widehat{\xi}$. The embedding $\xi \longrightarrow \widehat{\xi}$ is isometric because each $\rho(s)$ is.

Suppose now that $f \in H_0$ and $t \in S$ and consider the function f_t defined by $f_t = \overline{\omega(x,t)}f(xt)$ for $x \in S$. If $s \in S$ is admissible for f, then normality implies that st is also admissible for f, and since $xt \in Sst$, for any $x \in Ss$,

$$f_t = \overline{\omega(x,t)}f(xt) = \overline{\omega(x,t)}\omega(xt(st)^{-1}, st)\rho(xt(st)^{-1})f(st) =$$
$$\overline{\omega(x,t)}\omega(xs^{-1}, st)\rho(xs^{-1})f(st) =$$
$$\overline{\omega(x,t)\omega(s,t)}\omega(xs^{-1}, s)\omega(xs^{-1}s, t)\rho(xs^{-1})f(st) =$$
$$\overline{\omega(s,t)}\omega(xs^{-1}, s)\rho(xs^{-1})f(st) =$$
$$\omega(xs^{-1}, s)\rho(xs^{-1})f_t(s)$$

which shows that the same s is admissible for f_t; in particular $f_t \in H_0$.

Evaluating the inner product at a point s admissible for both f and g, we obtain

$$\langle f_t, g_t\rangle = \langle f_t(s), g_t(s)\rangle = \left\langle \overline{\omega(s,t)}f(st), \overline{\omega(s,t)}g(st)\right\rangle = \langle f, g\rangle\,;$$

thus, $\rho'(t)f = f_t$ for $t \in S$ defines an isometry $\rho'(t)$ on H'.

If $\xi \in H$, then

$$(\rho'(t)\widehat{\xi})(x) = \rho'(t)\rho(x)\xi = (\rho(x))_t\xi = \overline{\omega(x,t)}\rho(xt)\xi =$$
$$\overline{\omega(x,t)}\omega(x,t)\rho(x)\rho(t)\xi = \rho(x)\rho(t)\xi = \widehat{\rho(t)\xi}(x)$$

for $x \in S$, so $\rho'(t)$ restricts to $\rho(t)$ on the copy of H inside H'. Furthermore,

$$\rho'(s)\rho'(t)f(x) = \overline{\omega(x,s)}\rho'(t)f(xs) = \overline{\omega(x,s)\omega(xs,t)}f(xst) =$$
$$\overline{\omega(x,st)\omega(s,t)}f(xst) = \overline{\omega(s,t)}\rho'(st)f(x)$$

for all $x \in S$ and $f \in H_0$

Thus ρ' is a ω-representation of S by isometries and it remains to prove that these isometries are in fact unitaries. Let $t \in S$ and suppose that s is admissible for $g \in H_0$. Consider the function defined by

$$g_{t^{-1}}(x) = \begin{cases} \omega(xt^{-1}, t)g(xt^{-1}), & \text{if } x \in St \\ 0, & \text{otherwise} \end{cases}$$

Then st is admissible for $g_{t^{-1}}$: if $x \in Sst$, then $xt^{-1} \in Ss$ is admissible for g and

$$g_{t^{-1}}(x) = \omega(xt^{-1}, t)g(xt^{-1}) = \omega(xt^{-1}, t)\omega(xt^{-1}s^{-1}, s)\rho(xt^{-1}s^{-1})g(s) =$$

$$\omega(xt^{-1}s^{-1}, st)\omega(s,t)\rho(xt^{-1}s^{-1})g(s) =$$
$$\omega(x(st)^{-1}, st)\rho(x(st)^{-1})g_{t^{-1}}(st)$$

so $g_{t^{-1}} \in H_0$. Since

$$\rho'(t)g_{t^{-1}}(x) = \overline{\omega(x,t)}g_{t^{-1}}(xt) = \overline{\omega(x,t)}\omega(x,t)g(x) = g(x)$$

for $x \in S$, $\rho'(t)$ is surjective for every $t \in S$. Thus ρ' is a unitary ω-representation of the subsemigroup S on H', which finishes the proof of (i).

To prove (ii), assume $f \in H_0$ and fix s admissible for f. Then for $x \in Ss$,

$$\rho'(s)(f)(x) = \overline{\omega(x,s)}f(xs) = \overline{\omega(x,s)}\omega(xss^{-1}, s)\rho(xss^{-1})f(s) =$$
$$\rho(x)(f(s)) = \widehat{f(s)}(x)$$

Hence $f(x) = (\rho'(s)^*\widehat{f(s)})(x)$ for x in the admissible set Ss, which implies $f = \rho'(s)^*\widehat{f(s)}$ in H'. Since H_0 is dense in H', (ii) follows. □

For the rest of this section, G will denote a group, ω a multiplier of G and S a normal, generating subsemigroup of G.

The following result is a generalization of Naimark-Sz.-Nagy's theorem of characterization of positive definite functions (Corollary 2.6, [1]), which can be obtained by taking $\omega \equiv 1$.

Theorem 10. *([6]) Let H be a Hilbert space and φ a ω-positive definite map on G with values in $B(H)$. Then there are a Hilbert space H', an operator $T \in B(H, H')$ and a unitary ω-representation ρ of G on H' such that $\varphi(x) = T^*\rho(x)T$, for all $x \in G$. Moreover, H' is the closed linear span of the set $\bigcup_x \rho(x)TH$.*

Proof. Let V be a minimal Kolmogorov decomposition of φ and set $H' = H_V$. Let $x, y, z \in G$. Then it is easy to verify that
$\omega(x^{-1}z^{-1}, zx)\omega(z,x)\omega(x^{-1}, y) = \omega(x^{-1}z^{-1}, zy)\omega(z,y)\omega(x^{-1}, x)$ and it follows from this that

$$V(zx)^*V(zy) = \omega(x^{-1}z^{-1}, zx)\overline{\omega(x^{-1}z^{-1}, zy)}\varphi(x^{-1}z^{-1}zy) =$$
$$= \omega(x^{-1}, x)\overline{\omega(x^{-1}, y)\omega(z,x)}\omega(z,y)\varphi(x^{-1}y) = \overline{\omega(z,x)}\omega(z,y)V(x)^*V(y)$$

which can be written $\omega(z,x)V(zx)^*\overline{\omega(z,y)}V(zy) = V(x)^*V(y)$. Hence, the map $x \longmapsto \overline{\omega(z,x)}V(zx)$ is another minimal Kolmogorov decomposition for φ. Consequently, there is a unique unitary $\rho(z) \in B(H')$ such that $\rho(z)V(x) = \overline{\omega(z,x)}V(zx)$, for all $x \in G$ (by Lemma 1.4, [1]). Since we have

$$\rho(y)\rho(z)V(x) = \overline{\omega(y, zx)\omega(z,x)}V(yzx) =$$

$$\overline{\omega(y,z)\omega(yz,x)}V(yzx) = \overline{\omega(y,z)}\rho(yz)V(x)$$

and the set $\bigcup_x V(x)H$ has dense linear span in H' (by minimality of V), therefore $\rho(yz) = \omega(y,z)\rho(y)\rho(z)$. Thus, the map $\rho : x \longmapsto \rho(x)$ is a projective unitary representation of G with ω as associated multiplier.

Set $T = V(e)$. Then $T^*\rho(x)T = \overline{\omega(x,e)}V(e)^*V(xe) = V(e)^*V(x) = \varphi(x)$. Also, $\rho(x)TH = V(x)H$ and therefore H' is the closed linear span of the set $\bigcup_x \rho(x)TH$. □

The projective representation ρ is called a *dilation* of φ.

Theorem 11. *([6]) Let H be a Hilbert space and let $\rho\colon S \to B(H)$ be a projective isometric representation with associated multiplier the restriction of ω to S. Then there is a unique extension ρ' of ρ to G having the following properties :*

(1) $\rho'(xs) = \omega(x,s)\rho'(x)\rho(s)$ for all $x \in G$ and $s \in S$;

(2) $\rho'(x)^* = \omega(x^{-1},x)\rho'(x^{-1})$ for all $x \in G$.

Moreover, ρ' is ω-positive definite.

Proof. Since S is a normal generating subsemigroup of G, the uniqueness of ρ' is clear.

To prove the existence of ρ', suppose that $x = s^{-1}t$, $s,t \in S$, because S generates G and set $\rho'(x) = \omega(s^{-1},t)\overline{\omega(s^{-1},s)}\rho(s)^*\rho(t)$. We show that ρ' is well defined. Suppose that we can also write $x = u^{-1}v$, where $u,v \in S$. Then $ut = u(su^{-1}v) = (usu^{-1})v$ and since $usu^{-1} \in S$ (by the normality of S) and ρ is a projective isometric representation with the multiplier ω, we have

$$\rho(ut) = \rho((usu^{-1})v) \Longrightarrow \omega(u,t)\rho(u)\rho(t) = \omega(usu^{-1},v)\rho(usu^{-1})\rho(v).$$

However,

$$\rho((usu^{-1})u) = \rho(us) \Longrightarrow \omega(usu^{-1},u)\rho(usu^{-1})\rho(u) = \omega(u,s)\rho(u)\rho(s),$$

so $\overline{\omega(u,s)}\omega(usu^{-1},u)\rho(u)^*\rho(usu^{-1})\rho(u) = \rho(s)$ and therefore,

$$\omega(u,s)\overline{\omega(usu^{-1},u)}\rho(u)^*\rho(usu^{-1})^*\rho(u) = \rho(s)^*.$$

Hence,

$$\rho(s)^*\rho(t) = \omega(u,s)\overline{\omega(usu^{-1},u)}\rho(u)^*\rho(usu^{-1})^*\rho(u)\rho(t) =$$
$$\omega(u,s)\overline{\omega(usu^{-1},u)\omega(u,t)}\omega(usu^{-1},v)\rho(u)^*\rho(usu^{-1})^*\rho(usu^{-1})\rho(v) =$$
$$\omega(u,s)\overline{\omega(usu^{-1},u)\omega(u,t)}\omega(usu^{-1},v)\rho(u)^*\rho(v) \Longrightarrow$$

$$\omega(s^{-1},t)\overline{\omega(s^{-1},s)}\rho(s)^*\rho(t) =$$
$$\omega(s^{-1},t)\overline{\omega(s^{-1},s)}\omega(u,s)\overline{\omega(usu^{-1},u)\omega(u,t)}\omega(usu^{-1},v)\rho(u)^*\rho(v)$$

It remains to verify that

$$\omega(s^{-1},t)\overline{\omega(s^{-1},s)}\omega(u,s)\overline{\omega(usu^{-1},u)\omega(u,t)}\omega(usu^{-1},v) = \omega(u^{-1},v)\overline{\omega(u^{-1},u)} \quad (2.2)$$

Since $t = su^{-1}v$, the relation (2.2) becomes:

$$\omega(s^{-1},su^{-1}v)\overline{\omega(s^{-1},s)}\omega(u,s)\overline{\omega(usu^{-1},u)\omega(u,su^{-1}v)}\omega(usu^{-1},v) = \\ = \omega(u^{-1},v)\overline{\omega(u^{-1},u)} \quad (2.3)$$

By Definition 1, we have:

$$\omega(s^{-1},su^{-1}v) = \omega(s^{-1},su^{-1})\omega(s^{-1}su^{-1},v)\overline{\omega(su^{-1},v)} =$$
$$\omega(s^{-1},su^{-1})\omega(u^{-1},v)\overline{\omega(su^{-1},v)}$$
$$\omega(usu^{-1},u) = \omega(u,s)\omega(su^{-1},u)\overline{\omega(u,su^{-1})}$$
$$\omega(usu^{-1},v) = \omega(su^{-1},v)\omega(u,su^{-1}v)\overline{\omega(u,su^{-1})}$$

Hence, the relation (2.3) becomes:

$$\omega(s^{-1},su^{-1})\overline{\omega(s^{-1},s)\omega(su^{-1},u)} = \overline{\omega(u^{-1},u)}, \quad (2.4)$$

taking into account that the range of ω is contained in the unit circle $\mathbf{T}$.

By Definition 1, we get

$$\omega(s^{-1},su^{-1})\omega(s,u^{-1}) = \omega(s^{-1},s).$$

So the relation (2.4) becomes :

$$\omega(s^{-1},su^{-1})\overline{\omega(s^{-1},su^{-1})\omega(s,u^{-1})\omega(su^{-1},u)} = \overline{\omega(u^{-1},u)} \iff$$
$$\omega(s,u^{-1})\omega(su^{-1},u) = \omega(u^{-1},u) \iff$$
$$\omega(s,u^{-1}u)\omega(u^{-1},u) = \omega(u^{-1},u) \text{ true by Definition 1}$$

Since $x = s^{-1}t$ and ρ is a projective representation with the multiplier ω, the conditions (1) and (2) can be easily verified using Definition 1 and the definition of ρ'.

It remains to show that ρ' is ω-positive definite. Thus, if $x_1,\ldots,x_n \in G$, we must show positivity of the operator matrix (V_{ij}), where

$$V_{ij} = \omega(x_i^{-1},x_i)\overline{\omega(x_i^{-1},x_j)}\rho'(x_i^{-1}x_j).$$

We claim that there is an element $s \in S$ such that $sx_1, \ldots, sx_n \in S$. To prove this, write $x_i = v_i u_i^{-1}$, where $u_i, v_i \in S$. Then, for $s = u_1 \ldots u_n$, we have $sx_i = u_1 \ldots u_i(u_{i+1} \ldots u_n v_i)u_i^{-1}$, so $sx_i \in S$ as required.

Consequently, for some elements $s, t_1, \ldots, t_n \in S$, we have $x_i = s^{-1}t_i$; hence, since $\omega(t_i^{-1}s, s^{-1}t_j) = \overline{\omega(t_i^{-1}, s)}\omega(t_i^{-1}, t_j)\omega(s, s^{-1}t_j)$ (by Definition 1), we have

$$V_{ij} = \omega(t_i^{-1}s, s^{-1}t_i)\overline{\omega(t_i^{-1}s, s^{-1}t_j)}\rho'(t_i^{-1}t_j) =$$

$$\omega(t_i^{-1}s, s^{-1}t_i)\overline{\omega(t_i^{-1}s, s^{-1}t_j)}\omega(t_i^{-1}, t_j)\overline{\omega(t_i^{-1}, t_i)}\rho(t_i)^*\rho(t_j) =$$

$$\omega(s, s^{-1}t_i)\overline{\omega(s, s^{-1}t_j)}\rho(t_i)^*\rho(t_j).$$

Thus, $V_{ij} = V_i^* V_j$, where $V_i = \overline{\omega(s, s^{-1}t_i)}\rho(t_i)$. Hence, (V_{ij}) is positive. □

Theorem 12. *([6]) Let H be a Hilbert space and $\rho \colon S \to B(H)$ a projective isometric representation with associated multiplier the restriction of ω to S. Then there are a Hilbert space H', an isometry $T \colon H \to H'$ and a unitary ω-representation $\varphi \colon G \to B(H')$ such that $T^*\varphi(s)T = \rho(s)$, for all $s \in S$. Moreover, H' is the closed linear span of the set $\bigcup_{x \in G} \varphi(x)T(H)$.*

Proof. We obtain the proof by applying Theorem 10 to the ω-positive map ρ' extending ρ that is given in Theorem 11. □

3 Dilation theory in the case of projective isometric representations on Hilbert spaces with unitary operator-valued multipliers

Theorem 13. *([3]) Let X be a non-empty set, let $\mathcal{M}$ be a von Neumann algebra, let $k \colon X \times X \to \mathcal{M}$ be a positive definite kernel and let V be a minimal Kolmogorov decomposition of k. Then there is a $*$-homomorphism $\phi \colon \mathcal{U}(\mathcal{M}') \to B(H_V)$ such that for any $x \in X$,*

$$V(x)a = \phi(a)V(x) \quad a \in \mathcal{U}(\mathcal{M}').$$

Moreover, for each $a \in \mathcal{U}(\mathcal{M}')$, $\phi(a)$ is unitary on H_V.

Theorem 14. *([3]) Let S be a semigroup and ϕ be the $*$-homomorphism given by Theorem 13. For each $\mathcal{U}(\mathcal{M})$-multiplier ω on S, $\phi(\omega)$ is a $\mathcal{U}(\mathcal{N})$-multiplier, where $\mathcal{N}$ is a von Neumann algebra generated by $\phi(\mathcal{U}(Z(\mathcal{M})))$ and $\phi(\omega)(s,t) = \phi(\omega(s,t))$ for any $s, t \in S$.*

Theorem 15. *([3]) Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H, let ω be a $\mathcal{U}(\mathcal{M})$-multiplier and let φ be a ω-positive definite map on G with values in $B(H)$. Then there are a Hilbert space H', an operator $T \in B(H, H')$ and a unitary*

$\phi(\omega)$-representation ρ of G on H' such that $\varphi(x) = T^*\rho(x)T$, for all $x \in G$, where the $*$-homomorphism ϕ is given as in Theorem 13. Moreover, H' is the closed linear span of the set $\bigcup_x \rho(x)TH$.

Proof. Let V be a minimal Kolmogorov decomposition of φ and set $H' = H_V$. Let $x, y, z \in G$. Then it is easy to verify that
$\omega(x^{-1}z^{-1}, zx)\omega(z,x)\omega(x^{-1}, y) = \omega(x^{-1}z^{-1}, zy)\omega(z,y)\omega(x^{-1}, x)$ and it follows from this that

$$V(zx)^*V(zy) = k(zx, zy) = \omega(x^{-1}z^{-1}, zx)\omega(x^{-1}z^{-1}, zy)^*\varphi(x^{-1}z^{-1}zy) =$$

$$= \omega(x^{-1}, y)^*\omega(z,x)^*\omega(z,y)\omega(x^{-1}, x)\varphi(x^{-1}y) = \omega(z,x)^*\omega(z,y)V(x)^*V(y)$$

which is equivalent to

$$V(x)^*V(y) = [V(zx)\omega(z,x)^*]^*V(zy)\omega(z,y)^*$$

Hence for each $z \in G$, the map $x \longmapsto V(zx)\omega(z,x)^*$ is another minimal Kolmogorov decomposition for φ. Consequently, there is a unique unitary $\rho(z) \in B(H')$ such that $\rho(z)V(x) = V(zx)\omega(z,x)^*$, for all $x \in G$ (by Lemma 1.4, [1]). Since we have

$$\rho(y)\rho(z)V(x) = \rho(y)V(zx)\omega(z,x)^* = V(yzx)\omega(y,zx)^*\omega(z,x)^* =$$

$$V(yzx)\omega(y,z)^*\omega(yz,x)^* = \rho(yz)V(x)\omega(y,z)^* = \rho(yz)\phi(\omega(y,z)^*)V(x)$$

and the set $\bigcup_x V(x)H$ has dense linear span in H' (by minimality of V), therefore $\rho(yz) = \phi(\omega(y,z))\rho(y)\rho(z), y, z \in G$.

Moreover, for any $x, y \in G$, $a \in \mathcal{U}(Z(\mathcal{M}))$, $h \in H$, we have, by Theorem 13,

$$\rho(y)\phi(a)V(x)h = \rho(y)V(x)ah = V(yx)\omega(y,x)^*ah =$$

$$\phi(a)V(yx)\omega(y,x)^*h = \phi(a)\rho(y)V(x)h$$

Therefore, for any $y \in G$, $\rho(y) \in \mathcal{N}'$ and $\rho\colon G \to \mathcal{N}'$ is a projective unitary $\phi(\omega)$-representation of G, where the von Neumann algebra $\mathcal{N}$ is given as in Theorem 14. Moreover,

$$V(e)^*\rho(x)V(e) = V(e)^*V(x) = k(e,x) = \varphi(x)$$

and $\rho(x)V(e)H = V(x)H$. By the minimality of V, the linear span of $\bigcup_x V(x)H$ is dense in H'. Hence, H' is the closed linear span of the set $\bigcup_x \rho(x)V(e)H$. Set $T = V(e)$ and the proof is completed. □

The projective unitary $\phi(\omega)$-representation ρ is called a *dilation* of ϕ.

Remark 16. *If in Theorem 15, the von Neumann algebra $\mathcal{M} = B(H)$ and $\phi(\omega) = \omega$ a $\mathbf{T}$-valued multiplier, we obtain Theorem 10.*

Theorem 17. *([3]) Let ω be a $\mathcal{U}(\mathcal{M})$-multiplier on G, let S be a normal generating subsemigroup of G and let $\rho \colon S \to \mathcal{M}$ be a projective isometric representation with associated $\mathcal{U}(\mathcal{M})$-multiplier the restriction of ω on S. Then there is a unique extension ρ' of ρ to G having the following properties :*

(1) $\rho'(xs) = \omega(x,s)\rho'(x)\rho(s)$ *for all* $x \in G$ *and* $s \in S$*;*

(2) $\rho'(x)^* = \omega(x^{-1},x)\rho'(x^{-1})$ *for all* $x \in G$*.*

Moreover, ρ' *is* ω*-positive definite.*

Proof. Since S is a normal generating subsemigroup of G, the uniqueness of ρ' is clear.

To prove the existence of ρ', suppose that $x = s^{-1}t$, $s, t \in S$, because S generates G and set $\rho'(x) = \omega(s^{-1},t)\omega(s^{-1},s)^*\rho(s)^*\rho(t)$. We show that ρ' is well defined. Suppose that we can also write $x = u^{-1}v$, where $u, v \in S$. Then $ut = u(su^{-1}v) = (usu^{-1})v$ and since $usu^{-1} \in S$ and ρ is a projective isometric representation with the multiplier ω, we have

$$\rho(ut) = \rho((usu^{-1})v) \Longrightarrow \omega(u,t)\rho(u)\rho(t) = \omega(usu^{-1},v)\rho(usu^{-1})\rho(v).$$

However,

$$\rho((usu^{-1})u) = \rho(us) \Longrightarrow \omega(usu^{-1},u)\rho(usu^{-1})\rho(u) = \omega(u,s)\rho(u)\rho(s),$$

so $\omega(u,s)^*\omega(usu^{-1},u)\rho(u)^*\rho(usu^{-1})\rho(u) = \rho(s)$.

Hence,

$$\rho(s)^*\rho(t) = \omega(u,s)\omega(usu^{-1},u)^*\rho(u)^*\rho(usu^{-1})^*\rho(u)\rho(t) =$$

$$\omega(u,s)\omega(usu^{-1},u)^*\omega(u,t)^*\omega(usu^{-1},v)\rho(u)^*\rho(usu^{-1})^*\rho(usu^{-1})\rho(v) =$$

$$\omega(u,s)\omega(usu^{-1},u)^*\omega(u,t)^*\omega(usu^{-1},v)\rho(u)^*\rho(v) \Longrightarrow$$

$$\omega(s^{-1},t)\omega(s^{-1},s)^*\rho(s)^*\rho(t) =$$

$$\omega(s^{-1},t)\omega(s^{-1},s)^*\omega(u,s)\omega(usu^{-1},u)^*\omega(u,t)^*\omega(usu^{-1},v)\rho(u)^*\rho(v)$$

As in the proof of Theorem 11, it can be verified the relation:

$$\omega(s^{-1},t)\omega(s^{-1},s)^*\omega(u,s)\omega(usu^{-1},u)^*\omega(u,t)^*\omega(usu^{-1},v) = \\ = \omega(u^{-1},v)\omega(u^{-1},u)^* \tag{3.1}$$

Since $x = s^{-1}t$ and ρ is a projective representation with the associated multiplier ω, it can be easily verified the conditions (1) and (2).

To prove that ρ' is ω-positive definite, we follow the proof of Theorem 11 and show the positivity of the operator matrix (V_{ij}), where

$$V_{ij} = \omega(x_i^{-1}, x_j)\omega(x_i^{-1}, x_i)^* \rho'(x_i^{-1} x_j),$$

for $x_1, \ldots, x_n \in G$. □

Theorem 18. *([3]) Let ω be a $\mathcal{U}(\mathcal{M})$-multiplier on G, let S be a normal generating subsemigroup of G and let $\rho\colon S \to \mathcal{M}$ be a projective isometric representation with associated $\mathcal{U}(\mathcal{M})$-multiplier the restriction of ω to S. Then there are a Hilbert space H', an isometry $T\colon H \to H'$ and a unitary $\phi(\omega)$-representation φ such that $T^*\varphi(s)T = \rho(s)$, for all $s \in S$. Moreover, H' is the closed linear span of the set $\bigcup_{x\in G} \varphi(x)TH$.*

Proof. We obtain the proof by applying Theorem 15 to the ω-positive map ρ' extending ρ that is given in Theorem 17. □

Remark 19. *If in Theorem 18, the von Neumann algebra $\mathcal{M} = B(H)$ and $\phi(\omega) = \omega$ a* **T***-valued multiplier, we obtain Theorem 12.*

4 Dilation theory in the case of projective isometric representations on Hilbert C^*-modules with T-valued multipliers

Now we give the generalizations of the notions and theorems in Sections 2 and 3 to Hilbert C^*-modules.

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner-product to take values in a C^*- algebra rather than in the field of complex numbers.

Definition 20. *A* pre-Hilbert A-module *is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle\cdot,\cdot\rangle : E \times E \to A$ which is $\mathbb{C}$ -and A-linear in its second variable and satisfies the following relations:*

1. $\langle\xi, \eta\rangle^* = \langle\eta, \xi\rangle$ *for every* $\xi, \eta \in E$;
2. $\langle\xi, \xi\rangle \geq 0$ *for every* $\xi \in E$;
3. $\langle\xi, \xi\rangle = 0$ *if and only if* $\xi = 0$.

We say that E is a Hilbert A-module *if E is complete with respect to the topology determined by the norm $\|\cdot\|$ given by $\|\xi\| = \sqrt{\|\langle\xi, \xi\rangle\|}$.*

Definition 21. *Let X be a nonempty set, let A be a C^*-algebra and let E be a right Hilbert A-module. A map $k\colon X\times X\to L_A(E)$ is a* positive definite kernel *if the matrix $(k(x_i,x_j))_{ij}$ in $M_n(L_A(E))$ is positive for every integer n and for all $x_1,\ldots,x_n\in S$, where $L_A(E)$ is the algebra of all adjointable module maps from E to E, i.e. the algebra of all module maps $T\colon E\to E$ for which there is a module map $T^*\colon E\to E$ such that $\langle T\xi,\eta\rangle=\langle\xi,T^*\eta\rangle$, for all $\xi,\eta\in E$.*

Definition 22. *([2]) If k can be written in the form $k(x,y)=V(x)^*V(y)$ for any $x,y\in X$, where V is a map from X to $L_A(E,E_V)$ for some right Hilbert A-module E_V, then k is positive definite. Such a map V is said to be the* Kolmogorov decomposition *for a kernel k. If the linear span of the set $\bigcup\limits_{x\in X}V(x)E$ is dense in E_V, then V is said to be* minimal.

Definition 23. *([2]) Let S be a semigroup. A* multiplier *on a semigroup S is a function $\omega\colon S\times S\to\mathbf{T}$ such that*

(i) $\omega(e,s)=\omega(s,e)=1$;

(ii) $\omega(s,t)\omega(st,u)=\omega(s,tu)\omega(t,u)$

for all $s,t,u\in S$, where $\mathbf{T}$ is the unit circle.

Definition 24. *A* projective isometric ω-representation of S *is a map $\rho\colon S\to L_A(E)$ having the following properties:*

(i) $\rho(s)$ *is an isometry and* $\rho(e)=1$;

(ii) $\rho(st)=\omega(s,t)\rho(s)\rho(t)$, *for all* $s,t\in S$.

Definition 25. *Let G be a discrete group and let ω be a multiplier on G. A map ρ from G into $L_A(E)$ is said to be* ω-positive definite *if the map $k\colon G\times G\to L_A(E)$ defined by $k(x,y)=\omega(x^{-1},x)\overline{\omega(x^{-1},y)}\rho(x^{-1}y)$ is positive definite. We define a* (minimal) Kolmogorov decomposition for *ρ to be a (minimal) Kolmogorov decomposition for k.*

The following theorem may be regarded as a generalization of Stinespring dilation theorem for a covariant completely positive map which determines a positive definite kernel (Theorem 2.4 and Example 2.2, [7]).

Theorem 26. *([2]) Let G be a group and let ω be a multiplier on G. If a map $\varphi\colon G\to L_A(E)$ is ω-positive definite, then there is a right Hilbert A-module F, $T\in L_A(E,F)$ and a unitary ω-representation ρ of G on F such that $\varphi(x)=T^*\rho(x)T$ for all $x\in G$. Moreover, F is the closed linear span of $\bigcup\limits_{x\in G}\rho(x)TE$.*

Proof. From Definition 25, the map $k\colon G\times G\to L_A(E)$ defined by $k(x,y)=\omega(x^{-1},x)\overline{\omega(x^{-1},y)}\varphi(x^{-1}y)$ is positive definite. By (Theorem 2.3, [7]), there is a minimal Kolmogorov decomposition $V\in L_A(E,E_V)$ for the map k. That is, V becomes a minimal Kolmogorov decomposition for φ by definition. Take $F=E_V$. For $x,y,z\in G$, it is not difficult to verify that $\omega(x^{-1}z^{-1},zx)\omega(z,x)\omega(x^{-1},y)=\omega(x^{-1}z^{-1},zy)\omega(z,y)\omega(x^{-1},x)$.

Then we obtain

$$V(zx)^*V(zy)=\omega(x^{-1}z^{-1},zx)\overline{\omega(x^{-1}z^{-1},zy)}\varphi(x^{-1}z^{-1}zy)=$$

$$\omega(x^{-1},x)\overline{\omega(x^{-1},y)\omega(z,x)}\omega(z,y)\varphi(x^{-1}y)=\omega(z,y)\overline{\omega(z,x)}V(x)^*V(y)$$

Hence, the map $x\longmapsto\overline{\omega(z,x)}V(zx)$ is another minimal Kolmogorov decomposition for φ. By (Theorem 2.3, [7]), there is a unitary $\rho(z)\in L_A(F)$ such that $\rho(z)V(x)=\overline{\omega(z,x)}V(zx)$ for all $x\in G$.

From a simple computation, we have $\rho(y)\rho(z)V(x)=\overline{\omega(y,z)}\rho(yz)V(x)$. Since V is minimal, the set $\bigcup_{x\in G}V(x)E$ is dense in F. Hence we have $\rho(yz)=\omega(y,z)\rho(y)\rho(z)$, which shows that the map $x\longmapsto\rho(x)$ is a projective unitary representation of G with ω as an associated multiplier. By taking $T=V(e)$, we obtain that $T^*\rho(x)T=\varphi(x)$ and $\rho(x)TE=V(x)E$ for all $x\in G$, which completes the proof. □

The following theorem may be considered as a generalization of Theorem 11.

Theorem 27. *([2]) Let S be a normal generating subsemigroup of a group G, let ω be a multiplier on G, let E be a right C^*-module over a C^*-algebra A and let $\rho\colon S\to L_A(E)$ be a projective isometric representation with associated multiplier the restriction of ω to S. Then there is a unique extension ρ' of ρ to G having the following properties :*

(1) $\rho'(xs)=\omega(x,s)\rho'(x)\rho(s)$ for all $x\in G$ and $s\in S$;

(2) $\rho'(x)^*=\omega(x^{-1},x)\rho'(x^{-1})$ for all $x\in G$.

Moreover, ρ' is ω-positive definite.

Proof. Since S is a normal generating subsemigroup of G, the uniqueness of ρ' is clear.

To show the existence of ρ', suppose that $x=s^{-1}t$, $s,t\in S$, because S generates G and set $\rho'(x)=\omega(s^{-1},t)\overline{\omega(s^{-1},s)}\rho(s)^*\rho(t)$. We have to show that the map ρ' is well-defined. For this it must be checked that for $x=s^{-1}t=u^{-1}v$, $\omega(s^{-1},t)\overline{\omega(s^{-1},s)}\rho(s)^*\rho(t)=\omega(u^{-1},v)\overline{\omega(u^{-1},u)}\rho(u)^*\rho(v)$.

Indeed, we have $ut=usx=usu^{-1}v$. Then the element $usu^{-1}\in S$ because of normality of S in G. Since the restriction of ρ to S is a projective ω-isometric representation, we have that

$\omega(u,t)\rho(u)\rho(t) = \rho(ut) = \rho(usu^{-1}v) = \omega(usu^{-1}, v)\rho(usu^{-1})\rho(v)$
However, we have the equality

$$\omega(usu^{-1}, u)\rho(usu^{-1})\rho(u) = \rho(us) = \omega(u,s)\rho(u)\rho(s), \tag{4.1}$$

so that

$$\rho(s)^* = \omega(u,s)\overline{\omega(usu^{-1}, u)}\rho(u)^*\rho(usu^{-1})^*\rho(u). \tag{4.2}$$

Hence, we obtain from equations (4.1) and (4.2) that

$$\omega(s^{-1}, t)\overline{\omega(s^{-1}, s)}\rho(s)^*\rho(t) =$$

$$= \omega(s^{-1}, t)\overline{\omega(s^{-1}, s)}\omega(u,s)\overline{\omega(usu^{-1}, u)}\omega(usu^{-1}, v)\overline{\omega(u,t)}\rho(u)^*\rho(v)$$

Since $t = su^{-1}v$ and the range of ω is contained in the unit circle $\mathbf{T}$, we have that

$$\omega(s^{-1}, t)\overline{\omega(s^{-1}, s)} =$$

$$\omega(s^{-1}, su^{-1}v)\overline{\omega(s^{-1}, s)}\omega(u,s)\overline{\omega(usu^{-1}, u)}\omega(usu^{-1}, v)\overline{\omega(u, su^{-1}v)}$$

Hence, ρ' is well-defined and it is a routine to check (1) and (2) (see the analogue Theorems in Section 2 and 3).

To show that ρ' is ω-positive definite, we follow the proof of Theorem 11. □

Corollary 28. *([2]) Let G, S and ω be as in Theorem 27. If $\rho\colon S \to L_A(E)$ is a projective isometric representation with the restriction of ω to S as the associated multiplier, then there are a right Hilbert A-module F, $T \in L_A(E,F)$ and a unitary ω-representation φ of G on F such that $\rho(s) = T^*\varphi(s)T$ for all $s \in S$. Moreover, F is the closed linear span of $\bigcup\limits_{x\in G}\varphi(x)TE$.*

Proof. The proof follows immediately from Theorem 26 and Theorem 27. □

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