A FIXED POINT THEOREM FOR UNIFORMLY LOCALLY CONTRACTIVE MAPPINGS IN A C-CHAINABLE CONE RECTANGULAR METRIC SPACE

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Abstract. Recently, Azam, Arshad and Beg [4] introduced the notion of cone rectangular metric spaces by replacing the triangular inequality of a cone metric space by a rectangular inequality. In this paper, we introduce the notion of c-chainable cone rectangular metric space and we establish a fixed point theorem for uniformly locally contractive mappings in such spaces. An example is given to illustrate our obtained result.

1 Introduction and Preliminaries

One of the simplest and most useful results in the fixed point theory is the Banach-Caccioppoli contraction mapping principle. This principle has been generalized in different directions in different spaces by mathematicians over the years.

Fixed point theory in K-metric and K-normed spaces was developed by Perov et al. [12, 16, 17], Mukhamadijev and Stetsenko [13], Vandergraft [23] and others. For more details on fixed point theory in K-metric and K-normed spaces, we refer the reader to fine survey paper of Zabrejko [25]. The main idea consists to use an ordered Banach space instead of the set of real numbers, as the codomain for a metric.

In 2007, Huang and Zhang [9] reintroduced such spaces under the name of cone metric spaces and reintroduced definition of convergent and Cauchy sequences in the terms of interior points of the underlying cone. They also proved some fixed point theorems in such spaces in the same work. After that, fixed point points in K-metric spaces have been the subject of intensive research (see [1, 2, 3, 7, 10, 18, 15, 19, 21, 22, 24] and others).

Following the idea of Branciari [5], Azam, Arshad and Beg [4] introduced the notion of cone rectangular metric spaces by replacing the triangular inequality by a rectangular inequality. They extended the Banach contraction principle to such spaces.

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In this paper, we introduce the notion of c-chainable cone rectangular metric space and we establish a fixed point theorem for uniformly locally contractive mappings in such spaces. The presented theorem can be considered as a generalization of the recent result obtained by P. Das and L. K. Dey [6] in the generalized metric space introduced by Branciari [5].

First, we start by recalling some basic definitions concerning cone rectangular metric spaces and preliminary results presented in [4, 9].

Let E always be a real Banach space equipped with the norm $\|\cdot\|$ and P be a subset of E. P is called a cone if and only if:

(i) P is closed, nonempty, and $P \neq \{0\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ on E with respect to P by:

$$x \leq y \Leftrightarrow y - x \in P, \text{ for all } x, y \in E.$$

We shall write $x < y$ if $x \leq y$ and $x \neq y$, while $x \ll y$ will stands for $y - x \in \operatorname{int}(P)$, where $\operatorname{int}(P)$ denotes the interior of P. The cone P is called normal if there exists $k > 0$ such that for all $x, y \in E$, we have:

$$0 \leq x \leq y \Rightarrow \|x\| \leq k\|y\|.$$

In this case, k is called the normal constant of P. Rezapour and Hamlbarani [19] proved that there aren't normal cones with normal constant $c < 1$ and for each $\nu > 1$ there are cones with normal constant $c > \nu$. Also, omitting the assumption of normality they obtain generalizations of some results of [9].

In the following we always suppose that E is a real Banach space and P is a cone in E with $\operatorname{int}(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P. We recall that the existence of fixed point in partially ordered sets has been investigated recently in [14] and references therein.

Definition 1. *[9] Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:*

(1) $0 \leq \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

Then ρ is called a cone metric on X, and (X, ρ) is called a cone metric space with respect to P.

Example 2. *[9] Let* $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 \,|\, x \geq 0,\, y \geq 0\}$, $X = \mathbb{R}$ *and* $\rho : X \times X \to E$ *defined by:*

$$\rho(x, y) = (|x - y|, \tau|x - y|),\ \forall\, x, y \in X,$$

where $\tau \geq 0$ *is a constant. Then* (X, ρ) *is a cone metric space.*

Definition 3. *[4] Let* X *be a nonempty set. Suppose the mapping* $d : X \times X \to E$ *satisfies:*

(a) $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$*;*

(b) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*

(c) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$ *for all* $x, y \in X$ *and for all distinct points* $w, z \in X \backslash \{x, y\}$ *[rectangular inequality].*

Then d is called a cone rectangular metric on X*, and* (X, d) *is called a cone rectangular metric space with respect to* P*.*

It is clear that any cone metric space is a cone rectangular metric space. The inverse is not true in general.

Example 4. *[11] Let* $E = \mathbb{R}^2$, $P = \{(x, y) \,|\, x, y \in \mathbb{R}, x, y \geq 0\}$ *and* $X = \{1, 2, 3, 4\}$. *Define* $d : X \times X \to E$ *by:*

$$\begin{aligned} d(x, x) &= (0, 0) \\ d(1, 2) &= d(2, 1) = (3, 9), \\ d(2, 3) &= d(3, 2) = d(1, 3) = d(3, 1) = (1, 3), \\ d(1, 4) &= d(4, 1) = d(2, 4) = d(4, 2) = d(3, 4) = d(4, 3) = (4, 12). \end{aligned}$$

Then (X, d) *is a cone rectangular metric space but* (X, d) *is not a cone metric space because it lacks the triangular inequality:*

$$(3, 9) = d(1, 2) > d(1, 3) + d(3, 2) = (2, 6).$$

Definition 5. *[4] Let* (X, d) *be a cone rectangular metric space,* $\{x_n\}$ *be a sequence in* X *and* $x \in X$*. If for every* $c \in E$ *with* $0 \ll c$ *there is* $N \in \mathbb{N}$ *such that for all* $n > N$, $d(x_n, x) \ll c$*, then* $\{x_n\}$ *is said to be convergent to* x *and* x *is a limit of* $\{x_n\}$*. We denote this by* $x_n \to x$ *as* $n \to +\infty$ *or* $\lim\limits_{n \to +\infty} x_n = x$*.*

Definition 6. *[4] Let* (X, d) *be a cone rectangular metric space and* $\{x_n\}$ *be a sequence in* X*. If for all* $c \in E$ *with* $0 \ll c$ *there is* $N \in \mathbb{N}$ *such that for all* $n > N$, $d(x_n, x_{n+m}) \ll c$*, then* $\{x_n\}$ *is called a Cauchy sequence in* (X, d)*. If every Cauchy sequence is convergent in* (X, d)*, then* (X, d) *is called a complete cone rectangular metric space.*

The following lemma has an important role in the proof of our result.

Lemma 7. *Let (X, d) be a cone rectangular metric space with respect to the cone P. Let $a, a_n \in P$ and $\{x_n\} \subset X$. Then, the following conditions hold:*

(i) If $a \leq a_n$ for every n and $\|a_n\| \to 0$ as $n \to +\infty$, then $a = 0$.

(ii) If $d(x_m, x_{m+n}) \leq a_m$ for every m, n and $\|a_m\| \to 0$ as $m \to +\infty$, then $\{x_n\}$ is a Cauchy sequence.

Proof.

(i) Since $a \leq a_n$ for every n, we have $a_n - a \in P$. By $\|a_n\| \to 0$ as $n \to +\infty$, and since P is closed, we deduce that $-a \in P$. We have $a \in P$ and $-a \in P$, then $a = 0$.

(ii) Let $0 \ll c$ and $I(0, r) = \{y \in E : \|y\| < r\}$ such that $c + I(0, r) \subset \text{Int}(P)$. Now, $\|a_m\| \to 0$ implies that there exists m_0 such that $\|a_m\| < r$, for every $m \geq m_0$, and so $a_m \in I(0, r)$. It follows, $-a_m \in I(0, r)$. Therefore, $c - a_m \in \text{int}(P)$ implies $d(x_m, x_{m+n}) \leq a_m \ll c$, for every n, and so (ii) holds.

□

Remark 8. *The reader should make attention to the difference between cone metric space and cone rectangular metric space.*

- *If (X, d) is a cone metric space and $\{x_n\}$ is a convergent sequence in (X, d), then the limit of $\{x_n\}$ is unique (see [9]-Lemma 2). However, when (X, d) is a cone rectangular metric space, it is not the case. A counter-example is given in [11].*

- *If (X, d) is a cone metric space and $\{x_n\}$ is a convergent sequence in (X, d), then $\{x_n\}$ is a Cauchy sequence in (X, d) (see [9]-Lemma 3). However, when (X, d) is a cone rectangular metric space, this result is not true in general. A counter-example is given in [11] (see also [20]).*

We also note that the relations $P + \text{int } P \subseteq \text{int } P$ and $t \text{ int } P \subseteq \text{int } P$ $(t > 0)$ always hold true.

2 Main result

In this section, we prove our main result. We first introduce the following definitions, adapted after the case of usual metric spaces [6].

Definition 9. *A cone rectangular metric space (X, d) is called c-chainable, for $0 \ll c$, if for every $x, y \in X$, there is a finite set of points $x = x_0, x_1, \ldots, x_n = y$, n depends on both x and y, such that $d(x_{i-1}, x_i) \ll c$, for $1 \leq i \leq n$.*

Definition 10. *Let (X,d) be a cone rectangular metric space, $0 \ll c$ and $\lambda \in (0,1)$. A mapping $T : X \to X$ is called (c,λ)-uniformly locally contractive if and only if:*

$$d(x,y) \ll c \Rightarrow d(Tx,Ty) \ll \lambda d(x,y), \text{ for all } x,y \in X.$$

Definition 11. *Let (X,d) be a cone rectangular metric space. We say that (X,d) is Hausdorff if and only if every convergent sequence in (X,d) has one and only one limit.*

Our main result is the following.

Theorem 12. *Let (X,d) be a $c/2$-chainable Hausdorff complete cone rectangular metric space with respect to a cone P, $0 \ll c$ and $\lambda \in (0,1)$. Let $T : X \to X$ be a (c,λ)-uniformly locally contractive mapping. Assume that*

$$d(x,y) \ll \frac{c}{2} \text{ and } d(y,z) \ll \frac{c}{2} \Rightarrow d(x,z) \ll c, \text{ for all } x,y,z \in X. \tag{2.1}$$

Then T has a unique fixed point in X.

Proof. By adopting arguments similar to those in Das and Dey ([6], Theorem 1), we prove the theorem in three steps.

• **Step I.** Choose $x \in X$ $(x \neq Tx)$. Since X is $c/2$-chainable, we can find finite number of points

$$x = x_0, x_1, x_2, \dots, x_{n-1}, x_n = Tx \tag{2.2}$$

such that

$$d(x_{i-1}, x_i) \ll \frac{c}{2} \text{ for all } i = 1,2,\dots,n.$$

Now, without loss of generality, we suppose that the points $x_1, x_2, \dots, x_{n-1}$ are distinct and different from x and Tx if $n > 2$. Thus, we show that

$$d(x,Tx) \ll n\,\frac{c}{2}. \tag{2.3}$$

Clearly, as condition (2.1) holds, (2.3) is obvious if $n = 1$ or $n = 2$, and so, we assume $n > 2$. We need to consider the following two cases.

Case-I. Let n be an odd number and put $n = 2m+1$, where $m \geq 1$. Then

$$d(x,Tx) \leq d(x,x_1) + d(x_1,x_2) + \cdots + d(x_{2m},Tx) \ll (2m+1)\frac{c}{2} = n\,\frac{c}{2}.$$

Case-II. Let n be an even number and put $n = 2m$, where $m \geq 2$. Then

$$d(x,Tx) \leq d(x,x_2) + d(x_2,x_3) + \cdots + d(x_{2m-1},Tx) \ll c + (2m-2)\frac{c}{2} = n\,\frac{c}{2}.$$

Hence (2.3) holds.

Since T is a $(c,\lambda)-$uniformly locally contractive mapping, we have:

$$d(Tx_{i-1},Tx_i) \ll \lambda\, d(x_{i-1},x_i) \ll \lambda\frac{c}{2} \text{ for all } i.$$

Consequently proceeding by induction, for each $m \in \mathbb{N}$, we have:

$$d(T^m x_{i-1},T^m x_i) \ll \lambda^m \frac{c}{2}.$$

Now, since by (2.1)

$$d(x_{i-1},x_{i+1}) \ll c,$$

then proceeding by induction, as T is a $(c,\lambda)-$uniformly locally contractive mapping, it follows

$$d(T^m x_{i-1},T^m x_{i+1}) \ll \lambda^m c.$$

Repeating the same arguments as above, one can easily show that

$$d(T^m x,T^{m+1}x) \ll \lambda^m n\frac{c}{2} \text{ for all } m \in \mathbb{N}. \tag{2.4}$$

• **Step II.** We note that if $T^m x = T^n x$ for some $m,n \in \mathbb{N}$, $m > n$, then put $p = m - n$ and $u = T^n x$, we have $T^p u = u$ and so $T^{kp}u = u$, for all $k \in \mathbb{N}$. Now taking the points u and Tu and proceeding as in step I we can obtain, for some fixed $n \in \mathbb{N}$,

$$d(T^m u,T^{m+1}u) \ll \lambda^m n\frac{c}{2} \text{ for all } m \in \mathbb{N}.$$

Then

$$d(u,Tu) = d(T^{kp}u,T^{kp+1}u) \ll \lambda^{kp} n\frac{c}{2}.$$

Since $\|\lambda^{kp} n\frac{c}{2}\| \to 0$ as $k \to +\infty$, then by (i) of Lemma 7 it follows $d(u,Tu) = 0$, and so $Tu = u$. Thus, we suppose $T^m x \neq T^n x$, for all $m,n \in \mathbb{N}$.
We are ready to show that $\{T^m x\}$ is a Cauchy sequence in X. Choose $k \in \mathbb{N}$ such that $k > 2$ and $\lambda^k < 1/n$, for n of (2.2). By (2.4), we get:

$$d(T^k x,T^{k+1}x) \ll \lambda^k n\frac{c}{2} \ll \frac{c}{2} \quad \text{and} \quad d(T^{k+1}x,T^{k+2}x) \ll \lambda^{k+1} n\frac{c}{2} \ll \frac{c}{2}.$$

Then, by (2.1) we have:

$$d(T^k x,T^{k+2}x) \ll c.$$

Let m be a positive integer such that $m > k$. We consider again two cases.

Case-I. If n is odd, put $n = 2l + 1$, where $l \geq 0$. Then we have:

$$\begin{aligned} d(T^m x,T^{m+n}x) &\leq d(T^m x,T^{m+1}x) + d(T^{m+1}x,T^{m+2}x) \\ &\quad + \cdots + d(T^{m+2l}x,T^{m+2l+1}x) \\ &\ll (\lambda^m + \lambda^{m+1} + \cdots + \lambda^{m+2l})n\frac{c}{2} \\ &\ll \frac{\lambda^m}{1-\lambda} n\frac{c}{2}. \end{aligned}$$

Case-II. If n is even, put $n = 2l$, where $l \geq 1$. In this case by (2.1) and (2.4), we have:

$$\begin{aligned} d(T^m x, T^{m+n}x) &\leq d(T^m x, T^{m+2}x) + d(T^{m+2}x, T^{m+3}x) \\ &\quad + \cdots + d(T^{m+2l-1}x, T^{m+2l}x) \\ &\ll \lambda^{m-k}c + (\lambda^{m+2} + \cdots + \lambda^{m+2l-1})n\frac{c}{2} \\ &\ll \lambda^{m-k}c + \frac{\lambda^{m+2}}{1-\lambda}n\frac{c}{2} = \frac{c\lambda^{m-k}}{2(1-\lambda)}(2 - 2\lambda + n\lambda^{k+2}). \end{aligned}$$

Coupling the two cases together, we have:

$$d(T^m x, T^{m+n}x) \ll \frac{c\lambda^{m-k}}{2(1-\lambda)} \max\{n\lambda^k, 2 - 2\lambda + n\lambda^{k+2}\}.$$

Since $\lambda \in [0, 1)$, $\|\frac{c\lambda^{m-k}}{2(1-\lambda)} \max\{n\lambda^k, 2 - 2\lambda + n\lambda^{k+2}\}\| \to 0$ as $m \to +\infty$. Then, by (ii) of Lemma 7, we deduce that $\{T^m x\}$ is Cauchy in X. By the completeness of X, there exists $u \in X$ such that

$$\lim_{m\to+\infty} T^m x = u.$$

Now, as a uniformly locally contractive mapping is continuous, and since (X, d) is Hausdorff, we obtain:

$$T(u) = T(\lim_{m\to+\infty} T^m x) = \lim_{m\to+\infty} T^{m+1}x = u.$$

Thus, u is a fixed point of T.

• **Step III.** To prove the uniqueness, let us assume that v is another fixed point of T, i.e. $Tv = v$. Since X is $c/2$-chainable, we can find a $c/2$-chain

$$u = x_0, x_1, x_2, \ldots, x_n = v.$$

Then using the same arguments as in step I, one can conclude that

$$d(u, v) = d(T^m u, T^m v) \ll \lambda^m n\frac{c}{2}, \text{ for all } m \in \mathbb{N}.$$

Finally, since $\|\lambda^m n\frac{c}{2}\| \to 0$ as $m \to +\infty$, by (i) of Lemma 7, we get $d(u, v) = 0$, which implies $u = v$. This makes end to the proof. □

Example 13. *Let $E = \mathcal{M}_{n\times n}(\mathbb{R})$ be the space of real matrix of order $n \geq 1$. Let $P \subset E$ be the cone defined by:*

$$P := \{M = (a_{ij})_{1\leq i,j\leq n} \mid a_{ij} \geq 0, \forall\, i, j\}.$$

Let $X = \{1,2,3,4\}$ *and* $d : X \times X \to E$ *be defined by:*

$$\begin{cases} d(1,2) = d(2,1) = 0.25\, I_n, \\ d(1,3) = d(3,1) = 0.1\, I_n, \\ d(2,3) = d(3,2) = 0.1\, I_n, \\ d(1,4) = d(4,1) = 0.2\, I_n, \\ d(2,4) = d(4,2) = 0.2\, I_n, \\ d(3,4) = d(4,3) = 0.2\, I_n, \\ d(x,x) = 0 \text{ for all } x \in X, \end{cases}$$

where I_n *is the identity matrix. Further, let* $T : X \to X$ *be the mapping defined by:*

$$Tx = \begin{cases} 3 & \text{if} \quad x \in \{1,2,3\}, \\ 1 & \text{if} \quad x = 4. \end{cases}$$

In this case, (X, d) *is not a cone metric space with respect to* P *since*

$$d(1,2) = 0.25\, I_n > d(1,3) + d(3,2) = 0.2\, I_n.$$

However, it is easy to see that (X, d) *is* $c/2$*-chainable cone rectangular metric space with respect to* P*,* $c = 0.44\, I_n$ *and satisfies condition (2.1). Moreover, we have that* T *is a* (c, λ)*-uniformly locally contractive mapping, with* $\lambda = 3/4$*. Applying Theorem 12, we obtain that* T *has a unique fixed point, that is* $x = 3$*.*

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