INTUITIONISTIC FUZZY GROUPOIDS AND ACTIONS

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ABSTRACT: *The purpose of this short note is to introduce a notion of (T,S)-intuitionistic fuzzy subgroupoid that allows to study within a unified framework various intuitionistic fuzzy structures such as intuitionistic fuzzy sets, intuitionistic fuzzy subgroup and (T, S)- indistinguishability operators. We also propose a notion of (T, S) - intuitionistic fuzzy action of a groupoid defined as a (T,S)-fuzzy subgroupoid.*

KEY WORDS: *groupoid, (T, S)-intuitionistic fuzzy subgroupoid, intuitionistic fuzzy action.*

1. TERMINOLOGY AND NOTATION

The term groupoid has different meanings. In this paper we refer to the notion of groupoid in the sense of Brandt [4] (small category with inverses). More precisely, by a (crisp) groupoid we mean a set G, together two maps: a partially defined multiplication

$$(x, y) \to xy \ [: G^{(2)} \subset G \times G \to G],$$

and an inverse map

$$x \to x^{-1} \ [: G \to G],$$

satisfying the following properties:

1. $(x^{-1})^{-1} = x$ for all $x \in G$.
2. If $(x,y) \in G^{(2)}$ and $(y,z) \in G^{(2)}$, then (xy, z), $(x,yz) \in G^{(2)}$ and $(xy)z=x(yz)$.
3. For all $x \in G$, (x, x^{-1}), $(x^{-1}, x) \in G^{(2)}$, and if $(x, y) \in G^{(2)}$ (respectively, $(z,x) \in G^{(2)}$), then $y = x^{-1}(xy)$ (respectively, $z = (zx)x^{-1}$).

The maps r and d on a groupoid G, defined by $r(x) = xx^{-1}$ and respectively, $d(x) = x^{-1}x$, are called the range and respectively the domain map. It is easy to see that $(x,y) \in G^{(2)}$ if and only if $d(x)=r(y)$. The maps r and d have a common image $r(G)=d(G)$ called the unit space of G. The unit space is denoted $G^{(0)}$.

The concept of fuzzy set was introduced by Zadeh [12] and concept of intuitionistic fuzzy set is due to Atanassov [1]. Afterwards, the theory of fuzzy and intuitionistic fuzzy sets was extended to other algebraic structure: groups, equivalence relation, group actions etc. In [5] we proposed a unifying approach to the above mentioned fuzzy structures through the (Brandt) groupoids and in [6] we tried to fuzzify not only the subset of the groupoid but also the groupoid operations.

In this paper we introduce a notion of intuitionistic fuzzy subgroupoid that generalizes intuitionistic fuzzy sets, intuitionistic fuzzy subgroup and indistinguishability operators. We also introduce a notion of intuitionistic fuzzy action (of a groupoid on a set) defined as a intuitionistic fuzzy subgroupoid. The notions of fuzzy action [3], intuitionistic fuzzy action [8] and [11] are obtained as particular cases.

In this paper $I = [0,1]$, $T: I \times I \to I$ is a t-norm, i.e. a function $T: I \times I \to I$ which satisfies the following properties:

1. $T(a, b) = T(b, a)$ for all $a, b \in I$;
2. $T(a, b) \le T(c, d)$ if $a \le c$ and $b \le d$;

3. $T(a, T(b, c)) = T(T(a, b), c)$ for all $a, b, c \in I$;
4. $T(a, 1) = a$ for all $a \in I$,

and $S: I \times I \to I$ is a t-conorm, i.e. a function $S: I \times I \to I$ which satisfies the following properties:

1. $S(a, b) = S(b, a)$ for all $a, b \in I$;
2. $S(a, b) \le S(c, d)$ if $a \le c$ and $b \le d$;
3. $S(a, S(b, c)) = S(S(a, b), c)$ for all $a, b, c \in I$;
4. $S(a, 0) = a$ for all $a \in I$.

(see [7] for various examples of t-norms).

We write

$I \times^* I = \{(a,b) \in I \times I : a + b \le 1\}$

and for $(a_1,b_1), (a_2,b_2) \in I \times^* I$

$(a_1,b_1) \le^* (a_2,b_2)$ iff $a_1 \le a_2$ and $b_1 \ge b_2$,

$(a_1,b_1) \ge^* (a_2,b_2)$ iff $a_1 \ge a_2$ and $b_1 \le b_2$.

2. INTUITIONISTIC GROUPOIDS

Definition of the intuitionistic groupoid: Let G be groupoid, $T : I \times I \to I$ be a t-norm and $S: I \times I \to I$ be a t-conorm. A function

$$\gamma = (\gamma_\mu, \gamma_\nu) : G \to I \times^* I$$

is said to be (T,S)-fuzzy intuitionistic subgroupoid of G if the following conditions are satisfied

i. $\gamma(xy) \ge^* (T(\gamma_\mu(x), \gamma_\mu(y)), S(\gamma_\nu(x), \gamma_\nu(y)))$ for all $(x, y) \in G^{(2)}$.

ii. $\gamma(x^{-1}) \ge^* \gamma(x)$ for all $x \in G$.

iii. $\gamma(r(x)) \ge^* \gamma(x)$ for all $x \in G$.

It follows easily from the definition that

1. $\gamma(x^{-1}) = \gamma(x)$ for all $x \in G$, because $(x^{-1})^{-1} = x$. Hence if we denote by inv the inversion on G, then $inv[\gamma](x) = \sup\{\gamma(y): y^{-1} = x\} = \gamma(x^{-1}) = \gamma(x)$.
2. $\gamma(d(x)) \ge^* \gamma(x)$ for all $x \in G$ (indeed, if we replace x with x^{-1} in iii, then $\gamma(r(x)) \ge^* \gamma(x^{-1}) = \gamma(x)$).
3. $\gamma(r(x)) = \gamma(xx^{-1})$
$\ge^* (T(\gamma_\mu(x),\gamma_\mu(x^{-1})), S(\gamma_\nu(x),\gamma_\nu(x^{-1})))$
$= (T(\gamma_\mu(x),\gamma_\mu(x)), S(\gamma_\nu(x),\gamma_\nu(x)))$
for all $x \in G$.
4. $\gamma(d(x)) = \gamma(x^{-1}x)$
$\ge^* (T(\gamma_\mu(x^{-1}),\gamma_\mu(x)), S(\gamma_\nu(x^{-1}),\gamma_\nu(x)))$
$= (T(\gamma_\mu(x),\gamma_\mu(x)), S(\gamma_\nu(x),\gamma_\nu(x)))$
for all $x \in G$.
5. $(T(\gamma_\mu(x),\gamma_\mu(d(x))), S(\gamma_\nu(x),\gamma_\nu(d(x)))) \le^* \gamma(xd(x)) = \gamma(x)$ for all $x \in G$.
6. $(T(\gamma_\mu(r(x)),\gamma_\mu(x)), S(\gamma_\nu(r(x)),\gamma_\nu(x))) \le^* \gamma(r(x)x) = \gamma(x)$ for all $x \in G$.
7. If $T(a,b) = T_{min}(a,b) = \min\{a, b\}$ and $S(a,b) = S_{max}(a, b) = \max\{a, b\}$ for all $a, b \in I$, then condition iii in the definition of intuitionistic groupoid is automatically satisfied.

Let us highlight a few structures which fit in the definition of intuitionistic groupoid.

If G is a group, then G can be viewed as groupoid with $G^{(2)} = G \times G$ and $G^{(0)} = \{e\}$ (the neutral element of G). Also $\gamma : G \to I \times^* I$ is a (T, S)-fuzzy subgroupoid of G if and only if

1. $\gamma(xy) \ge^* (T(\gamma_\mu(x), \gamma_\mu(y)), S(\gamma_\nu(x), \gamma_\nu(y)))$ for all $x,y \in G$
2. $\gamma(x^{-1}) = \gamma(x)$ for all $x \in G$
3. $\gamma(e) \ge^* \gamma(x)$ for all $x \in G$

For $T(a,b) = T_{min}(a,b) = \min\{a, b\}$ and $S(a,b) = S_{max}(a, b) = \max\{a, b\}$, this notion of (T, S)-fuzzy subgroupoid of a group G coincide with the notion of intuitionistic fuzzy subgroup in [2] (and for S=0, fuzzy subgroup [10]).

A set X can be regarded as a groupoid under the operations: $xx = x$, $x^{-1} = x$. Then $\gamma = (\gamma_\mu, \gamma_\nu) : X \to I \times^* I$ is a (S,T)-fuzzy intuitionistic subgroupoid of X if and only if $\gamma = \{(x, \gamma_\mu(x), \gamma_\nu(x), x \in X\}$ is an intuitionistic fuzzy set on X.

If $G = X \times X$, then G can be seen as a groupoid (the pair groupoid) under the operations: $(u, v)(v,w) = (u, w)$, $(u,v)^{-1} = (v,u)$.

A function $E = (E_\mu, \gamma_\nu) : X \times X \to I \times^* I$ is a (T,S) -fuzzy intuitionistic subgroupoid of $X \times X$ if and only if the following conditions are satisfied

1. $E(u,v) \ge^* (T(E_\mu(u,w), E_\mu(w,v)), S(E_\nu(u,w), E_\nu(w,v)))$ for all $u,v,w \in X$.
2. $E(u,v) = E(v,u)$ for all $u,v \in X$.
3. $E(u,u) \ge^* E(u,v)$ for all $u,v \in X$.

If E satisfies in addition the condition $E(u,u) = (1,0)$ for all $u \in X$, then E is a (T, S)-

indistinguishability operator on X in the sense of [9].

3. INTUITIONISTIC ACTIONS AS INTUITIONISTIC GROUPOIDS

We say a groupoid G acts (to the left) on a set X if there is $\rho : X \to G^{(0)}$ (called a momentum map) and a map

$$(g, x) \mapsto g \cdot x$$

from

$$\{(g, x) \in G \times X: d(g) = \rho(s)\}$$

to X, called left action, such that:

1. $\rho(g \cdot x) = r(g)$ for all $g \in G$ and $x \in X$ such that $d(g) = \rho(x)$.
2. $\rho(x) \cdot x = x$ for all $x \in X$.
3. If $(g, h) \in G^{(2)}$, $x \in X$ and $d(h) = \rho(x)$, then $(gh) \cdot x = g \cdot (h \cdot x)$.

In the same manner, we define a right action of G on X, using a map $\sigma : X \to G^{(0)}$ and a map

$$(x, g) \mapsto x \cdot g$$

from

$$\{(x, g) \in X \times G : \sigma(x) = r(g)\}$$

to X satisfying the following conditions:

1. $\sigma(x \cdot g) = d(g)$ for all $g \in G$ and $x \in X$ such that $r(g) = \sigma(x)$.
2. $x \cdot \sigma(x) = x$ for all $x \in X$.
3. If $(g, h) \in G^{(2)}$, $x \in X$ and $r(g) = \sigma(x)$, then $x \cdot (gh) = (x \cdot g) \cdot h$.

If $(g, x) \mapsto g \cdot x$ is a left action of a groupoid G on a set X with the momentum map ρ then

$$G[X, \rho] = \{(x, g, y) \in X \times G \times X: d(g) = \rho(y), r(g) = \rho(x)\}$$

is a groupoid under the operations:

$$(x, g_1, y)(y, g_2, z) = (x, g_1g_2, z)$$
$$(x, g, y)^{-1} = (y, g^{-1}, x).$$

Furthermore,

$$G \ltimes X = \{(g \cdot x, g, x) \in X \times G \times X: d(g) = \rho(y)\}$$

is a subgroupoid of G[X, ρ].

Similarly, if $(x, g) \mapsto x \cdot g$ is a right action of a groupoid G on a set X with the momentum map σ then

$$G[X, \sigma] = \{(x, g, y) \in X \times G \times X: d(g) = \sigma(y), r(g) = \sigma(x)\}$$

is a groupoid under the operations:

$$(x, g_1, y)(y, g_2, z) = (x, g_1g_2, z)$$
$$(x, g, y)^{-1} = (y, g^{-1}, x).$$

and

$$X \rtimes G = \{(x, g, x \cdot g) \in X \times G \times X: r(g) = \sigma(x)\}$$

is a subgroupoid of G[X, σ].

Every groupoid G acts to the left, respectively, right on its unit space $G^{(0)}$ by

$$g \cdot d(g) = r(g)$$

respectively,

$$r(g) \cdot g = d(g).$$

(In these cases $\rho(u) = \sigma(u) = u$ for all $u \in G^{(0)}$.). In addition

$$g \mapsto (r(g), g, d(g))$$

is a groupoid isomorphism from G to $G \ltimes G^{(0)}$, as well as, to $G^{(0)} \rtimes G$.

Definition of the intuitionistic fuzzy action: Let $T : I \times I \to I$ be a t-norm and $S: I\times I \to I$ be a t-conorm. A (T, S) - intuitionistic fuzzy left (respectively, right) action of a groupoid G on a set X with momentum map $\rho: X \to G^{(0)}$ (respectively, $\sigma: X \to G^{(0)}$) is a (T, S) - intuitionistic fuzzy subgroupoid of G[X, ρ] (respectively, G[X,σ]).

It follows from the definition that a function $\gamma = (\gamma_\mu, \gamma_\nu) : G[X, \rho] \to I \times^* I$ is a (T, S) - intuitionistic fuzzy left action of G on X with momentum map $\rho : X \to G^{(0)}$ if and only if the following conditions are satisfied

i. $\gamma(x, g_1g_2, z) \geq^* (T(\gamma_\mu(x, g_1, y), \gamma_\mu(y, g_2, z)), S(\gamma_\nu(x, g_1, y), \gamma_\nu(y, g_2, z)))$ for all $x, y, z \in X$, $g_1, g_2 \in G$ such that $\rho(x) = r(g_1)$, $\rho(y) = d(g_1) = r(g_2)$ and $\rho(z) = d(g_2)$.

ii. $\gamma(y, g^{-1}, x) \geq^* \gamma(x, g, y)$ for all for all $x, y \in X$, $g \in G$ such that $\rho(x) = r(g)$ and $\rho(y) = d(g)$.

iii. $\gamma(x, \rho(x), x) \geq^* \gamma(x, g, y)$ for all $x, y \in X$, $g \in G$ such that $\rho(x) = r(g)$ and $\rho(y) = d(g)$.

If $\gamma = (\gamma_\mu, \gamma_\nu) : G[X, \rho] \to I \times I$ is a (T, S) - intuitionistic fuzzy left action of G on X, then $\gamma_\mu(y, g, x)$ can be viewed as the degree to which y is the result of the action of g on x (i.e. $g \cdot x$). Also $\gamma_\nu(y, g, x)$ can be interpreted as the degree to which y and $g \cdot x$ differ.

G acts upon itself by either left (or right) translation (multiplication). Thus if we take into consideration, for instance the left action of G upon itself and the groupoid

$$G[G, r] = \{(x, g, y) \in G \times G \times G: d(g) = r(y), r(g) = r(x)\},$$

a (T, S) - intuitionistic fuzzy left (respectively, right) action of a groupoid G on G with momentum map $r : G \to G^{(0)}$, i.e. a (T, S) - intuitionistic fuzzy subgroupoid $\gamma = (\gamma_\mu, \gamma_\nu) : G[G, r] \to I \times I$, can be regarded as a fuzzification of the groupoid partially defined operation. More precisely, $\gamma_\mu(y, g, h)$ can be interpreted as the degree to which y is the product gh, and $\gamma_\nu(y, g, h)$ can be interpreted as the degree to which y and gh differ.

We end this section by showing that fuzzy actions in the sense of [3] and the intuitionistic fuzzy actions in the sense of [8] and [11], are particular cases of a (T, S) - intuitionistic fuzzy left action of a group G seen as groupoid.

Let G be a group acting to the left on a set X. Seen as a groupoid action, it has the momentum map $\rho: X \to \{e\}$ where e is neutral element of G. Hence in this case

$$G[X, \rho] = \{(x, g, y) \in X \times G \times X: d(g) = \rho(y), r(g) = \rho(x)\} = X \times G \times X.$$

Therefore a (T, S) - intuitionistic fuzzy left action of a group G on a set X is a (T, S) - intuitionistic fuzzy subgroupoid of $X \times G \times X$, i.e. a function

$$\gamma = (\gamma_\mu, \gamma_\nu) : X \times G \times X \to I \times^* I$$

satisfying the following conditions

i. $\gamma(x, gh, z) \geq^* (T(\gamma_\mu(x, g, y), \gamma_\mu(y, h, z)), S(\gamma_\nu(x, g, y), \gamma_\nu(y, h, z)))$ for all x, y, z $\in$ X and g, h $\in$ G.

ii. $\gamma(y, g^{-1}, x) \geq^* \gamma(x, g, y)$ for all for all $x, y \in X$, $g \in G$.

iii. $\gamma(x, e, x) \geq^* \gamma(x, g, y)$ for all x, $y \in X$, $g \in G$.

If we take S=0, γ_μ such that $\gamma_\mu(x, e, x) = 1$ for all $x \in X$, and if we define

$$\alpha(g,x,y) = \gamma_\mu(y, g, x)$$

for all $x, y \in X$, $g \in G$, then $\alpha: G \times X \times X \to I$ is a T –fuzzy action in the sense of [3]. Indeed,

1. $T(\alpha(hg, x, y), \alpha(g, x, z)) = T(\gamma_\mu(y, hg, x), \gamma_\mu(z, g, x)) = T(\gamma_\mu(y, hg, x), \gamma_\mu(x, g^{-1}, z)) \leq \gamma_\mu(y, (hg)g^{-1} z) = \alpha(h, z, y)$ for all x, y, $z \in X$ and $g, h \in G$.
2. $T(\alpha(g, x, z), \alpha(h, z, y)) = T(\gamma_\mu(z, g, x), \gamma_\mu(y, h, z)) = T(\gamma_\mu(y, h, z), \gamma_\mu(z, g, x)) \leq \gamma_\mu(y, hg, x) = \alpha(hg, x, y)$ for all x, y, $z \in X$ and $g, h \in G$.
3. $\alpha(e,x,x) = \gamma_\mu(x, e, x) = 1$ for all $x \in X$.

If $\gamma = (\gamma_\mu, \gamma_\nu)$ such that $\gamma_\mu(x, e, x) = 1$, $\gamma_\nu(x, e, x) = 0$ for all $x \in X$, and if we define

$$\alpha(g,x,y) = \gamma(y, g, x)$$

for all $x, y \in X$ and $g \in G$, then

$$\alpha: G \times X \times X \to I \times^* I$$

is a (T, S)-intuitionistic fuzzy action in the sense of [8] and [11] ($\alpha(x_g, y) = \gamma(y, g, x)$).

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