MATLAB SIMULATIONS OF THE SOLUTION OF A LINEAR QUADRATIC CONTROL PROBLEM FOR A CLASS OF STOCHASTIC FRACTIONAL SYSTEMS WITH MARKOVIAN JUMPS

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ABSTRACT: In this paper we apply the results in [7] concerning a finite-horizon, linear, quadratic optimal control problem for class of fractional order systems (FOS) with Markovian jumps to simulate the behaviour of the state variable of the system under the action of an optimal control input. We provide here some MATLAB algorithms for the computation of the state variable of the system and we illustrate its behaviour by plotting various instances of it. Unlike [7], where the simulations concern only the value of the optimal cost, in this paper we simulate the response of the system to both an optimal control action and no control action. As far as we know, such simulations are new for this type of systems.

KEY WORDS: ***fractional order systems,*** *discrete-time Riccati equation of control, optimal control, stochastic equations*

1. INTRODUCTION

Fractional calculus (FC) began to become more popular among the scientists due to its new applications in various fields of economic interest. For example, linear quadratic (LQ) optimal control problems represent an important branch of the control theory of linear fractional order systems (LFOS) and have many practical applications (see [1-3],[7-8] and the references therein). At this moment these issues are insufficiently addressed and any new result helps the development of the field. In [7] was solved recently a finite horizon LQ optimal control problem for infinite dimensional LFOS with Markovian jumps. Using a state expanded linear form of the fractional system (similar to the one in [5] and [8]) and an associated class of Riccati type equation, [7] provides the theoretic formulas of the optimal control and of the optimal cost. In this paper we shall use the results from [7] to simulate the response of a finite dimensional LFOS to the action of the optimal control. All simulations are made in MATLAB and the source codes of the programs are presented in this paper.

2. PRELIMINARIES

In this section we present the mathematical framework and the known theoretical results.

Let $\alpha \in (0,2)$ and $h > 0$ be fixed. For any $j \in \mathbf{N}$, $\binom{\alpha}{j}$ denotes the generalized binomial coefficient (see [7], [8]) and

$$\Delta^{[\alpha]} x_{k+1} = \frac{1}{h^{\alpha}} \sum_{j=0}^{k+1} (-1)^j \binom{\alpha}{j} x_{k+1-j}, h > 0$$

is the discrete-time version of the Grünwald-Letnikov operator (see for e.g. [3-4]).

In the sequel we shall consider the discrete-time fractional system with control

(1) $\Delta^{[\alpha]} x_{k+1} = \mathsf{A}_k(r_k)x_k + \mathsf{B}_k(r_k)u_k, k \in \mathrm{N}$

(2) $x_0 = x \in \mathrm{R}^d$,

where

- $\{r_k\}_{k\in\mathbf{N}}$ is a homogeneous Markov chain on a complete probability space $(\Omega, \mathbf{F}, P)$, having a countably finite or infinite state space $\mathbf{Z}$, $P(r_k = i) > 0, k \in \mathbf{N}, i \in \mathbf{Z}$ and the transition probability matrix $Q = \{q_{ij} = P(r_{n+1} = j \,|_{r_n = i})$ for all $n \in \mathbf{N}\}_{(i,j)\in \mathbf{Z}\times\mathbf{Z}}$.
- for a fixed $k \in \mathbf{N}$, $\mathsf{A}_k(i)$, $\mathsf{B}_k(i)$, $i \in \mathbf{Z}$ are bounded sequences of real matrices of dimensions $d \times d$ and $d \times m$, respectively.
- the control sequence $u = \{u_k\}_{k\in\mathrm{N}}$ belongs to a class of admissible controls $\mathbf{U}^a$ formed by all sequences u with the property that $u_k \in L^2(\Omega, R^m)$ is $\{r_i, 0 \le i \le k-1\}$-measurable for all $k \in \mathrm{N}^*$.

As in [7], our linear quadratic *optimal control problem associated with (1) consists* in minimizing the cost functional

(3) $I_{x,N,i}(u) = E[x_N^T S(r_N) x_N |_{r_n = i}] + \sum_{k=0}^{N-1} E[x_k^T C_k^T(r_k) C_k(r_k) x_k + u_k^T K_k(r_k) u_k |_{r_0 = i}]$

over the class $\mathbf{U}_N^a$ of admissible controls formed by all finite sub-sequences $\boldsymbol{u}_N = \{\boldsymbol{u}_k\}_{k\in\{0,1,\dots,N\}}$ of $\mathbf{U}^a$.

The coefficients $C_k(i)$, $K_k(i)$, $S(i)$ of the cost functional $I_{x,N,i}(u)$ are real matrices of dimensions $\boldsymbol{d} \times \boldsymbol{p}$, $\boldsymbol{m} \times \boldsymbol{m}$ and $\boldsymbol{d} \times \boldsymbol{d}$, respectively, which satisfy the condition that that there are $\delta_k, \delta > 0$ such that

$$u^T K_k(i) u \ge \delta_k u^T u, x^T S(i) x \ge \delta x^T x$$

for all $i \in \mathbf{Z}$ and $k \in \mathbf{N}$. Let

$A_k^0(i) = h^\alpha \mathsf{A}_k(i) + \alpha I_H, c_i := (-1)^i \begin{pmatrix} \alpha \\ i+1 \end{pmatrix}$,

$B_k(i) = h^\alpha \mathsf{B}_k(i)$ for all $k \in \mathbf{N}, i \in \mathbf{Z}$

and let E be the linear operator defined by

$$\mathsf{E}(P(i)) = \sum_{j\in\mathbf{Z}} q_{ij} P(j), i \in \mathbf{Z}.$$

for any bounded sequence $\{P(i)\}_{i\in\mathbf{Z}}$ of symmetric and nonnegative matrices.

As it follows from [7] system (1)-(2) can be equivalently rewritten as an expanded state linear discrete-time system with Markovian jumps

(4) $X_{k+1} = \mathbf{A}_k(r_k) X_k + \mathbf{B}_k(r_k) u_k$
$X_0 = (x_0, 0, \dots 0) \in R^{dN}, \ k = 1,..,N$

where x_0 is the initial value of system (1)-(2),

$$\mathbf{A}_k(i) = \begin{pmatrix} A_k^0(i) & c_1 I_{\mathrm{R}^d} & \dots & c_{N-1} I_{\mathrm{R}^d} \\ I_{\mathrm{R}^d} & 0 & . & 0 \\ . & I_{\mathrm{R}^d} & . & . \\ . & . & I_{\mathrm{R}^d} & 0 \\ 0 & . & 0 & 0 \end{pmatrix},$$

$$\mathbf{B}_k(i) = \begin{pmatrix} B_k(i) \\ 0 \\ . \\ 0 \end{pmatrix} \text{and } X_k^T = \Big(\underset{1}{x_k}, \underset{2}{x_{k-1}}, \dots, x_0, 0, .., \underset{N}{0}\Big).$$

The following backward discrete-time Riccati equation (see [6]) plays a key role in solving the above LQ optimal control problem:

$$\begin{aligned} (5) \quad P_k(i) = {} & \mathbf{A}_k^*(i) \mathsf{E}(P_{k+1})(i) \mathbf{A}_k(i) \\ & + \mathbf{C}_k^*(i) \mathbf{C}_k(i) - [\mathbf{A}_k^*(i) \mathsf{E}(P_{k+1})(i) \mathbf{B}_k(i)] \\ & \cdot [K_k(i) + \mathbf{B}_k^*(i) \mathsf{E}(P_{k+1})(i) \mathbf{B}_k(i)]^{-1} \\ & \cdot [\mathbf{B}_k^*(i) \mathsf{E}(P_{k+1})(i) \mathbf{A}_k(i)], i \in \mathbf{Z}, k \in \{0,1,..,N-2\} \\ & P_N(i) = \mathbf{S}(i). \end{aligned}$$

The sequences $\mathbf{C}_k(i)$ and $\mathbf{S(i)}$ are defined as in [7] by $\mathbf{C}_k(i)(v_0, v_1, \dots, v_{N-1})^T = C_k(i) v_0^T$, $\mathbf{S(i)}(v_0, \dots, v_{N-1})^T = (S(i) v_0^T, 0, \dots, 0)^T$ for all $(v_0, v_1, \dots, v_{N-1}) \in \mathrm{R}^{dN}$.

The next theorem is the main result from [7] and gives a formula for the computation of the optimal control gain.

Theorem 1. *If* $\{P_n\}_{n=0,..,N-1}$ *is the unique solution of the Riccati equation (5) and* $W_k, k = 0,..,N$ *is defined by*

$\mathbf{K}_k(P)(i) = K_k(i) + \mathbf{B}_k^*(i)\mathsf{E}(P)(i)\mathbf{B}_k(i)$

$W_k(i) = -[\mathbf{K}_k(P_{k+1})(i)]^{-1} \cdot [\mathbf{B}_k^*(i)\mathsf{E}(P_{k+1})(i)\mathbf{A}_k(i)]$ *then the control sequence that minimizes the cost functional* $I_{x,N,i}(u)$ *is*

$\hat{u} = \{\hat{u}_0 = W_0(r_0)X_0, \ldots, \hat{u}_N = W_N(r_N)X_N\}$

and the optimal cost is

$\min_{u \in \mathbf{U}_{0,N-1}^a} I_{x,N,i}(u) = \langle P_0(i)X_0, X_0 \rangle.$

3. MATLAB SIMULATIONS

In this section we present the algorithms which provide the state variable of the system under the control action. For the case of LFOS with multiplicative white noise and no Markovian jumps see [5].

Let us consider a time-invariant, version of the LFOS with control where $\alpha = \frac{1}{2}, h = 2, d = m = 2, p = 3, x_0 = x = \begin{pmatrix} 10 \\ 20 \end{pmatrix}, \mathbf{Z} = \{1,2\},$

$\mathsf{A}_k(1) = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \mathsf{A}_k(2) = \begin{pmatrix} -1 & 1 \\ 3 & 2 \end{pmatrix}, Q = \begin{pmatrix} 1/3 & 2/3 \\ 2/5 & 3/5 \end{pmatrix}$ *(Transition matrix),*

$K_k(1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, K_k(2) = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix},$

$B_k(i) = \frac{1}{i}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, C_k(i) = \frac{3}{i^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, S_k(i) = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix}$ and

and $r_0 = \begin{pmatrix} 1 & 2 \\ 0.3 & 0.7 \end{pmatrix}$ is the initial distribution of the Markov chain.

In what follows the functions:

matrA(h,alf,A,i,d,N); matrB(d,i,m,N), matrC(i,d,N,p) and **matrS(i,d,N)** define the coefficients $\mathbf{A}_k$, B_k, C_k and S of the Riccati equation (5), **cj(alf,j)** defines the coefficient c_i and function **erond(Z,i, P,X)** defines the operator E.

The next source code provides function **matrA(h, alf,A,i,d,N)** in the time invariant case and gives matrix $\mathbf{A}_k$ starting from matrix A_k, which was transmitted as a parameter A to the function.

```
function [rez] = matrA(h, alf,A,i,d,N)
I=eye(d);
A0=h^alf*A(:,:,i)+alf*I;
Z= zeros(d);
B0=I;
for i=1:N-1
   A0=[A0 I*cj( alf,i )];
   B0=[B0 Z];
end
A0=[A0;B0];
for i=3:N
   B0=[Z B0];
   B0=B0(:,1:d*N)
   A0=[A0;B0]
end
rez=A0;
end
```

The next function **LantMarkov(x,n, p, Q, m)** generates the first n elements of the Markov chain $r = \{r_n\}$ with the state space *{0, 1, 2,...,Z}*, the transition matrix *Q* and the initial distribution *p*. Function **FctRep(p,n)** computes the cumulative distribution function of a random variable with the probability distribution *p*.

Function **SimVAD(x,p,F,n)** generates a random variable having the vector values x and the cumulative distribution function F.

```
function [rez]= LantMarkov(x,n, p, Q, m)
 F=FctRep(p,n);
 r(1) =SimVAD(x, p,F,n);
 i = r(1);
 for k = 2 : m
% the i-th line of the transition matrix Q
 p = Q(i, :);
 F=FctRep(p,n);
r(k) = SimVAD(x,p,F,n);
i = r(k);
end
rez= r;

function [rez]=FctRep(p,n)
F(1)=p(1);
for i=2:n
   F(i)=F(i-1)+p(i);
end
rez=F
end
function [rez] = SimVAD(x,p,F,n)
```

```
k=1;
u=rand;
g=F(1);
while u>g
   k=k+1
   g=F(k);
end
rez=x(k);
end
end
```

Function **OptimCo1(N)** receives as parameter the number of iterations N, computes the solution P_n, n=1,..,N of the Riccati equation (5) and returns the coefficient $\mathbf{A}_k(i)-\mathbf{B}_k(i)W_k(i)$ of the response X_k of the system (4) to the action of the optimal control provided by Theorem 1.

```
function [rez]= OptimCo1(N)
Z=2;
alf=1/2;
A(:,:,1)=[1 2; 0 3];
A(:,:,2)=[-1 1; 3 2];
d=2;
h=2;
p=3;
m=2;
K(:,:,1)=[2 0; 0 2];
K(:,:,2)=[4 2; 2 4];
Q=[1/3 2/3; 2/5 3/5];
x0=[1 ;1];
for i=1:Z
A00(:,:,i)=matrA(h, alf,A,i,d,N);
B(:,:,i)=matrB(d,i,m,N);
C(:,:,i)=matrC(i,d,N,p);
end;
S=matrS(d,N);

% here it is computed  P(:,:,i,N+1) which
% denotes the component P_{N-1}(i) of the
% solution of the Riccati equation (5)

for i=1:Z
P(:,:,i,N+1)=C(:,:,i)'*C(:,:,i)+
A00(:,:,i)'*S*A00(:,:,i)-
A00(:,:,i)'*S*B(:,:,i)*inv(K(:,:,1)+B(:,:,i)'*S*
B(:,:,i))*B(:,:,i)'*S*A00(:,:,i)
end

% here it is computed  P(:,:,i,k) which denotes
% the  component P_{k-1}(i) of the solution of the
%  Riccati equation (5).

for k=N:-1:1
  for i=1:Z
  P(:,:,i,k)=C(:,:,i)'*C(:,:,i)+
A00(:,:,i)'*erond(Z,i,Q,P(:,:,:,k+1))*A00(:,:,i
)                                              -
A00(:,:,i)'*erond(Z,i,Q,P(:,:,:,k+1))*B(:,:,i)
*inv(K(:,:,1)+B(:,:,i)'
*erond(Z,i,Q,P(:,:,:,k+1))
*B(:,:,i))*B(:,:,i)'*erond(Z,i,Q,P(:,:,:,k+1))
*A00(:,:,i);
 end
end
for k=N:-1:1
  for i=1:Z
  WB(:,:,i,k)=A00(:,:,i)-B(:,:,i)*inv(K(:,:,1)
+B(:,:,i)'*erond(Z,i,Q,P(:,:,:,k+1))*B(:,:,i))*
B(:,:,i)'*erond(Z,i,Q,P(:,:,:,k+1))*A00(:,:,i);
  end
end
rez=WB
```

The following two algorithms generate 20 instances of the random variable x_n, n = 1,.., N, representing the response of the system (1) in the cases when the controlled gain is either optimal or absent.

Algorithm 1(generates the first component of x_n)

```
N=6; Z=2;
% p is the initial distribution of the Markov chain
p=[0.3 0.7];
% x is the state space of the Markov chain
 x=1:Z;
% Q is the transition matrix of the Markov chain
Q=[1/3 2/3; 2/5 3/5];
U=zeros(1:2)';
% here is defined X the initial value of the
% expanded state system (4)
X=[10 ;20];
for i=1:N-1
X=[X;U]
end
Y=X
% there are generated 20 instances of xn, the state
% variable of system (1)
 for i=1:20
```

```
r=LantMarkov(x,2, p, Q, N)
% LantMarkov(x,2, p, Q, N) generates vector r
% representing the first N components of a
% Markov chain
WB=OptimCo1(N)
%WB is the coefficient of Xn from the exanded
%state system (4) in the case when the optimal
%control is defined as in Theorem 1
for k=1:N
X=WB(:,:,r(k),1)*X;
end
% the resultin value X represents the N-th step
 % solution of system (4) and X=(x1,x2,..,xN),
% where x1,x2,..,xN are state states of system (1)
xx(1)=X(1)
for k=2:N
   xx(k)=X(1+2*(k-1))
end
% xx returns the first components of x1,x2,..,xN
% these components are ploted here with a dash-
% dot, red line
plot(1:N,xx,'-.r*');
hold on
for i=1:Z
WW (:,:,i)=matrA(h, alf,A,i,d,N);
end;
% WW collects the coefficients of system (4) in
% the  absence of control
X=Y
for k=1:N
X=WW(:,:,r(k))*X;
end
xx(1)=X(1)
for k=2:N
   xx(k)=X(1+2*(k-1))
end
% xx returns the first components of x1,x2,..,xN
% these components are plotted here with a solid
% blue line
plot(1:N,xx,'blue');
end
```

Algorithm 2(generates the second component of x_n)

```
N=6; Z=2;U=zeros(1:2)';
% here it is defined the initial value of the
% expanded state system (4)
X=[10 ;20];
for i=1:N-1
X=[X;U];
end
Y=X;
% p is initial distribution of the Markov chain
p=[0.3 0.7];
% x is the state space of the Markov chain
x=1:Z;
% Q is the transition matrix of the Markov chain
Q=[1/3 2/3; 2/5 3/5];
% there are generated 20 instances of xn, the state
% variable of system (1)
for i=1:20
% the first N components of the Markov chain
are %generated in vector r
r=LantMarkov(x,2, p, Q, N)
% LantMarkov(x,2, p, Q, N) generates vector r
% representing the first N components of a
% Markov chain
WB=OptimCo1(N)
%WB is the coefficient of Xn from the exanded
%state system (4) in the case when the optimal
%control is defined as in Theorem 1
for k=1:N
X=WB(:,:,r(k),2)*X;
end
for k=1:N
   xx(k)=X(2*k)
end
% xx returns the first components of x1,x2,..,xN
% these components are ploted here with a dash-
% dot, red line
plot(1:N,xx,'-.r*');hold on;
for i=1:Z
WW (:,:,i)=matrA(h, alf,A,i,d,N);
end;
% WW collects the coefficients of system (4) in
% the  absence of control
X=Y
for k=1:N
X=WW(:,:,r(k))*X;
end
for k=1:N
   xx(k)=X(2*k)
end
% xx returns the first components of x1,x2,..,xN
% these components are plotted here with a solid
% blue line
plot(1:N,xx,'blue');
```

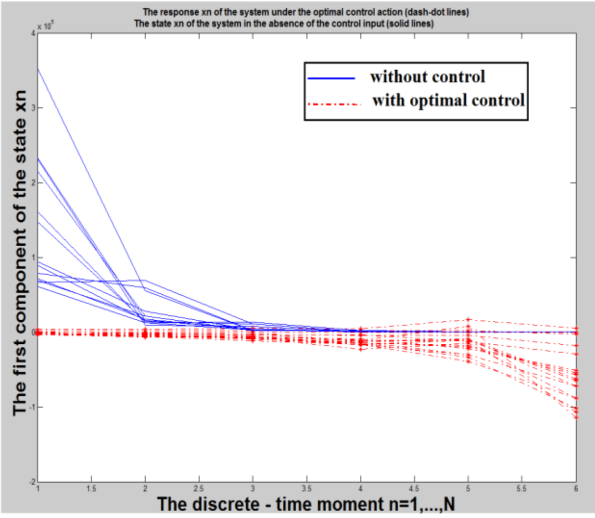

Figure 1. The first component of the state x_n.

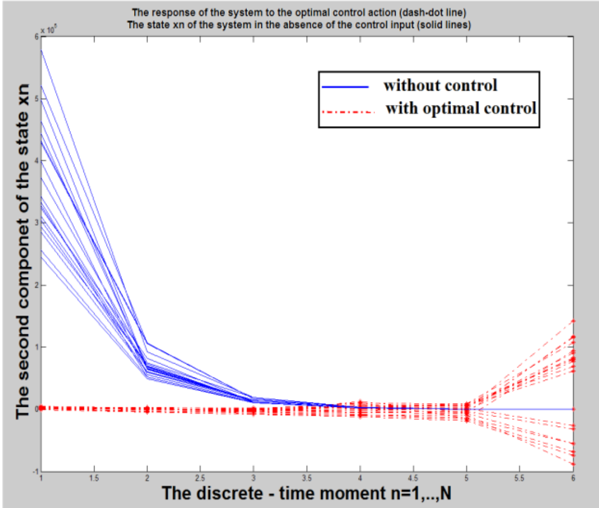

Figure 2. The second component of the state x_n.

4. CONCLUSION

This paper presents two computer algorithms which provides the response of a class of LFOSs to the optimal control input which represents the solution of a finite horizon optimal linear quadratic control problem solved in [7]. We hope that these algorithms will be helpful to all those who have to simulate the behaviour of fractional linear systems with Markovian jumps. Such algorithms seem to be new in the literature.

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