GROUPOIDS AND IRREVERSIBLE DISCRETE DYNAMICAL SYSTEMS I

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Abstract. *The purpose of this paper is to provide a formal approach based on groupoids for studying certain discrepancies between computational output and theoretical expectations in the analysis of the orbit space associated to an irreversible dynamical system.*

Keywords: dynamical system; groupoid; equivalence relation; computational analysis of dynamical.

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1. INTRODUCTION

The temporal evolution of a real world system can mathematically be described by a dynamical system. Classically, the continuous-time evolution is given by an ordinary differential equation of the form $\frac{dx}{dt}$=F(x) (F satisfying the Lipschitz existence condition), where x is state-valued function. On the other hand, if time is assumed to go on continuously but just single instances of time are taken into account, then the mathematical model is a discrete dynamical system. The mathematical setting for a discrete-time dynamical system is a space X and a map $\varphi:X\to X$. The space X is the phase space (the space of all possible states of the system) and the map φ defines time evolution - the change of the states over one time step: the state $x\in X$ at time t = 0 evolves into $\varphi(x)$ at t = 1, $\varphi(\varphi(x))$ at t = 2, etc. Consequently, $\varphi^n(x)$ is the state of the system at time t = n if x is the state of the system at time t = 0. Also this type of dynamical system naturally arises when an ordinary differential equation is integrated by an explicit numerical scheme.

There is a rich interplay between dynamical systems theory and computational analysis of dynamical systems. In this paper we take advantage of the framework of groupoids in order to study at a formal level the discrepancies between orbit computation using floating point arithmetic and theoretical expectations. More precisely, we introduce a groupoid associated to an irreversible dynamical system and to an equivalence relation on the phase space. The study of computational output versus theoretical expectations in the analysis of the orbit space will be replace by a comparative study of this groupoid and of the original groupoid associated to the dynamical system as in [3] and [4].

2. GROUPOIDS ASSOCIATED TO IRREVERSIBLE DYNAMICAL SYSTEMS

A groupoid is a set G, together with a distinguished subset $G^{(2)}\subset G\times G$, and two maps: a product map $(\gamma_1, \gamma_2) \to \gamma_1\gamma_2$ [:$G^{(2)} \to G$], and an inverse map $\gamma \to \gamma^{-1}$ [:$G \to G$], such that the following relations are satisfied:

(1) $(\gamma^{-1})^{-1} = \gamma$

(2) If $(\gamma_1,\gamma_2)\in G^{(2)}$ and $(\gamma_2,\gamma_3)\in G^{(2)}$, then $(\gamma_1\gamma_2, \gamma_3)$, $(\gamma_1,\gamma_2\gamma_3) \in G^{(2)}$ and $(\gamma_1\gamma_2)\, \gamma_3= \gamma_1(\gamma_2, \gamma_3)$.

(3) $(\gamma, \gamma^{-1})\in G^{(2)}$, and if $(\gamma_1, \gamma)\in G^{(2)}$, then $(\gamma_1\gamma)\gamma^{-1} = \gamma_1$.

(4) $(\gamma^{-1}, \gamma)\in G^{(2)}$, and if $(\gamma, \gamma_1)\in G^{(2)}$, then $\gamma^{-1}(\gamma\gamma_1) = \gamma_1$.

The maps r and d on G, defined by the formulae $r(\gamma) =\gamma\gamma^{-1}$ and $d(\gamma) =\gamma^{-1}\gamma$, are called the range and the source (domain) maps. It follows easily from the definition that they have a common image called the unit space of G, which is denoted $G^{(0)}$. The fibres of the range and the source maps are denoted $G^x =r^{-1}(\{x\})$ and $G_x =d^{-1}(\{x\})$, respectively. For x and y in $G^{(0)}$, (r,d)-fibre is $G_y^x = G^x\cap G_y$. It is easy to see that G_x^x is a group, called the isotropy group at x, and will be denoted G(x).

The relation x~y if and only if there is $\gamma\in G$ such that $r(\gamma)=x$ and $d(\gamma)$ is an equivalence relation on $G^{(0)}$. Its equivalence classes are called orbits. The graph of this equivalence relation

$$R=\{(x,y)\in G^{(0)} \times G^{(0)} : \text{there is } \gamma\in G \text{ such that } r(\gamma)=x \text{ and } d(\gamma)\}$$

can be regarded as a groupoid, under the operations:

$$(x,y)(y,z) = (x,z)$$
$$(x,y)^{-1} = (y,x)$$

R is called the principal groupoid associated with G. We denote by $(r,d):G\to R$, the map defined by

$$(r,d)(x)=(r(x),d(x)) \text{ for all } x\in G.$$

A topological groupoid consists of a groupoid G and a topology compatible with the groupoid structure i.e. the inverse and multiplication are continuous maps (the topology on $G^{(2)}$ is induced from G×G endowed with the product topology).

Notation 2.1. Let X be a topological space, $\varphi: X \to X$ a function and E be the graph of an equivalence relation on X. Let us denote be $G(X, \varphi, E)$ the set:

$$G(X, \varphi, E) =\{(x,k,y)\in X\times \mathbf{Z}\times X: \text{there is } n\in \mathbf{Z} \text{ such that } n+k\geq 0 \text{ and for all } m\geq n\ (\varphi^{m+k}(x), \varphi^{m}(y))\in E\},$$

where **Z** is the group of integers.

We endow $G(X, \varphi, E)$ with the subspace topology coming from $X\times\mathbf{Z}\times X$, where **Z** has the discrete topology. Under the operations

$$(x, n, y)(y, m, z) = (x, n+m, y)$$
$$(x, n, y)^{-1} = (y, -n, x)$$

$X\times\mathbf{Z}\times X$ is a topological groupoid. In the following the unit space of the groupoid $X\times\mathbf{Z}\times X$

$$\{(x,0,x),\ x\in X\}$$

will be identified with X.

Proposition 2.2. Let $\varphi: X \to X$ be a function, E be the graph of an equivalence relation on X and

$$G(X, \varphi, E) =\{(x,k,y)\in X\times\mathbf{Z}\times X: \text{ there is } n\in\mathbf{Z} \text{ such that } n+k\geq 0 \text{ and for all } m\geq n\ (\varphi^{m+k}(x), \varphi^{m}(y))\in E \}.$$

Then

1. $G(X, \varphi, E)$ is a subgroupoid of $X\times\mathbf{Z}\times X$ having the same unit space.
2. If X is a topological space and $G(X, \varphi, E)$ is endowed with the induced topology from $X\times\mathbf{Z}\times X$, then $G(X, \varphi, E)$ is a topological groupoid.

Proof. If $(x, k_1, y), (y, k_2, z) \in G(X, \varphi, E)$, then there are n_1 and n_2 such that $n_1+k_1 \geq 0$, $n_2+k_2 \geq 0$, and for all $m \geq \max(n_1-k_2, n_2)$,

$$\left(\varphi^{m+k_2+k_1}(x), \varphi^{m+k_2}(y)\right)\in E \text{ and } \left(\varphi^{m+k_2}(y), \varphi^{m}(z)\right)\in E.$$

Consequently, if $n_0 = \max(n_1-k_2, n_2)$, $n_0+k_2+k_1\geq n_1 + k_1\geq 0$ and for every $m\geq n_0$, $\left(\varphi^{m+k_2+k_1}(x), \varphi^{m}(z)\right)\in E$. Hence $(x, k_1+k_2, z)\in G(X, \varphi, E)$. If $(x,k,y)\in G(X, \varphi, E)$, then there is $n\in\mathbf{Z}$ such that $n+k\geq 0$ and for all $m\geq n$ $(\varphi^{m+k}(x), \varphi^{m}(y))\in E$. Let $n_1=\max(n+k,k)$. Then $n_1-k\geq 0$ and for all $m\geq n_1$ we have $\left(\varphi^{m-k+k}(x), \varphi^{m-k}(y)\right)\in E$ and consequently, $\left(\varphi^{m-k}(y), \varphi^{m}(x)\right)\in E$. Thus $(y,-k,x)\in G(X, \varphi, E)$.

Examples 2.3.

1. Let fl(x) denote the floating point number approximating x and let $\varphi: \mathbf{R} \to \mathbf{R}$ be a function. Let us define an equivalence relation E on R:

$y_1 \sim y_2$ if and only if $fl(y_1) = fl(y_2)$ or

there are x_1, x_2 such that $fl(x_1) = fl(x_2)$, $fl(\varphi(x_1)) = fl(y_1)$ $fl(\varphi(x_2))=fl(y_2)$.

Then $(x, fl(x))$, $(\varphi(fl(x)), \varphi(x))\in E$ and $(fl(\varphi(fl(x))), \varphi(x))\in E$ for all x.

The study of computational output versus theoretical expectations in the analysis of the orbit space could be replace by a comparative study of the groupoid $G(X, \varphi, E)$ and of the groupoid $G(X, \varphi) =\{(x,k,y)\in X\times\mathbf{Z}\times X: \text{there is } n\in\mathbf{Z} \text{ such that } n+k\geq 0\ \varphi^{m+k}(x) = \varphi^{m}(y)\}$ introduced in [3] (let us notice that if φ^* is an approximation for φ such that $fl(\varphi(fl(x)))=fl(\varphi^*(fl(x))$ for all x, then $(\varphi^{n}(x), \varphi^{*n}(fl(x))\in E$ for all x and all $n\in\mathbf{Z}$, $n\geq 1$).

2. To study a long term dynamical behavior of a discrete dynamical system (X, φ) we can use the groupoid $G(X, \varphi, E)$ where E is defined bellow assuming that the phase space is endowed with a uniform structure US:

x ~ y if and only if

for each $V\in US$ there is $n_V\in\mathbf{Z}$, $n_V\geq 0$ such that $(\varphi^{m}(x), \varphi^{m}(y))\in V$ for all $m\geq n_V$.

If (X, d) is a metric space and US is the uniform structure associated to the metric then

x ~ y if and only if $\lim_{n\to\infty} d(\varphi^{n}(x),\varphi^{n}(y))=0$.

3. If the space X is endowed with a uniform structure US and $E=\cap\{V, V\in US\}$ then $G(X,\varphi,E)$ and the groupoid $G(X, P, H, \theta, US)$ introduced in [2] coincide (where $xn=\theta(x,n)=\varphi^n(x)$, $P=\mathbf{N}$, $H=\mathbf{Z}$).

4. If $E = \Delta = \{ (x, x) : x \in X \}$, $\varphi:X\to X$, then $G(X,\varphi,E)= G(X, \varphi)$ (the groupoid introduced in [3]).

Notation 2.4. Let $\varphi: X \to X$ be a function, E be the graph of an equivalence relation on X and

$$G(X, \varphi, E) =\{(x,k,y)\in X\times\mathbf{Z}\times X: \text{ there is } n\in\mathbf{Z} \text{ such that } n+k\geq 0 \text{ and for all } m\geq n\ (\varphi^{m+k}(x), \varphi^{m}(y))\in E \}.$$

For each $x\in X$, let us denote by

$$H_x^x=\{k \in \mathbf{Z}: \text{there is } n\in\mathbf{Z} \text{ such that } n+k\geq 0 \text{ and for all } m\geq n\ (\varphi^{m+k}(x), \varphi^{m}(x))\in E \}$$

Let k_x be the smallest positive $k\in H_x^x$ if such k exists, and $k_x=0$ otherwise.

The next propositions will be used to characterize the transported topology (introduced in [1]) from $G(X,\varphi,E)$ to its principal groupoid.

Proposition 2.5. With the notation 2.4, for every $x,y\in X$, let

$$G_y^x=\{\gamma\in G(X, \varphi, E): r(\gamma)=x \text{ and } d(\gamma)=y\}.$$

(we identified the unit space of $G(X, \varphi, E)$ with X). Then

1. For every $x\in X$, $G_x^x =\{(x, k_x t, x): t\in\mathbf{Z}\}$.
2. For every $\gamma\in G(X, \varphi, E)$, $k_{r(\gamma)} = k_{d(\gamma)}$.
3. For every $\gamma\in G(X, \varphi, E)$ with the property that $k_{r(\gamma)} = 0$ (and consequently, $k_{d(\gamma)}=0$), there is a unique $k_{r(\gamma),d(\gamma)}\in\mathbf{Z}$ such that
$$G_{d(\gamma)}^{r(\gamma)}=\{(r(\gamma), k_{r(\gamma),d(\gamma)}, d(\gamma))\}.$$
Moreover $k_{d(\gamma),r(\gamma)} = - k_{r(\gamma),d(\gamma)}$.
4. For every $\gamma\in G(X, \varphi, E)$ with the property that $k_{r(\gamma)} \neq 0$, there is $k\geq 0$ such that $(r(\gamma),k,d(\gamma))\in G_{d(\gamma)}^{r(\gamma)}$

Proof. For each $x\in X$,

$$G_x^x=\{(x, k, x) \in X \times \mathbf{Z} \times X : \text{ there is } n\in\mathbf{Z} \text{ such that } n+k\geq 0 \text{ and for all } m\geq n\ (\varphi^{m+k}(x), \varphi^{m}(x))\in E \}$$

$$=\{(x, k, x) \in X \times \mathbf{Z} \times X : k\in H_x^x\}$$

$=\{x\}\times H_x^x\times\{x\}$, is the isotropy group at x associated to the groupoid $G(X, \varphi, E)$. Since H_x^x is a subgroup of $\mathbf{Z}$, it follows that there is an integer $k_x\geq 0$ such that $H_x^x=k_x\mathbf{Z}$ ($k_x = 0$ iff $H_x^x=\{0\}$ and k_x is the smallest positive $k\in H_x^x$ otherwise). Thus $G_x^x =\{(x, k_x t, x): t\in\mathbf{Z}\}$.

For every $\gamma\in G(X, \varphi, E)$, $H^{r(\gamma)}_{r(\gamma)}$ and $H^{d(\gamma)}_{d(\gamma)}$ are isomorphic. Consequently, $k_{r(\gamma)} = k_{d(\gamma)}$.

Let $\gamma\in G(X, \varphi, E)$ be such that $k_{r(\gamma)} = 0$. Let us assume by contradiction that there are $k_1\neq k_2$ such that $\gamma_1=(r(\gamma), k_1, d(\gamma))\in G^{r(\gamma)}_{d(\gamma)}$ and $\gamma_2=(r(\gamma), k_2, d(\gamma))\in G^{r(\gamma)}_{d(\gamma)}$. Then

$$(r(\gamma), k_1- k_2 ,r(\gamma)) = \gamma_1\gamma_2^{-1}\in G^{r(\gamma)}_{r(\gamma)}=\{(r(\gamma),0,r(\gamma))\}.$$

Hence $k_1 - k_2 = 0$, which is in contradiction with $k_1\neq k_2$. Consequently, there is a unique $k_{r(\gamma),d(\gamma)}\in\mathbf{Z}$ such that

$$G^{r(\gamma)}_{d(\gamma)}=\{(r(\gamma), k_{r(\gamma),d(\gamma)}, d(\gamma))\}.$$

Obviously, $G^{d(\gamma)}_{r(\gamma)}=\left(G^{r(\gamma)}_{d(\gamma)}\right)^{-1}=\{(d(\gamma), -k_{r(\gamma),d(\gamma)}, r(\gamma))\}$.

Let $\gamma\in G(X, \varphi, E)$ be such that $k_{r(\gamma)} \neq 0$. Then there is $m\in\mathbf{Z}$ such that $(r(\gamma),m,d(\gamma))\in G^{r(\gamma)}_{d(\gamma)}$. Let $t\in\mathbf{Z}$, $t\geq 0$ be such that $m+tk_{r(\gamma)} \geq 0$. It is easy to see that $(r(\gamma),k+tk_{r(\gamma)},d(\gamma))\in G^{r(\gamma)}_{d(\gamma)}$.

Notation 2.6. With the notation 2.4, for every $x,y\in X$, let

$$G^{x}_{y}=\{\gamma\in G(X, \varphi, E): r(\gamma)=x \text{ and } d(\gamma)=y\}.$$

and let $\gamma\in G(X, \varphi, E)$. If $k_{r(\gamma)} = 0$ (and consequently, $k_{d(\gamma)}=0$), let us denote by $k_{r(\gamma),d(\gamma)}$ the unique $k\in\mathbf{Z}$ such that

$$G^{r(\gamma)}_{d(\gamma)}=\{(r(\gamma), k, d(\gamma))\}.$$

If $k_{r(\gamma)} \neq 0$ let us denote by $k_{r(\gamma),d(\gamma)}$ the smallest nonnegative number k with the property that $(r(\gamma), k, d(\gamma))\in G^{r(\gamma)}_{d(\gamma)}$.

For each x, let n_x be the smallest nonnegative integer n, $n+k_x\geq 0$ satisfying $(\varphi^{k_x+m}(x),\varphi^m(x))\in E$ for all $m\geq n$.

For every equivalent units $x,y\in X$, let $n_{x,y}$ be the smallest nonnegative integer n satisfying $(\varphi^{k_{x,y}+m}(x),\varphi^m(y))\in E$, for all $m\geq n$.

Proposition 2.7. With the notations 2.4 and 2.6, we have

1. If $\gamma\in G(X, \varphi, E)$ and $k_{r(\gamma)} \neq 0$, then
$$G^{r(\gamma)}_{d(\gamma)}=\{(r(\gamma), k_{r(\gamma),d(\gamma)} + k_{r(\gamma)} t, d(\gamma)), t\in\mathbf{Z}\}.$$
2. If $\gamma\in G(X, \varphi, E)$, $k_{r(\gamma)} \neq 0$, then $k_{r(\gamma),d(\gamma)} \in \{0,1, \ldots, k_{r(\gamma)}-1\}$.
3. If $\gamma\in G(X, \varphi, E)$, then $k_{r(\gamma),d(\gamma)}=0 \iff k_{d(\gamma),r(\gamma)}=0 \iff$ there is $n\in\mathbf{N}$ such that for all $m\geq n$
$$(\varphi^m(r(\gamma)), \varphi^m(d(\gamma)))\in E.$$
4. If $\gamma\in G(X, \varphi, E)$ and $k_{r(\gamma),d(\gamma)}\neq 0$, then $k_{d(\gamma),r(\gamma)} = k_{r(\gamma)} - k_{r(\gamma),d(\gamma)}$.
5. For every equivalent units $x,y\in X$ with the property that $k_x\neq 0$, we have
$$n_{x,y} < k_x+\max(n_x,n_y).$$

Proof. Let $\gamma\in G(X, \varphi, E)$ be such that $k_{r(\gamma)} \neq 0$. Let $k_1 \in \mathbf{Z}$ and let k_0 be the remainder obtained by Division Theorem : $k_1 = k_{r(\gamma)} t + k_0$. Then

$$\gamma_0 = (r(\gamma), k_0, d(\gamma)) \in G_{d(\gamma)}^{r(\gamma)} \text{ if and only if } \gamma_1 = (r(\gamma), k_1, d(\gamma)) \in G_{d(\gamma)}^{r(\gamma)}.$$

Since $k_{r(\gamma)}$ is the smallest positive k with the property that $(r(\gamma),k,r(\gamma)) \in G_{r(\gamma)}^{r(\gamma)}$, it follows that k_0 is the smallest nonnegative number k having the property that $=(r(\gamma), k, d(\gamma)) \in G_{d(\gamma)}^{r(\gamma)}$. Thus $k_0 = k_{r(\gamma),d(\gamma)}$. Therefore $\gamma_1 = (r(\gamma), k_1, d(\gamma)) \in G_{d(\gamma)}^{r(\gamma)}$ if and only if there is $t\in\mathbf{Z}$ such that $k_1 = k_{r(\gamma)} t + k_{r(\gamma),d(\gamma)}$. Since $k_{r(\gamma),d(\gamma)}$. is the remainder, obviously, $k_{r(\gamma),d(\gamma)} \in \{0,1, \ldots, k_{r(\gamma)}-1\}$.

Moreover $k_1 = k_{r(\gamma)} t + k_{r(\gamma),d(\gamma)}$ implies $-k_1 = k_{r(\gamma)} (-t-1) + k_{r(\gamma)} - k_{r(\gamma),d(\gamma)} = k_{d(\gamma)} (-t-1) + k_{r(\gamma)} - k_{r(\gamma),d(\gamma)}$. If $k_{r(\gamma),d(\gamma)} \neq 0$, then $0 \leq k_{r(\gamma)} - k_{r(\gamma),d(\gamma)} < k_{r(\gamma)} = k_{d(\gamma)}$. Thus $k_{r(\gamma)} - k_{r(\gamma),d(\gamma)}$ is the remainder of the division of $-k_1$ by $k_{d(\gamma)}$. On the other hand

$$\gamma_1 = (r(\gamma), k_1, d(\gamma)) \in G_{d(\gamma)}^{r(\gamma)} \text{ if and only if } \gamma_1^{-1} = (d(\gamma), -k_1, r(\gamma)) \in G_{r(\gamma)}^{d(\gamma)}.$$

Consequently, $k_{d(\gamma),r(\gamma)} = k_{r(\gamma)} - k_{r(\gamma),d(\gamma)}$.

Let us consider two equivalent units $x,y\in X$ such that $k_x\neq 0$. Let us assume by contradiction that $n_{x,y} \geq k_x+\max(n_x,n_y)$. Hence $n_{x,y} - 1 \geq \max(n_x,n_y)$. Since for all $n \geq \max(n_x,n_y)$,

$$(\varphi^{k_{x,y}+n}(x), \varphi^{k_{x,y}+k_x+n}(x)) \in E, \ (\varphi^{k_x+n}(y), \varphi^{n}(y)) \in E,$$

it follows that for all $m \geq n_{x,y} - 1$,

$$(\varphi^{k_{x,y}+m}(x), \varphi^{k_{x,y}+k_x+m}(x)) \in E, \ (\varphi^{k_x+m}(y), \varphi^{m}(y)) \in E.$$

On the other hand for all $m \geq n_{x,y} - 1$, $k_x + m \geq n_{x,y}$ and therefore

$$(\varphi^{k_{x,y}+k_x+m}(x), \varphi^{k_x+m}(y)) \in E.$$

Consequently, $(\varphi^{k_{x,y}+m}(x), \varphi^{m}(y)) \in E$ for all $m \geq n_{x,y} - 1$, which is in contradiction with the choice of $n_{x,y}$. Thus $n_{x,y} < k_x+\max(n_x,n_y)$.

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