GROUPOIDS AND IRREVERSIBLE DISCRETE DYNAMICAL SYSTEMS II

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Abstract. The purpose of this paper is to study the topology of the orbit space of an irreversible discrete dynamical system \((X, \varphi)\) seen as a principal groupoid associated to the groupoid \(G(X, \varphi, E)\) introduced in [1] (where \(E\) is an equivalence relation on \(X\)).

Keywords: dynamical system; groupoid; equivalence relation; topology.

1. INTRODUCTION

We use the same notation and definitions as in [1]. The principal groupoid \(R\) associated with a groupoid \(G\) can be endowed with various topologies such as product topology [6] (the subspace topology on \(R\) induced from \(G^{(0)} \times G^{(0)}\), where \(G^{(0)}\) is endowed itself with the subspace topology coming from \(G\)) or quotient topology [6] (the finest topology on \(R\) with the property that \((r, d): G \to R\) is continuous). However under these topologies the fibres of \(R\) fail to have certain topological properties that the fibres of \(G\) had and in many cases the topological properties of fibres (endowed the subspace topology) are more important than the properties of the space. In [2] we introduced a topology on \(R\) such that the maps \(d_x: G^x \to R^x\) are continuous open maps, where \(d_x\) is defined by \(d_x(\gamma) = d(\gamma)\) for all \(\gamma \in G\), so certain properties of \(G^x\) are transported on \(R^x\). We called that topology the transported topology from \(G\). The transported topology from \(G\) on \(R\) is finer than the quotient topology on \(R\) which is finer than the product topology on \(R\).

The purpose of this paper is to characterize the transported topology for the groupoid \(G(X, \varphi, E)\) introduced in [1].

2. THE PRINCIPAL GROUPOID ASSOCIATED TO \(G(X, \varphi, E)\)

Proposition 2.1. Let \(X\) be a locally compact space, \(\varphi: X \to X\) a function and \(E\) the graph of an equivalence relation on \(X\) satisfying the condition that for each net \((x_i)_{i \in I}\) in \(X\) converging to \(x \in X\), there is \(i_0 \in I\) and \(m_0 \in \mathbb{Z}\), \(m_0 \geq 0\) such that \((\varphi^m(x_i), \varphi^m(x)) \in E\) for all \(i \geq i_0\) and all \(m \geq m_0\). Then

\[ G(X, \varphi, E) = \{(x, k, y) \in X \times \mathbb{Z} \times X:\text{ there is } n \in \mathbb{Z}\text{ such that } n+k \geq 0\text{ and for all } m \geq n\ (\varphi^{m+k}(x), \varphi^{m}(y)) \in E\} \]

endowed with the subspace topology coming from \(X \times \mathbb{Z} \times X\), where \(\mathbb{Z}\) has the discrete topology, is a topological locally compact groupoid under the operations

\[(x, n, y)(y, m, z) = (x, n+m, y)\]
\[(x, n, y)^{-1} = (y, -n, x)\]

Proof. We proved in [1] \(G(X, \varphi, E)\) is a topological groupoid. Let \((x, k, y) \in G(X, \varphi, E)\), \(A_x\) be a compact neighborhood of \(x\) and \(B_y\) be a compact neighborhood of \(y\). Let us prove that \((A_x \times \{k\} \times B_y) \cap G(X, \varphi, E)\) is a compact neighborhood of \((x, k, y)\). Let \((x_i, k, y_i)_{i \in I}\) be a net in \((A_x \times \{k\} \times B_y) \cap G(X, \varphi, E)\). Since \(A_x\) (respectively, \(B_y\)) is compact there is a subnet
of \((x_i)_i\) (respectively, \((y_i)_i\)), also denoted \((x_i)_i\) (respectively, \((y_i)_i\)), converging to \(a \in A_X\), (respectively, \(b \in B_X\)). Let us show that \((a,k,b) \in G(X, \varphi, E)\). Let \(m_0 \geq 0\) and \(i_0 \in I\) be such that \(m_0 + k \geq 0\) and \((\varphi^{m+k}(x_i), \varphi^m(a), \varphi^m(b)) \in E\) for all \(i \geq i_0\) and all \(m \geq m_0\). Since \((x_i, k, y_i) \in G(X, \varphi, E)\), it follows that there is \(n_i \in \mathbb{Z}\) such that \(n_i + k \geq 0\) and for all \(m \geq n_i\), \((\varphi^{m+k}(x_i), \varphi^m(y_i)) \in E\). Let \(n_0 = \max(n_{i_0}, m_0)\). Then for all \(m \geq n_0\),

\[
(\varphi^{m+k}(a), \varphi^m(b)) = (\varphi^{m+k}(x_i), \varphi^m(a)) (\varphi^{m+k}(x_i), \varphi^m(y_i)) (\varphi^m(y_i), \varphi^m(b)) \in E.
\]

Thus \((a,k,b) \in G(X, \varphi, E)\), and therefore \((A_X \times \{k\} \times B_Y) \cap G(X, \varphi, E)\) is compact.

**Notation 2.2.** Let \(X\) be a topological space, \(\varphi: X \to X\) a function and \(E\) be the graph of an equivalence relation on \(X\). The principal groupoid associated to \(G(X, \varphi, E) = \{(x, k, y) \in X \times \mathbb{Z} \times X : \text{there is } n \in \mathbb{Z} \text{ such that } n + k \geq 0 \text{ and for all } m \geq n \ (\varphi^{m+k}(x), \varphi^m(y)) \in E\}\) is \(R(X, \varphi, E) = \{(x, y) \in X \times X : \text{there is } n, k \in \mathbb{Z} \text{ such that } n + k \geq 0 \text{ and for all } m \geq n \ (\varphi^{m+k}(x), \varphi^m(y)) \in E\}\).

Let us denote with \(\tau_G\) the subspace topology on \(G(X, \varphi, E)\) coming from \(X \times \mathbb{Z} \times X\), where \(\mathbb{Z}\) has the discrete topology.

Let us denote with \(\tau_R\) the topology on \(R(X, \varphi, E)\) transported topology from \(G(X, \varphi, E)\) (defined in [2]). Let us recall that a basis for the topology \(\tau_R\) is given by the family of sets \(\{U(F)\}_F\), where each \(F\) is a finite collection \(F\) of open subsets of \(G(X, \varphi, E)\) (i.e. \(F \subset \tau_G\)) and

\[
U(F) = \bigcap_{U \in F} \{r, d\}(U) = \bigcap_{U \in F} \{\{(x, y) : \text{there is } k \in \mathbb{Z} \text{ such that } (x, k, y) \in U\}. \}
\]

Let us denote by \(\tau_X\) the topology on \(X\), and let us notice that \(\tau_X\) coincides with the topology on \(X\) seen as unit space of \(G(X, \varphi, E)\) (under the identification \(x \to (x, 0x)\))

Let us denote by \(\tau_X(R)\) the topology on \(X\) seen as unit space of \(G(X, \varphi, E)\) (under the identification \(x \to (x, x)\)). The topology \(\tau_X(R)\) is finer that \(\tau_X\) [2].

**Proposition 2.3.** Let \(X\) be a topological space, \(\varphi: X \to X\) a function and \(E\) be the graph of an equivalence relation on \(X\). With the notations 2.4, 2.5 [1] and 2.2, if \((x_i)_{i \in I}\) is a net in \(X\) and \(x \in X\) such that \(k_x \neq 0\), then the following conditions are equivalent:

i) \((x_i)_{i \in I}\) converges to \(x\) with respect to \(\tau_X(R)\).

ii) \((x_i)_{i \in I}\) converges to \(x\) with respect to \(\tau_X\) and there is \(i_0\) such that for all \(i \geq i_0\), \(k_{x_i} \neq 0\) and \(k_x \mid k_{x_i}\) (\(k_{x_i}\) divides \(k_x\)).

**Proof.** \((x_i)_{i \in I}\) converges to \(x\) with respect to \(\tau_X(R)\) if and only if \((x_i, x) \to (x, x)\) with respect to \(\tau_R\). Furthermore \((x_i, x) \to (x, x)\) with respect to \(\tau_R\) if and only if for every \(\gamma\) in
G(X,\varphi,E) with r(\gamma)=x and d(\gamma)=y and every subnet \( \{x_{i,j}, x_{i,j}\}_j \) of \( (x_i,x_i) \) there is a subnet \( \{x_{i,k}, x_{i,k}\}_k \) with the property that there are there are \( \gamma_k \in G(X, \varphi, E) \) with \( r(\gamma_k)=x_{i,k} \) and \( d(\gamma_k)=x_{i,k} \) such that \( \gamma_k \rightarrow \gamma \). Thus the following conditions are equivalent

a. \((x_i,x_i)\rightarrow(x,x)\) with respect to \( \tau_R \)

b. \((x_i)_{i \in I} \) converges to \( x \) with respect to \( \tau_X \) and for every \( k \) such that \((x, k, x) \) \( \in \) \( G(X,\varphi,E) \), there is \( i_k \) such that for all \( i \geq i_k (x_i, k, x_i) \) \( \in \) \( G(X,\varphi,E) \).

Let \( k=k_x \) (obviously, \((x,k_x,y)\) \( \in \) \( G(X,\varphi,E) \)). Moreover \((x_i, k_x, y_i) \) \( \in \) \( G(X,\varphi,E) \) if and only if there is an integer \( t \) such that \( k_x=t_i x_i \). Thus if \((x_i,k_x,y_i) \) \( \in \) \( G(X,\varphi,E) \), then \( k_x=0 \) and \( k_x \mid k_x \). Conversely, let us assume that there is an integer \( t \) such that \( k_x=t_i x_i \) and let \( k \) such that \((x, k, y) \) \( \in \) \( G(X,\varphi,E) \). Then there is an integer \( t \) such that \( k=t_i x_i \). Hence \( k=t_i x_i \) and therefore \((x_i, k, y_i) \) \( \in \) \( G(X,\varphi,E) \).

**Proposition 2.4.** Let \( X \) be a topological space, \( \varphi: X \rightarrow X \) a function and E be the graph of an equivalence relation on X. With the notations 2.4, 2.6 [1] and 2.2, if \((x_i)_{i \in I} \) is a net in \( X \) and \( x \in X \) such that \( k_x \neq 0 \), then the following conditions are equivalent:

i) \((x_i)_{i \in I} \) converges to \( x \) with respect to \( \tau_X(R) \).

ii) \((x_i)_{i \in I} \) converges to \( x \) with respect to \( \tau_X \)

**Proof.** As in the proof of the preceding proposition, \((x_i)_{i \in I} \) converges to \( x \) in \( \tau_X(R) \) if and \((x_i)_{i \in I} \) converges to \( x \) in \( \tau_X \) and for all \( k \) with the property that \((x, k, x) \) \( \in \) \( G(X,\varphi,E) \), there is \( i_k \) such that for all \( i \geq i_k (x_i, k, x_i) \) \( \in \) \( G(X,\varphi,E) \). If \( k_x=0 \) and \((x, k, x) \) \( \in \) \( G(X,\varphi,E) \), then \( k=0 \). Since \((x_i,0,x_i)\) \( \in \) \( G(X,\varphi,E) \) for all \( i \) \( \in \) \( I \), it follows that \((x_i)_{i \in I} \) converges to \( x \) in \( \tau_X(R) \).

**Proposition 2.4.** Let \( X \) be a topological space, \( \varphi: X \rightarrow X \) a function and E be the graph of an equivalence relation on X. With the notations 2.4, 2.6 [1] and 2.2, if \((x_i,y_i)_{i \in I} \) is a net in \( R(X,\varphi,E) \) and \((x,y)\) \( \in \) \( R(X,\varphi,E) \) such that \( k_x \neq 0 \), then the following conditions are equivalent:

i) \((x_i,y_i)_{i \in I} \) converges to \((x,y)\) with respect to \( \tau_R \).

ii) \((x_i)_{i \in I} \) converges to \( x \) with respect to \( \tau_X \), \((y_i)_{i \in I} \) converges to \( y \) with respect to \( \tau_X \) and there is \( i_0 \) such that for all \( i \geq i_0 \),

\[ k_x \neq 0, k_x | k_x (k_x \text{ divides } k_x) \text{ and } k_x | k_{x,y} - k_{x,y} \]

**Proof.** \((x_i,y_i)_{i \in I} \) converges to \((x,y)\) with respect to \( \tau_R \) if and only if for every \( \gamma \) in \( G(X,\varphi,E) \) with \( r(\gamma)=x \) and \( d(\gamma)=y \) and every subnet \( \{x_{i,j}, y_{i,j}\}_j \) of \( (x_i,y_i) \) there is a subnet \( \{x_{i,k}, y_{i,k}\}_k \) with the property that there are there are \( \gamma_k \in G(X, \varphi, E) \) with \( r(\gamma_k)=x_{i,k} \) and \( d(\gamma_k)=y_{i,k} \) such that \( \gamma_k \rightarrow \gamma \) [2].

i) \( \Rightarrow \) ii) If \((x_i,y_i) \) is a net in \( R(X, \varphi, E) \) and if \((x_i,y_i) \rightarrow(x,y) \) with respect to \( \tau_R \), then \( r(x_i,y_i) \rightarrow r(x,y)=x \) and \( d(x_i,y_i) \rightarrow d(x,y)=y \) with respect to \( \tau_X(R) \). Hence there is \( i_1 \) such that for all \( i \geq i_1, k_{x,y}=0, k_{x,i} | k_x (k_x \text{ divides } k_x) \). If \( \gamma=(x, k_{x,y}, y) \), then \( \gamma \in G(X,\varphi,E) \), \( r(\gamma)=x \) and \( d(\gamma)=y \). Thus there is \( i_2 \) such that for all \( i \geq i_2, (x_i, k_{x,y}, y_i) \in G(X,\varphi,E) \). Hence there is \( i_1 \in \mathbb{Z} \) such that
Therefore \( k_{x_i} = k_{x_i, y_i} + t_i k_{x_i} \).

\[ k_{x,y} = k_{x_i, y_i} + t_i k_{x_i} \]

\( \therefore \) If \((x, k, y) \in G(X, \varphi, E)\) imply \((x_i, k, y_i) \in G(X, \varphi, E)\), for large enough \(i\).

**Proposition 2.5** Let \(X\) be a topological space, \(\varphi: X \to X\) a function and \(E\) be the graph of an equivalence relation on \(X\). With the notations 2.4, 2.6 [1] and 2.8, if \((x_i, y_i)_{i \in I}\) is a net in \(R(X, \varphi, E)\) and \((x, y) \in R(X, \varphi, E)\) such that \(k_x = 0\), then the following conditions are equivalent:

\[ (x_i, y_i)_{i \in I} \text{ converges to } (x, y) \text{ in } R(X, \varphi, E). \]

\[ (x_i)_{i \in I} \text{ converges to } x \text{ with respect to } \tau_X, (y_i)_{i \in I} \text{ converges to } y \text{ in } \tau_X, \text{ there is } i_0 \text{ such that for all } i \geq i_0, \]

\[ (x_i = 0 \text{ and } k_{x_i, y_i} = k_{x,y}) \text{ or } (k_{x_i} \neq 0 \text{ and } k_{x_i} \mid k_{x,y} - k_{x_i, y_i}) \]

**Proof.** \(i) \Rightarrow ii)\) If \((x_i, y_i)_{i \in I}\) is a net in \(R(X, \varphi, E)\) and if \((x_i, y_i) \to (x, y)\) with respect to \(\tau_R\), then \(r(x_i, y_i) \to r(x, y) = x\) and \(d(x_i, y_i) \to d(x, y) = y\) with respect to \(\tau_X(R)\) and consequently with respect to \(\tau_X\). If \(\gamma = (x, k_{x,y}, y)\), then \(\gamma \in G(X, \varphi, E)\), \(r(\gamma) = x\) and \(d(\gamma) = y\). Thus there is \(i_2\) such that for all \(i \geq i_2\), \((x_i, k_{x,y}, y_i) \in G(X, \varphi, E)\). Hence there is \(i_0 \in \mathbb{Z}\) such that

\[ k_{x,y} = k_{x_i, y_i} + t_i k_{x_i} \]

Therefore if \(k_{x_i} = 0\), \(k_{x_i} \mid k_{x,y} - k_{x_i, y_i}\). If \(k_{x_i} = 0\), then \(k_{x_i, y_i} = k_{x,y}\).

\(ii) \Rightarrow i)\) It can be easily prove that \(ii)\) and \((x, k, y) \in G(X, \varphi, E)\) imply \((x_i, k, y_i) \in G(X, \varphi, E)\), for large enough \(i\).

**BIBLIOGRAPHY**


