A NEW REPRESENTATION RESULT FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH INFINITE MARKOV JUMPS AND MULTIPLICATIVE NOISE

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Abstract. *In this paper we give a new representation of the conditional mean square of the solutions for a class of stochastic differential linear equations with infinite Markov jumps (SDELMs) and multiplicative noise. The obtained result is related to the solutions of two Lyapunov type differential equations defined on ordered Banach spaces of sequences of bounded operators.*

Keywords: seqences, matrix, subspace;

1. INTRODUCTION

In the last decades, the SDELMs with and without multiplicative noise have attracted the interest of the researchers [5], [6] and led to new applications in modern queuing network theory [4] or in the study of safety-critical and high integrity systems (see [1] and the references therein.) As in the discrete time-case (see for e.g [9], [8]), the representation of the conditional mean square of the solutions for SDELMs play an important role in studying different stability and optimal control problems ([8], [5], [6], [1]). So, in this paper we establish a new representation result based on the solution properties of some Lyapunov type equations associated with the discussed SDELMs.

2. NOTATIONS

Let $\mathbf{Z}$ be an interval of integers, which may be finite or infinite. Let $\mathbf{R}^n$ be the n-dimensional Euclidian space of real numbers and let $M_{n\times m}(\mathbf{R})$ be the real normed linear space of all $n\times m$ matrices with real entries; if $m=n$ we will write $M_n(\mathbf{R})$ instead of $M_{n\times n}(\mathbf{R})$. Let $l^{\mathbf{Z}}_{M_{n\times m}(\mathbf{R})}$ be the space of all $\mathbf{Z}$-sequences $g=\{g_{[i]}\in M_{n\times m}(\mathbf{R})\}_{i\in\mathbf{Z}}$ with the property that $\|g\|_{\mathbf{Z}} := \sup_{i\in\mathbf{Z}}\|g_{[i]}\| < \infty$. It can be shown by using a standard procedure that $l^{\mathbf{Z}}_{M_{n\times m}(\mathbf{R})}$ is a real Banach space when endowed with the usual term-wise addition, the real scalar multiplication and the norm $\|\cdot\|_{\mathbf{Z}}$. The Banach subspace of $l^{\mathbf{Z}}_{M_n(\mathbf{R})}$ formed by all sequences $g=\{g_{[i]}\}_{i\in\mathbf{Z}}$ of symmetric matrices $g_{[i]}, i\in\mathbf{Z}$ will be denoted by $l^{\mathbf{Z}}_{S_n(\mathbf{R})}$. An element $g\in l^{\mathbf{Z}}_{M_n(\mathbf{R})}$ is said to be positive, and we write $g\geq 0$, iff $g_{[i]}$ is a nonnegative matrix ($g_{[i]}\geq 0$) for all $i\in\mathbf{Z}$. If I_n is the identity matrix from $M_n(\mathbf{R})$, then $\Phi=(...I_n, I_n, I_n,...)$ is an element of $l^{\mathbf{Z}}_{M_n(\mathbf{R})}$.

Let us consider the linear subspace $H_n^{\mathbf{Z}}$ of $l^{\mathbf{Z}}_{M_n(\mathbf{R})}$ formed by all sequences $\{P_{[i]}\}_{i\in\mathbf{Z}}$ with the property $\|P\|_2=\sqrt{\sum_{i\in\mathbf{Z}} TrP_{[i]}^T P_{[i]}}<\infty$, where TrA is the trace of the matrix $A\in M_n(\mathbf{R})$ and the the superscript ^{T} denotes the transpose. It is not difficult to see that $H_n^{\mathbf{Z}}$ is a Hilbert space with the inner product $\langle D,F\rangle_2=\sum_{i\in\mathbf{Z}} TrF_{[i]}^T D_{[i]}, D,F\in H_n^{\mathbf{Z}}$. Analogously, we define $N_n^{\mathbf{Z}}$, the linear subspace of $l^{\mathbf{Z}}_{M_n(\mathbf{R})}$ formed by all sequences $\{P_{[i]}\}_{i\in\mathbf{Z}}$ with the property $\|P\|_1=\sum_{i\in\mathbf{Z}} Tr\sqrt{P_{[i]}^T P_{[i]}}<\infty$. (We recall that, if $A\in M_n(\mathbf{R})$ is a nonnegative matrix, then $\sqrt{A}$ is the unique nonnegative matrix defined by $A=\sqrt{A}\sqrt{A}$). By a standard way it follows that $N_n^{\mathbf{Z}}$ is a Banach space.

Moreover, since there are $n_1,n_2>0$ such that

$n_1 Tr\sqrt{X^T X}\le\sqrt{TrX^T X}\le n_2 Tr\sqrt{X^T X}$

for all $X\in M_n(\mathbf{R})$ it follows that the linear spaces $N_n^{\mathbf{Z}}$ and $H_n^{\mathbf{Z}}$ coincide. In what follows we will denote by $\overline{\Gamma}$ the adjoint operator of any operator $\Gamma\in L(H_n^{\mathbf{Z}})$.

Let $T>0$. If B is an arbitrary Banach space, then we denote by $C([0,T],B)$ the space of all mappings $G:[0,T]\to B$ that are continuous. Also $C^1([0,T],B)$ denotes the subspace of $C([0,T],B)$ of all continuously differentiable mappings G on $(0,T)$ (i.e. G is differentiable on $(0,T)$ and G' is continuous on $(0,T)$). The product $t\in J\to G(t)(X(t))\in l^{\mathbf{Z}}_{M_{n\times p}(\mathbf{R})}$ of any two functions $G: J\to L\left(l^{\mathbf{Z}}_{M_n(\mathbf{R})}, l^{\mathbf{Z}}_{M_{n\times p}(\mathbf{R})}\right)$ and $X: J\to l^{\mathbf{Z}}_{M_n(\mathbf{R})}$ will be often denoted shortly $G(t,X(t))$. In this case we will write $G(t,X(t))(i)$ for the i -th component of $G(t,X(t))$.

Let $w(t)=(w_1(t),w_2(t),..,w_r(t)), t\in\mathbf{R}_+$ ($\mathbf{R}_+=\{t\in\mathbf{R}, t\ge 0\}$) be a standard r dimensional Wiener process (see [3]) on a complete probability space $(\Omega,\mathbf{F},P)$. For each $t\ge 0$, we denote by $\mathbf{F}_t$ the smallest σ -algebra which contains all sets $M\in\mathbf{F}$ with $P(M)=0$ and with respect to which all random vectors $\{w(s)\}_{s\le t}$ are measurable. Let $\eta(t), t\in\mathbf{R}_+$ be a right continuous, homogeneous Markov chain with the state space $\mathbf{Z}$ and a stationary standard transition probability matrix function $\{P_t(i,j)\}_{i,j\in\mathbf{Z}}$ defined by

$$P_t(i,j)=P(\eta(t+\tau)=j\mid\eta(\tau)=i)=\begin{cases}\lambda_{ij}t+o_{ij}(t), i\ne j\\ 1+\lambda_{ii}t+o_{ii}(t), i=j\end{cases},$$

for all $0\le\tau.$ Here $\Lambda=(\lambda_{ij})_{i,j\in\mathbf{Z}}$, is the infinitesimal matrix of the Markov process; it is known that $\lambda_{ij}\ge 0$ for $i\ne j$ and $\lambda_{ii}<0$. We also assume that:

1. $\eta(t)$ is conservative and stable, i.e. there is $c\in\mathbf{R}_+$ such that $\sum_{j\in\mathbf{Z}, j\ne i}\lambda_{ij}=-\lambda_{ii}\le c$ for all $i\in\mathbf{Z}$;
2. there is $c_1\in\mathbf{R}_+$ such that $\sum_{j\in\mathbf{Z}, j\ne i}\lambda_{ij}\le c_1$ for all $i\in\mathbf{Z}$;
3. the σ -algebras $\mathbf{F}_t$ and $\mathbf{G}_t=\sigma(\eta(\tau), 0\le\tau\le t)$ are independent for every $t\ge 0$.

2. MAIN RESULTS

We consider the class of stochastic differential equations

$$dx(t)= A_0(t,\eta(t))x(t)dt+\sum_{k=1}^{r}A_k(t,\eta(t))x(t)dw_k(t), t\geq t_0, x(t_0)=x_0\in\mathbf{R}^n, \tag{1}$$

where $A_k\in C_b(\mathbf{R}_+, l^{\mathbf{Z}}_{M_n(\mathbf{R})}), A_k(t)=\{A_k(t,i)\}_{i\in\mathbf{Z}}$ for all $k=0,1..,r$.

It is known that under the above hypotheses there is a unique continuous solution $x(t)=x(t,t_0,x)$, $t\geq t_0$, of (1). Let us denote $A(t,i)=A_0(t,i)+\frac{\lambda_{ii}}{2}I_n$ and, for all $i\in\mathbf{Z}$ and $X\in l^{\mathbf{Z}}_{S_n(\mathbf{R})}$ and $t\in\mathbf{R}_+$, we define the linear operators on $L(l^{\mathbf{Z}}_{S_n(\mathbf{R})})$:

$$\Pi_1(t,X)(i)=\sum_{k=1}^{r}A_k^T(t,i)X(i)A_k(t,i)+\sum_{j\in\mathbf{Z},\, j\neq i}\lambda_{ij}X(j),$$

$$\Gamma_1(t,X)(i)=\sum_{k=1}^{r}A_k(t,i)X(i)A_k^T(t,i)+\sum_{j\in\mathbf{Z},\, j\neq i}\lambda_{ji}X(j),$$

$$G(t,X)(i)=A^T(t,i)X(i)+X(i)A(t,i)+\Pi_1(t,X)(i) \tag{2}$$

$$\overline{G}(t,Y)(i)=A(t,i)Y(i)+Y(i)A^T(t,i)+\Gamma_1(t,Y)(i). \tag{3}$$

It is not difficult to see that $G(t),\overline{G}(t)\in L(l^{\mathbf{Z}}_{S_n(\mathbf{R})})$ and their restrictions to $H_n^{\mathbf{Z}}$ and $N_n^{\mathbf{Z}}$, respectively, remain linear and bounded operators. In addition $G,\overline{G}\in C_b(\mathbf{R}_+,B)$, where $B=L(l^{\mathbf{Z}}_{S_n(\mathbf{R})}), L(H_n^{\mathbf{Z}}), L(N_n^{\mathbf{Z}})$. It is not difficult to see that the adjoint operator of $G(t)$ (as a linear and bounded operator from $L(H_n^{\mathbf{Z}})$) is exactly the restriction of $\overline{G}(t)$ to $L(H_n^{\mathbf{Z}})$.

We associate with (1) the following Lyapunov equations:

$$\frac{d}{dt}X(t,i)+G(t,X(t))(i)=0 \tag{4}$$

$$\frac{d}{dt}Y(t,i)=\overline{G}(t,Y(t))(i). \tag{5}$$

The equation (4) with the initial condition $X(s)=D\in l^{\mathbf{Z}}_{S_n(\mathbf{R})}$ has a unique solution $X(t,s;D)\stackrel{not}{=}U(t,s)(D)\in C^1([s,\infty),l^{\mathbf{Z}}_{S_n(\mathbf{R})})$ [7]. The mapping $(t,s)\to U(t,s)\in L(l^{\mathbf{Z}}_{S_n(\mathbf{R})})$ is an evolution operator on $l^{\mathbf{Z}}_{S_n(\mathbf{R})}$ having the property $\frac{\partial U(t,s)}{\partial s}=U(t,s)G(s)$ [7]. It is called the evolution operator generated by $G\in C_b(\mathbf{R}_+,l^{\mathbf{Z}}_{S_n(\mathbf{R})})$. Let $D\in H_n^{\mathbf{Z}}$. An easy computation shows that $U^*(t_0,t)(D)$ is the unique solution of (5) with the final condition $Y(t_0)=D$. Now let $\{V(t,s)\}_{0\leq s\leq t}$ the evolution operator generated by the mapping $\overline{G}\in C_b(\mathbf{R}_+,l^{\mathbf{Z}}_{S_n(\mathbf{R})})$ (see [7]). Since $\overline{G}\in C_b(\mathbf{R}_+,L(H_n^{\mathbf{Z}}))$, it follows that $V(t,s)(D)=U^*(t_0,t)(D)$ for all $D\in H_n^{\mathbf{Z}}$, by the uniqueness of the solution. Analogously we can deduce that $V(t,s)(D)\in N_n^{\mathbf{Z}}$ for all $D\in N_n^{\mathbf{Z}}$.

Further we consider the element of $H_n^{\mathbf{Z}}\cap N_n^{\mathbf{Z}}$ defined by ${}_{i,x}P_{[j]}=0$, if $i\neq j$ and ${}_{i,x}P_{[j]}=x\otimes x$. We get the following.

Lemma 1. For all $0 \le s \le t, i \in \mathbf{Z}$ and $x \in \mathbf{R}^n$ we have

$$\langle U(t,s)(\Phi)(i)x, x\rangle = \left\| V(t,s)\left({}_{i,x}P\right)\right\|_1$$

.

Proof. Let $\Phi(m) \in H_n^{\mathbf{Z}}, m \in \mathbf{N}^*$, defined by $\Phi(m)_{[i]} = \begin{cases} I_n, |i| \le m, \\ 0, |i| > m, \end{cases}$. Obviously $\Phi(1) \le \Phi(2) \le ... \le \Phi(m) \le ... \le \Phi$. By Lemma 2 from [9]we have

$$\left\| V(t,s)\left({}_{i,x}P\right)\right\|_1 = \lim_{m\to\infty} \left\langle \Phi(m), V(t,s)\left({}_{i,x}P\right)\right\rangle_2 .$$

From (5), it follows that $V(t,s)\left({}_{i,x}P\right) = U^*(s,t)\left({}_{i,x}P\right)$ and therefore

$$\lim_{m\to\infty} \left\langle \Phi(m), V(t,s)\left({}_{i,x}P\right)\right\rangle_2 = \lim_{m\to\infty} \left\langle \Phi(m), U^*(s,t)\left({}_{i,x}P\right)\right\rangle_2$$

$$= \lim_{m\to\infty} \sum_{j\in\mathbf{Z}} Tr\left({}_{i,x}P\right)(j) U(s,t)(\Phi(m))(j) = Tr\left({}_{i,x}P\right)(i) U(s,t)(\Phi) = \langle U(t,s)(\Phi)(i)x, x\rangle.$$

The conclusion follows.

For all $H \in l^{\mathbf{Z}}_{S_n(\mathbf{R})}$ and $0 \le t_0 \le s$ we define the mapping $T(s,t_0) : l^{\mathbf{Z}}_{S_n(\mathbf{R})} \to l^{\mathbf{Z}}_{S_n(\mathbf{R})}$, $\langle T(s,t_0)(H)(i)x, x\rangle = E\left[\langle H(\eta(s))x(s), x(s)\rangle \mid_{\eta(t_0)=i}\right]$, where $i \in \mathbf{Z}$ and $x \in \mathbf{R}^n$. Note that $T(s,t_0)$ is well defined, because $\sup_{i\in\mathbf{Z}} E\left[\langle H(\eta(s))x(s), x(s)\rangle \mid_{\eta(t_0)=i}\right] \le \|H\|_{\mathbf{Z}} \sup_{i\in\mathbf{Z}} E\left[\|x(s)\|^2 \mid_{\eta(t_0)=i}\right]$and $E\left[\|x(s)\|^2 \mid_{\eta(t_0)=i}\right] < K$, where K does not depends on i. (The last inequality follows by arguing as for the proof of Theorem 37 from [3]). Moreover, it follows easily that $T(s,t_0)$ is a linear and bounded operator on $l^{\mathbf{Z}}_{S_n(\mathbf{R})}$ and $T(s,t_0)(H) \ge 0$ for all $H \in l^{\mathbf{Z}}_{S_n(\mathbf{R})}, H \ge 0$ (we will say that $T(s,t_0)$ is a positive operator).

Theorem 1. For all $0 \le s \le t, i \in \mathbf{Z}$ and $x \in \mathbf{R}^n$ we have

$$E\left[\|x(s)\|^2 \mid_{\eta(t_0)=i}\right] = \left\langle T(s,t_0)(\Phi)_{[i]} x, x\right\rangle = \left\| V(s,t_0)\left({}_{i,x}P\right)\right\|_1 .$$

Proof. Applying Ito's formula (see Theorem 37 in [3]) for the function $v(t,x,i) = \langle H(i)x, x\rangle, t \in \mathbf{R}, x \in \mathbf{R}^n$ and $i \in \mathbf{Z}$ and the stochastic process $x(t,t_0,x)$ we get

$$E[\langle H(\eta(s))x(s), x(s)\rangle \mid_{\eta(t_0)=i}] - \langle H(i)x_0, x_0\rangle =$$

$$E[\int_{t_0}^{s} 2\langle H(\eta(t))x(t), A_0(t,\eta(t))x(t)\rangle + \sum_{k=1}^{r} \left\langle A_k^T(t,\eta(t)) H(\eta(t)) A_k(t,\eta(t))x(t), x(t)\right\rangle +$$

$$+ \sum_{j\in\mathbf{Z}} \langle H(\eta(t))x(t), x(t)\rangle \lambda_{\eta(t)j} \mid_{\eta(t_0)=i}]dt.$$

Hence $\langle T(s,t_0)(H)(i)x, x\rangle - \langle H(i)x, x\rangle = \int_{t_0}^{s} \left\langle T(t,t_0)[A_0^T(t)H + HA_0(t) + \Pi_1(t,H)(i)]x, x\right\rangle dt$.

Differentiating with respect to s we get $\frac{dT(s,t_0)}{ds} = T(s,t_0)G(s), T(t_0,t_0)(H) = H$. If

$H, D \in H_n^{\mathbf{Z}}$ we have $\frac{d\left\langle T^*(s,t_0)(D),H\right\rangle_2}{ds} = \left\langle \overline{G}(s)T^*(s,t_0)(D),H\right\rangle, T^*(t_0,t_0)(D) = D$ and we deduce that $T^*(s,t_0) = V(t,s)$. On the other hand, let $\{D(m)\}_{m\in\mathbf{N}} \subset l_{S_n(\mathbf{R})}^{\mathbf{Z}}$ be an increasing and bounded sequence with $D_{[i]}(x) = \lim_{m\to\infty} D(m)_{[i]}(x),$ for all $i \in \mathbf{Z}$ and $x \in H$. Since $T(s,t_0)$ is a positive operator, it follows that $T(s,t_0)D(m) \le T(s,t_0)D(m+1), m \in \mathbf{N}$. Thus, the definition of $T(s,t_0)D(m)$ and the Monotone convergence theorem imply that

$$\lim_{m\to\infty}\left\langle T(s,t_0)(D(m))_{[i]}x,x\right\rangle = \left\langle T(s,t_0)(D)_{[i]}x,x\right\rangle$$

for all $i \in \mathbf{Z}$ and $x \in \mathbf{R}^n$. Now it is clear that $T(s,t_0)(D(m))_{[i]}$ converges to $T(s,t_0)(D)_{[i]}$ for all $i \in \mathbf{Z}$. Replacing D_m and D with Φ_m and Φ , respectively, and using Lemma 1, we get successively $\left\langle T(s,t_0)(\Phi)_{[i]}x,x\right\rangle = \lim_{m\to\infty}\left\langle T(s,t_0)(\Phi(m))_{[i]}x,x\right\rangle = \left\| V(s,t_0)\left({}_{i,x}P\right)\right\|_1$.
The conclusion follows.

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