AN EXISTENCE RESULT FOR SYSTEMS THAT APPEAR IN APPLIED SCIENCES

Teacher Assistant PhD Dragoş-Pătru **COVEI**
Constantin Brancuşi University of Tg-Jiu, Gorj, România

Abstract. *Many of the engineering developments often occur as a result of the intersection between scientific communities with system of differential equations. They, although discussed in many papers from the specialized literature, still are not completely solved. The state of fact is expected because practical applications reveal complexity and aspects that allow new approaches. From this point of view and from these reasons such problems shall continue to be characteristic feature of activities and tasks taken by scientific research, of major significance in mathematics. Related to the context and with a rough processing of the information taken from many bibliographic titles, this article develops or generalizes certain previous results regarding systems of the Schrödinger type.*

Keywords: Entire solutions; Bounded solutions; Large solutions; Schrödinger systems type.

1. INTRODUCTION AND THE MAIN RESULT

Let $x \in \mathbb{R}^N$ and $r := |x|$. Consider the following quasilinear elliptic system

$$\begin{cases} \Delta_p u_1 + h_1(r)|\nabla u_1|^{p-1} = a_1(r)\, f_1(u_1(r),...,u_d(r)) \\ \qquad ... \\ \Delta_p u_d + h_d(r)|\nabla u_d|^{p-1} = a_d(r)\, f_d(u_1(r),...,u_d(r)) \end{cases} \tag{1.1.}$$

where Δ_p is the p-laplacian operator defined by

$$\Delta_p u = \text{div}\left(|\nabla u|^{p-2}\nabla u\right) \text{ for } 1 < p < \infty,$$

$N \geq 3$, $d \geq 1$ is integer, $h_{j_{j=\overline{1,d}}}$, $a_{i_{i=\overline{1,d}}}$: $[0,\infty) \to [0,\infty)$ are radial continuous functions and $f_{i_{i=\overline{1,d}}}$ satisfy the following hypotheses

(C1) $f_{i_{i=\overline{1,d}}}$: $[0,\infty)^d \to [0,\infty)$ are continuous in each variable;

(C2) $f_{i_{i=\overline{1,d}}}$ are increasing on $[0,\infty)^d$ in each variable.

The topic of the existence of solutions to the elliptic system (1.1.) with boundary blow-up is of interest to many researchers (see [1]-[7] and their references).

The questions of solutions existence in the case of problem (1.1.) has received an increased interest with Zhang-Liu recent paper [7]. In [7] the author considered problem (1.1.) when $d = p = 2$ and $h_{i_{i=\overline{1,d}}}(r) = 1$. Our purpose is to generalize the results of [7] to a larger class of systems (1.1.).

We introduce the following notations

$$i=\overline{1,d} \qquad H_i(r):=r^{N-1}e^{\int_0^r h_i(t)dt}$$
$$A_i(\infty):=\lim_{r\to\infty}A_i(r), \quad A_i(r)=\int_0^r\left(\frac{1}{H_i(t)}\int_0^t H_i(s)a_i(s)ds\right)^{1/(p-1)}dt.$$

The following is our main result.

Theorem 1. *(i) Assume that* $A_{i_{i=\overline{1,d}}}(\infty)=\infty$ *and*

$$\lim_{s\to\infty}\frac{\sum_{i=1}^{d}\left(f_i(s,...,s)\right)^{1/(p-1)}}{s}=0. \tag{1.2.}$$

Then the system (1.1.) has infinitely many positive entire large solutions.

(ii) Furthermore, if $A_{i_{i=\overline{1,d}}}(\infty)<\infty$ *and*

$$\sup_{s\geq 0}\left[\sum_{i=1}^{d}\left(f_i(s,...,s)\right)^{1/(p-1)}\right]<\infty$$

then the system (1.1.) has infinitely many positive entire bounded solutions.

We observe from [6] that such coupled nonlinear Schrődinger systems (1.1.) arise in the description of several physical phenomena such as the propagation of pulses in birefringent optical fibers and Kerr-like photorefractive media.

2. Proof of the Theorem 1. We first see that radial solutions of (1.1.) are solutions $(u_1,...,u_d)$ of the differential equations system

$$\begin{cases}(p-1)\left|u_1'(r)\right|^{p-2}u_1''+\frac{N-1}{r}u_1'(r)^{p-1}+h_1(r)\left|u_1'(r)\right|^{p-1}=a_1(r)f_1(u_1(r),...,u_d(r)),\\ \qquad\qquad ...\\ (p-1)\left|u_d'(r)\right|^{p-2}u_d''+\frac{N-1}{r}u_d'(r)^{p-1}+h_d(r)\left|u_d'(r)\right|^{p-1}=a_d(r)f_d(u_1(r),...,u_d(r)).\end{cases} \tag{2.1.}$$

Since the radial solutions of (1.1.) are solutions of the differential equations system (2.1.) it follows that the radial solutions of (1.1.) with

$$u_1(0)=\beta_1,...,u_d(0)=\beta_d$$

where $\beta_{i_{i=\overline{1,d}}}$ may be any non-negative numbers, satisfy:

$$\begin{cases}u_1(r)=\beta_1+\int_0^r\left(\frac{1}{H_1(t)}\int_0^t H_1(s)a_1(s)f_1(u_1(s),...,u_d(s))ds\right)^{1/(p-1)}dt,\\ ...\\ u_d(r)=\beta_d+\int_0^r\left(\frac{1}{H_d(t)}\int_0^t H_d(s)a_d(s)f_d(u_1(s),...,u_d(s))ds\right)^{1/(p-1)}dt.\end{cases} \tag{2.2.}$$

Define $u_1^0=\beta_1,...,u_d^0=\beta_d$ for $r\geq 0$.

Let $\{u_j^k\}_{j=\overline{1,d}}^{k\ge 1}$ be a sequence of functions on $[0,\infty)$ given by

$$\begin{cases} u_1^{k+1}(r)=\beta_1+\int_0^r\left(\frac{1}{H_1(t)}\int_0^t H_1(s)a_1(s)\,f_1\left(u_1^k(s),...,u_d^k(s)\right)ds\right)^{1/(p-1)}dt, \\ ... \\ u_d^{k+1}(r)=\beta_d+\int_0^r\left(\frac{1}{H_d(t)}\int_0^t H_d(s)a_d(s)\,f_d\left(u_1^k(s),...,u_d^k(s)\right)ds\right)^{1/(p-1)}dt. \end{cases} \tag{2.3.}$$

We remark that, for all $r\ge 0$, $j=\overline{1,d}$ and $k\in N$ we have $u_j^k(r)\ge\beta_j$. Moreover $\{u_j^k\}_{j=\overline{1,d}}^{k\ge 1}$ are non-decreasing sequence on $[0,\infty)$ such that

$$u_i^k(r)\le u_i^{k+1}(r)\le\left[f_i\left(\sum_{j=1}^d u_j^k(r),...,\sum_{j=1}^d u_j^k(r)\right)\right]^{1/(p-1)}A_i(r),\ i=\overline{1,d}. \tag{2.4.}$$

Let $R>0$ be arbitrary. It is easy to see that (2.4.) implies

$$\sum_{i=1}^d u_i^k(R)\le\sum_{i=1}^d\beta_i+\sum_{i=1}^d\left[f_i\left(\sum_{j=1}^d u_j^k(R),...,\sum_{j=1}^d u_j^k(R)\right)\right]^{1/(p-1)}\sum_{i=1}^d A_i(R),k\ge 1,$$

and so

$$1\le\frac{\sum_{i=1}^d\beta_i}{\sum_{i=1}^d u_i^k(R)}+\frac{\sum_{i=1}^d\left[f_i\left(\sum_{j=1}^d u_j^k(R),...,\sum_{j=1}^d u_j^k(R)\right)\right]^{1/(p-1)}\sum_{i=1}^d A_i(R)}{\sum_{i=1}^d u_i^k(R)},k\ge 1. \tag{2.5.}$$

Moreover, taking into account the monotonicity of $\left(\sum_{i=1}^d u_i^k(R)\right)_{k\ge 1}$, there exists

$$L(R):=\lim_{k\to\infty}\sum_{i=1}^d u_i^k(R).$$

We prove that $L(R)$ is finite. Indeed, if not, we let $k\to\infty$, in (2.5.) and the assumption (1.2.) leads us to a contradiction. Since $u_i^k(R)$ are increasing functions, it follows that the map $L:(0,\infty)\to(0,\infty)$ is nondecreasing and $\sum_{i=1}^d u_i^k(r)\le\sum_{i=1}^d u_i^k(R)\le L(R),\forall r\in[0,R],\forall k\ge 1.$

Thus the sequences $\left(u_i^k(R)\right)_{i=\overline{1,d}}^{k\ge 1}$ are bounded from above on bounded sets. We now define the following quantities $u_i(r):=\lim_{k\to\infty}u_i^k(r)$ for all $r\ge 0$ and $i=\overline{1,d}$. Then $u_{i_{\overline{i=1,d}}}$ is a positive solution of (2.2.).

Next, we show that $u_{i_{\overline{i=1,d}}}$, is a large solution of (2.2.). Let us remark that by (2.3.) we have the following estimate

$$u_i(r)\ge\beta_i+\left[f_i(\beta_1,...,\beta_d)\right]^{1/(p-1)}A_i(r),\text{ for all } r\ge 0 \text{ and } i=\overline{1,d}.$$

It follows from the assumption f_i are positive functions and $A_i(\infty)=\infty$, that $u_{i_{i=\overline{1,d}}}$ is a large solution of (2.2.) and so $u_{i_{i=\overline{1,d}}}$ is a positive entire large solution of (1.1.). Thus any large solution of (2.2.) provides a positive entire large solution of (1.1.) with $u_i(0)=\beta_{i_{i=\overline{1,d}}}$. Since $\beta_{i_{i=\overline{1,d}}} \in (0,\infty)$ was chosen arbitrarily, it follows that (1.1.) has infinitely many positive entire large solutions.

(ii) Assume that $\sup_{s\geq 0}\left[\sum_{i=1}^{d}\left(f_i(s,...,s)\right)^{1/(p-1)}\right]<\infty$ holds, then by (2.5.) we have

$$L(R) := \lim_{k\to\infty}\sum_{i=1}^{d} u_i^k(R)<\infty.$$

On the other hand $\sum_{i=1}^{d} u_i^k(r)\leq \sum_{i=1}^{d} u_i^k(R)\leq L(R),\ \forall r\in[0,R], \forall k\geq 1$. So the sequences $\left(u_i^k(R)\right)_{i=\overline{1,d}}^{k\geq 1}$ are bounded from above on bounded sets.

Let $u_i(r) := \lim_{k\to\infty} u_i^k(r)$ for all $r\geq 0$ and $i=\overline{1,d}$. Then $u_i(r)$, $\left(i=\overline{1,d}\right)$ is a positive solution of (2.2.). It follows from (2.4.) that $u_i(r)$, $\left(i=\overline{1,d}\right)$ is bounded, which implies that (1.1.) has infinitely many positive entire bounded solutions. This concludes the proof of **Theorem** 1.

We will finish the article with some remarks. For this we note

$$F(\infty) := \lim_{r\to\infty} F(r),\ F(r)=\int_a^r \left(\sum_{j=1}^{d} f_j(s,...,s)\right)^{-1/(p-1)} ds;\ r\geq a>0.$$

Then, we can easily see from [1,2] that the following holds:

Remark 1. *Assume that (C1)-(C2) hold and that* $F(\infty)=\infty$ *. Then the system (1.1.) possesses at least one positive radial solution* $(u_1,...,u_d)$ *. If, in addition,* $A_{j_{j=\overline{1,d}}}(\infty)<\infty$ *, the positive radial solution* $(u_1,...,u_d)$ *is bounded. Moreover, when* $A_{j_{j=\overline{1,d}}}(\infty)=\infty$ *the positive solution* $(u_1,...,u_d)$ *is entire large solution, i.e.* $\lim_{r\to\infty} u_1(r)=...=\lim_{r\to\infty} u_d(r)=\infty.$

Remark 2. *Assume that (C1)-(C2) hold and that*

(C4) $F(\infty)<\infty;$

(C5) $A_{j_{j=\overline{1,d}}}(\infty)<\infty;$

(C6) there exists $\beta>\frac{a}{d}$ *such that*

$$\sum_{j=1}^{d} A_j(\infty)<F(\infty)-F(d\beta).$$

Then, the system (1.1.) possesses at least one positive bounded radial solution $(u_1,...,u_d)$ *satisfying*

$$\beta + f_j^{1/(p-1)}(\beta)A_j(r) \le u_j(r) \le F^{-1}\left(F(d\beta) + \sum_{j=1}^{d} A_j(r)\right),\ j = \overline{1,d}.$$

The author wish to say that the results from [1,2] are obtained independently that the results developed in [7].

Acknowledgement. This work was supported by CNCSIS - UEFISCDI, project number PN II - IDEI 1080/2008 No. 508/2009.

REFERENCES

[1] D.P. Covei, *Existence of entire radially symmetric solutions for a quasilinear system with d-equations*, Hacettepe Journal of Mathematics and Statistics, accepted manuscript, 2010.
[2] D.P. Covei, *Radially and non-radially symmetric solutions for a Schrodinger semilinear system type with d-equations*, Funkcialaj Ekvacioj-Serio Internacia, accepted manuscript, 2010.
[3] E. Giarrusso, *On blow up solutions of a quasilinear elliptic equation*, Mathematische Nachrichten, Volume 213 (2000), Pages 89--104.
[4] A. V. Lair, *A necessary and sufficient condition for the existence of large solutions to sublinear elliptic systems*, Journal of Mathematical Analysis and Applications, Volume 365, Issue 1, Pages 1-428 (1 May 2010).
[5] C. Liu and Z. Yang, *Existence of large solutions for quasilinear elliptic problems with a gradient term*, Applied Mathematics and Computation 192 (2007) 533--545.
[6] A. Ghanmi, H. Maagli, V. Radulescu and N. Zeddini, *Large and bounded solutions for a class of nonlinear Schrodinger stationary systems*, Analysis and Applications, Vol. 7, No. 4 (2009) 391--404.
[7] X. Zhang and L. Liu, *The existence and nonexistence of entire positive solutions of semilinear elliptic systems with gradient term*, Journal of Mathematical Analysis and Applications, Volume 371, Issue 1, 1 November 2010, Pages 300-308.