FRÉCHET-MARINESCU’S DERIVATIVE IN THE MATHEMATICAL MODELING OF DYNAMIC SYSTEMS

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Abstract. The paper presents an application of the functional analysis, especially of differential calculus in linear topological locally convex spaces leading to formulae representing the evolution of states in dynamical systems with infinite fading memory.

Keywords: Fréchet-Marinescu’s differential, locally convex topology, fading memory systems.

1. A TOPOLOGY AND A DIFFERENTIAL CALCULUS IN THE THEORY OF SYSTEMS WITH INFINITE MEMORY

Fréchet’s derivability in topological linear normed spaces is well known and used not only by mathematicians but by many specialists of other scientific domains, from physics to economy and sociology.

The inputs into dynamical systems with finite memory could be considered as belonging to a normed Banach’s space and so the mathematical model can use the Fréchet’s differential calculus.

An approach of the study of continuous systems with infinite memory, using an adequate locally convex space, is described in the following lines.

In locally convex spaces the topology is not given by a norm, but a family of semi-norms. Unlike a norm, a semi-norm can be equal to zero in points \( x \neq 0 \) also, a semi-norm that is zero only for \( x=0 \) is a norm. A linear topological locally convex space \( X \) is structured by a family of semi-norms \( A = \| \cdot \|_\alpha \alpha \in A \).

A differential calculus in a theory of the dynamic systems with infinite memory was founded considering the inputs into the system as being \( m \)-dimensional vector functions \( u(\tau) \), with \( \tau \) the time variable, \( \tau \epsilon (\infty, t] \), \( t \) the present moment; \( x(t) \) is a parameter of the system state at the present moment, represented by a real number; a constitutive real functional \( F \) gives the value of \( x(t) \) corresponding to whole (globally) history of inputs; we write this correspondence in the following form, that suggests us the integration operation:

\[
x(t) = \int_{\tau=\infty}^{t} F[u(\tau)]
\]

(1.1)

A variable \( s \) opposite to the sense of the time variable \( \tau \) sense is more convenient sometimes, so we also introduce the following notations instead of \( u(\tau), \tau \epsilon (\infty,J) \):

\[
u'(s) = u(t-s), \quad s = t-\tau, \quad s \epsilon [0, +\infty)
\]

(1.2)

So, the constitutive equation written for the state \( x(t) \) will be the following:
\[ x(t) = \lim_{s \to \infty} F[x(t)] \quad (1.3) \]

Definition: A system \( S \) with infinite fading memory is a system for that the relationship between the input history \( u'(s), s \in [0, +\infty) \), and its state \( x(t) \) at the present moment \( t \) is given by the formula (1.3), where the values of \( u'(s) \) taken for \( s > 0 \), close to the value zero, bring a much more important contribution to the present state \( x(t) \) than those values of \( u'(s) \) taken for large values of \( s \), that is when \( s \to +\infty \).

To have concrete results for a mathematical modeling of systems with infinite fading memory, it is necessary to impose: 1) a topology on the set of the functions \( u'(s) \) admissible as input histories of the system \( S \); 2) certain properties of the constitutive functional \( F \).

For the first requirement, we introduce the following:

Definition: The space of admissible input histories till the present moment \( t \) is a linear topological locally convex space \( \Omega' \) of vector functions \( u'(s) \) defined on \([0, +\infty)\), \( u'(s) = (u'(s), u'_2(s), \ldots, u'_m(s)) \), which have any order derivatives, and the topology of \( \Omega' \) is given by a family of semi-norms linked by the moment \( t \):

\[ |u'|_\lambda = \sup_{s \in [0, \lambda]} |u'(s)| \quad \text{where:} \quad |u'(s)| = \left[ \sum_{k=1}^{m} u_k^2(t-s) \right]^{1/2} \quad (1.4) \]

According to this definition, a neighbourhood \( V_{\lambda,\delta}[u] \) of the function \( u' = u'(s) \) is formed by all the functions \( v' = v'(s) \) belonging to \( \Omega' \) and accomplishing the condition \( |v' - u'|_\lambda < \delta \) for certain numbers \( \lambda > 0, \delta > 0 \).

A real functional \( F \) defined on the locally convex space \( \Omega' \) is continuous on \( \Omega' \) if for every function \( u' \) from \( \Omega' \) and for any \( \varepsilon > 0 \), arbitrary small, there exist \( \lambda(\varepsilon) \) and \( \delta(\varepsilon) \) so that for any \( v' \in V_{\lambda(\varepsilon),\delta(\varepsilon)}[u] = \left\{ v': |v' - u'|_\lambda < \delta(\varepsilon) \right\} \), the following inequality has place: \( |F[u'] - F[v']| < \varepsilon \).

An important and justificatory result for the modelling of systems with infinite fading memory is the theorem that affirms: if the constitutive functional \( F \) of a system \( S \) is defined and continuous on the linear topological locally convex space \( \Omega' \) and the input history \( u'(s) \) belongs to \( \Omega' \), then the correspondence given by the formula (1.3) will describe the present state of a system with infinite fading memory (a demonstration in [5]).

In a mathematical approach of systems with infinite memory we used the hypothesis of the differentiability in the Fréchet – Marinescu’s sense for the locally convex spaces, a generalization of the Fréchet’s differential in normed spaces, introduced by G. Marinescu [3]. This means that a semi-norm \( |.|_\lambda \) exists so to have a formula of the following type:

\[ F[u'(s) + h(s)] = F[u'(s)] + \int_{s=0}^{\infty} F[u'(s)] h(s) + \omega(u'; h) \quad (1.5) \]
Here the functional $\frac{\partial F}{\partial s} [u^i|h(s)]$ is a linear functional on its argument $h(s) \in \mathcal{O}$ and represents the Fréchet–Marinescu’s differential of the constitutive functional $F$ in the point $u^i$ of the set $\mathcal{O}$; the term $\varphi(u^i;h)$, similarly to the rest from the Taylor’s formula for real functions, tends to zero faster than the value of the semi-norm $|h|_{\mathcal{J}}$ of the function $h(s)$.

The theorem proving the possibility to represent the value of the functional $\frac{\partial F}{\partial s} [u^i|h(s)]$ by integral operations was demonstrated [5] using only the differentiability property in locally convex spaces. The theorem of integral representation asserts the existence of a number $\lambda > 0$ and of $m$ real functions $a_1(s), a_2(s), \ldots, a_m(s)$, whose squares have Lebesgue’s integral on the interval $[0, \lambda]$, so that the following equality takes place:

$$\frac{\partial F}{\partial s} [u^i|h(s)] = \int_{s=0}^{\lambda} \sum_{0=k=1}^{m} a_k(s)h_k'(s)ds + \varphi(u^i;h) \tag{1.6}$$

Here $h(s)=(h_1(s), h_2(s), \ldots, h_m(s))$; the functions $h_k'(s), k=1,2,\ldots, m$, are the first order derivatives of the functions $h_k(s)$ and $h(s)=0$ for $s > \lambda$; the functions $a_k(s)$ depend on the input history $u^i(s)$, $s > \lambda$, and do not depend on $h(s)$. The length $\lambda$ of the integration interval is imposed by the differentiability condition accomplished by the constitutive functional, so that the ratio between $\varphi(u^i;h)$ and the semi-norm $|h|_{\mathcal{J}}$ tends to zero when the function $h(s)$ tends to zero in the topology of $\mathcal{O}$.

An immediate consequence of the theorem and of the formula 1.6 is the expression of the difference between two values of the constitutive functional $F$:

$$\int_{s=0}^{\lambda} \sum_{0=k=1}^{m} a_k(s)h_k'(s)ds + \varphi(u^i;h) \tag{1.7}$$

The absolute value of $\varphi(u^i;h)$ is smaller than the semi-norm $|h|_{\mathcal{J}}$ and it will tend to zero when $h$ tends to the function zero, faster than $|h|_{\mathcal{J}}$. The difference between the values of two states, those from the moments $t$ and $t - \lambda$, representing the evolution, can be written ([5]) in the form:

$$x(t) - x(t - \lambda) = \int_{t-\lambda}^{t} \sum_{0=k=1}^{m} a_k(\tau)h_k'(\tau)d\tau + \varphi(u^i;h) \tag{1.8}$$

A more complete formula considers the inputs into the system on an interval prior to the moment $t - \lambda$; it was demonstrated also [5]:

$$x(t) - x(t - \lambda) = \sum_{k=1}^{m} a_k(t)h_k(t) + \sum_{k=1}^{t-\lambda} \int q_k(\tau)h_k'(\tau)d\tau + \int_{t-\lambda}^{t} \int_{t-\mu}^{t} p_{jk}(\tau, \theta)e_j(\theta)h_k(\tau)d\theta + \varphi(u^i;h;\xi) \tag{1.9}$$
2. INTERPRETATIONS AND APPLICATIONS

Obviously, the results here presented have a theoretical importance, but they constitute a mathematical accreditation of a lot of formulae which are used in many and diverse domains, from thermodynamics to biology. A part of mathematical results has found applications, especially to demonstrate formulae which were used empirically.

Here we enumerate only some cases which can make use of the mathematical formulae presented above.

1. The constitutive equation of the tension in an incompressible fluid in the visco-elasticity theory of second order: the mathematical description of the behavior of some classes of materials, inquired by the mechanics of continuously deformable media, uses integral formulae which are rigorously demonstrated in base of the differential calculus in locally convex spaces [1]. In this case the functions \( a_k, q_k, p_{jk} \) reflects properties of the material, and \( h_k(\tau), e_j(\tau) \), describe the history of deformations. A corollary of the formula 1.9, applied to the internal energy value of a thermodynamic system, allows us to write a formula analogue to the second principle for reversible processes [2]. But in the same time, we can conclude, thinking at the general representation 1.3, that the reversible processes are only an approximation of the reality. Also, the formula 1.9, as a link between internal energy, mechanical work and entropy could be taken into consideration for a second order thermodynamics [2].

2. In the cognition theory, as a foundation of the mathematical expression of the observation operator, that is the “functional paradigm”, how Robert Vallée [6] named the integral correspondence from the time functions which are both histories of the evolutions of a certain object and of the capacity of reception of its observer, to the image of this object that the observer has at a fixed moment. The acceptance of the functional paradigm by the integral formula proposed by Vallée,

\[
\eta(t) = \int_{t_0}^{t} w(t-\tau)\xi(\tau)d\tau
\]

of the correspondence from the function \( \xi(\tau), \tau \in (t_0, t) \), representing the history of the evolution of the observed object, to the image \( \eta(t) \) that the observer has about this object at the instant \( t \), finds a rigorous demonstration based on the theory using locally convex spaces, the Fréchet- Marinescu’s differential and formulae above presented [4].

3. The formula (1.9) with the complementary term \( \omega_2(u';h';e) \) could be interpreted in the terms of synergy and chaos [6]. The synergy of the systems finds its mathematical expression in the double integrals, where the causes (inputs) of different nature, \( h(\tau), e(\theta) \) multiply each other, without acting simultaneously. On the other hand, the complementary term \( \omega_2(u';h';e) \) hides the mathematical chaos, that is the imprecision of the \( x(t) \) calculus; it is a functional, and it is possible to apply again the differential calculus and obtain a better approximation of the value \( x(t) \).
REFERENCES