EXISTENCE AND UNIQUENESS OF THE SOLUTIONS FOR A CLASS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH INFINITE MARKOVIAN JUMPS

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Abstract: *In this paper we discuss existence and uniqueness problems for the solutions of a class of infinite dimensional stochastic differential equations (SDEs) with infinite Markovian jumps (MJs). The term "infinity" used with the notion of Markov process means that the state space of the Markov process is countably infinite. This type of equations model different real world processes which experience abrupt changes of their states.*

Keywords: Stochastic differential equations, Ito integral, Markov processes, countably infinite state space

1. INTRODUCTION

Recently, the SDEs with infinite MJs have attracted the interest of researchers, due to the new areas where they apply. We mention here the telecommunications and the economic field (see [1], [3], [4] and the references therein). In this paper we give sufficient conditions for the existence and the uniqueness of the solutions for this class of stochastic differential equations. The obtained result is an infinite-dimensional version of Theorem 5.1.1 from [5], which solve a similar problem for finite dimensional SDEs without jumps.

2. PRELIMINARIES

Let $(\Omega,\mathcal{F},P)$ be a probability space and let $\mathcal{Z}$ be an interval of integers, which may be finite or infinite. Assume that $\eta(t), t\in\mathbf{R}_+$ is a right continuous homogeneous Markov chain on Ω with the state space $\mathcal{Z}$ and the infinitesimal matrix $\Lambda=(\lambda_{ij})_{i,j\in\mathcal{Z}}$ having the property that $\lambda_{ij}\geq 0$ for all $i,j\in\mathcal{Z},\ j\neq i$ and there are the constants $c,c_1\in\mathbf{R}_+$ such that $\sum_{j\in\mathbf{Z},\, j\neq i}\lambda_{ij}=-\lambda_{ii}\leq c,\ \sum_{j\in\mathbf{Z},\, j\neq i}\lambda_{ji}\leq c_1, i\in\mathcal{Z}$. It is known that the transition probabilities $\{p_{i,j}(t)\}_{i,j\in\mathcal{Z}}$ satisfy the first system of Kolmogorov equations

$$p'_{ij}(t)=\sum_{k\in\mathbf{Z}}\lambda_{ik}p_{kj}(t)$$

$$p_{ij}(0)=\delta_{i,j}, t\geq 0$$

Let $r\in\mathbf{N}^*=\mathbf{N}/\{0\}$ be given. We consider a standard $r-$dimensional Wiener process on Ω, $w(t)=(w_k(t))_{k=\overline{1,r}}, t\in\mathbf{R}_+$ such that the σ-algebras $\sigma(w(s), 0\leq s\leq t)$ and $\mathcal{G}_t=\sigma(\eta(s), 0\leq s\leq t)$ are independent.

Let us denote by $\mathcal{F}_t, t\geq 0$ the smallest σ-algebra containing all sets $M\in\mathcal{F}$ with the property $P(M)=0$ and with respect to which all random vectors $w(s), s\leq t$ are measurable. It is easy to see that the Wiener process $\{w(t)\}_{t\ge 0}$ is adapted to the filtration $\mathcal{H}_t := \mathcal{F}_t \vee \mathcal{G}_t, t \ge 0$ and for every $s \ge 0$, the process $\left(w(t+s)-w(t)\right)_{t\ge 0}$ is independent of the σ- algebra $\mathcal{H}_t$. Such a σ-algebra is called an admissible filtration for the Wiener process $w(t)$. We note that $\mathcal{H}_t$ contains all P-neglijable sets from $\mathcal{F}$.

In this paper, the mean (expectation) of ξ will be denoted by $E(\xi)$. For any $i \in \mathcal{Z}$, $E(\xi|_{\eta(t)=i})$ is the conditional mean of the integrable random variable ξ on the event $\eta(t)=i$.

Denoting by $\mathcal{H}_{t_+}$ the normal filtration $\cap_{h\ge 0}\mathcal{H}_{t+h}$, we have the following.

Proposition 1. *The Wiener process $w(t)$ is $\mathcal{H}_{t_+}$ adapted and $\sigma(w(t+h)-w(t), h>0)$ is $\mathcal{H}_{t_+}$ independent for all $t \ge 0$.*

Proof. The first assertion of the theorem is a consequence of the inclusion $\mathcal{H}_t \subset \mathcal{H}_{t_+}$. We shall prove that the process $\sigma(w(t+h)-w(t), h>0)$ is $\mathcal{H}_{t_+}$ -independent for all $t \ge 0$. For any $n \in \mathbf{N}^*$ we set $C_0(\mathbf{R}^n) = \{f : \mathbf{R} \to \mathbf{R}, f$ is continuous and vanishes to infinity$\}$. It is known that the σ-algebra generated by $C_0(\mathbf{R}^n)$ coincides with the Borel σ-algebra $\mathcal{B}(\mathbf{R}^n)$. Moreover, if $f, g \in C_0(\mathbf{R}^n)$ then $fg \in C_0(\mathbf{R}^n)$.

Let $n \in \mathbf{N}^*, t \ge 0$ and $h_1 < h_2 < ... < h_n$ be arbitrary, but fixed. We shall prove that, for any $\mathcal{H}_{t_+}$ -measurable and bounded random variable z and $f \in C_0(\mathbf{R}^n)$, we have

(1)
$$E\left(f\left(w(t+h_1)-w(t),...,w(t+h_n)-w(t)\right)z\right) = E\left(f\left(w(t+h_1)-w(t),...,w(t+h_n)-w(t)\right)\right)E(z).$$

Indeed, since the Wiener process has continuous trajectories we can apply the Lebesque Dominated Convergence Theorem to deduce that

$$E\left(f\left(w(t+h_1)-w(t),...,w(t+h_n)-w(t)\right)z\right) =$$
$$\lim_{q\to 0} E\left(f\left(w(t+h_1)-w(t+q),...,w(t+h_n)-w(t+q)\right)z\right)$$
$$= \lim_{\substack{q\to 0,\\ q>0}} E\left(E\left[f\left(w(t+h_1)-w(t+q),...,w(t+h_n)-w(t+q)\right)z|_{\mathcal{H}_{t_+}}\right]\right) = (*)$$

For any $q \in (0, h_1)$ the random variable $\left(w(t+h_1)-w(t+q),...,w(t+h_n)-w(t+q)\right)$ is $\mathcal{H}_{t+q}$ -independent and, therefore, it is $\mathcal{H}_{t_+}$ -independent, too. We get

$$E\left[f\left(w(t+h_1)-w(t+q),...,w(t+h_n)-w(t+q)\right)z|_{\mathcal{H}_{t_+}}\right] =$$
$$E\left[f\left(w(t+h_1)-w(t+q),...,w(t+h_n)-w(t+q)\right)\right]E(z|_{\mathcal{H}_{t_+}})$$

and

$$(*) = E\left(f\left(w(t+h_1)-w(t),...,w(t+h_n)-w(t)\right)\right)E(z).$$

Consider the linear space $\mathbf{H}$ of all bounded and measurable functions from $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ to $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$, which satisfies (1). We observe that $C_0(\mathbf{R}^n) \subset \mathcal{H}$. If $f : \mathbf{R}^n \to \mathbf{R}$ is bounded and there is a sequence of nonnegative functions f_n increasing pointwise to f, then, by the

Monotone Convergence Theorem of Lebesque, it follows that $f \in \mathcal{H}$. Obviously, the constant functions belong to $\mathcal{H}$. Applying a version of the monotone class theorem it follows that $\mathcal{H}$ contains the set of all bounded and $\mathcal{B}(\mathbf{R}^n)$ -measurable functions $f : \mathbf{R}^n \to \mathbf{R}$. Hence (1) holds for any bounded Borel function. The conclusion follows.
The above proposition shows that the filtration $\mathcal{H}_{t_+}$ and the Wiener process $w(t)$ satisfy the usual conditions from the Ito's integral theory. Therefore, in this paper the Ito integral will be considered with respect to this filtration and the Wiener process $w(t)$.
For any real separable Hilbert space H, we denote by $L^2_{\eta,w}([t_0,T],H)$ the space of all H-valued processes $X(t)$, $t \in [t_0,T]$, $t_0 \le T$ which are nonanticipative [5] with respect to the filtration $\mathcal{H}_{t_+}$ and have the property that $E\left(\int_{t_0}^{T} \|X(t)\|^2 dt\right) < \infty$; it is known that any $X \in L^2_{\eta,w}([t_0,T],H)$ is stochastically integrable on $[t_0,T]$ [6]. We note here that $\eta(t)$ is right continuous and adapted to $\mathcal{H}_{t_+}$; therefore, it is nonanticipative.

3. THE MAIN RESULT

In this section we assume the following hypothesis

(H1) i) The functions $a : \mathbf{R}_+ \times \mathcal{Z} \times H \to H, b_r : \mathbf{R}_+ \times \mathcal{Z} \times H \to H$ are such that for each $i \in \mathcal{Z}$, $a(\cdot,i,\cdot), b_k(\cdot,i,\cdot), k = 1,..,r$ is measurable with respect to $\mathcal{B}(\mathbf{R}_+ \times H)$, where $\mathcal{B}(\mathbf{R}_+ \times H)$ denotes the σ-algebra of Borel sets in $\mathbf{R}_+ \times H$.

ii) For each $T > 0$ there exists $\gamma(T) > 0$ such that, for all $t \in [0,T]$, $x_1, x_2 \in H$, $i \in \mathcal{Z}$,

$$\|a(t,i,x_1) - a(t,i,x_2)\|^2 \le \gamma(T)\|x_1 - x_2\|^2;$$
$$\|b_k(t,i,x_1) - b_k(t,i,x_2)\|^2 \le \gamma(T)\|x_1 - x_2\|^2, k = 1,..,r;$$
$$\|a(t,i,x)\|^2 \le \gamma(T)(1 + |x|^2); \|b_k(t,i,x)\|^2 \le \gamma(T)(1 + |x|^2), k = 1,..,r$$

Let now consider the stochastic system

(2) $$dx(t) = a(t,\eta(t),x(t))dt + \sum_{k=1}^{r} b_k(t,\eta(t),x(t))d_k w(t)$$

(3) $$x(t_0) = \xi, t_0 \ge 0, t \ge t_0$$

where ξ is a H valued and $\mathcal{H}_{t_{0+}}$ -measurable random variable on Ω such that $E\|\xi\|^2 < \infty$.

By a *solution* of (2)-(3) we mean a function $x(.) \in L^2_{\eta,w}([t_0,T],H)$ which satisfies the stochastic integral equation

(4) $$x(t) = \xi + \int_{t_o}^{t} a(s,\eta(s),x(s))ds + \sum_{k=1}^{r}\int_{t_o}^{t} b_k(s,\eta(s),x(s))dw_k(s).$$

Theorem 1 *Under the above hypotheses there is a unique continuous solution* $x(t) = x(t, x_0, \xi)$, $t \geq t_0$ *from* $L^2_{\eta,w}([t_0,T],H)$ *of (2)-(3). Moreover, we have*

(5) $$\sup_{t_0 \leq t \leq T} E\left(\|x(t)\|^2 \mid \eta(t_0)=i\right) \leq K\left(1 + E\left(\|\xi\|^2 \mid \eta(t_0)=i\right)\right),$$

where K *is a positive constant which depends on* $\gamma(T), T$ *and* r. *Here the uniqueness is understood in the sense that if* $x_1(.)$ *and* $x_2(.)$ *are two solutions of (2)-(3) from* $L^2_{\eta,w}([t_0,T],H)$, *then* $E\left(\|x_1(t) - x_2(t)\|\right) = 0$.

Proof First we shall prove the uniqueness. Let $x_1(.), x_2(.) \in L^2_{\eta,w}([t_0,T],H)$ be two solutions of (2)-(3). Taking into account (4), we get

(6) $$\|x_1(t) - x_2(t)\| \leq \int_{t_o}^{t} \|a(s,\eta(s),x_1(s)) - a(s,\eta(s),x_1(s))\| ds$$

$$+ \sum_{k=1}^{r} \left\| \int_{t_o}^{t} \left(b_k(s,\eta(s),x_1(s)) - b_k(s,\eta(s),x_2(s))\right) dw_k(s) \right\|.$$

It is known [2] that for any $\Phi(\cdot) \in L^2_{\eta,w}([t_0,T],H)$ we have

(7) $$E\left[\left\|\int_{t_o}^{t} \Phi(s)\, dw_k(s)\right\|\right]^2 \leq E\left[\left\|\int_{t_o}^{t} \Phi(s)\, dw_k(s)\right\|^2\right] = E\int_{t_o}^{t} \|\Phi(s)\|^2 ds.$$

From (6), (H1) and the above inequality we obtain successively

(8) $$E\left(\|x_1(t) - x_2(t)\|^2\right) \leq (1+r)\{t \int_{t_o}^{t} E\left(\|a(s,\eta(s),x_1(s)) - a(s,\eta(s),x_1(s))\|^2\right) ds +$$

$$\sum_{k=1}^{r} E\left[\left\|\int_{t_o}^{t} \left(b_k(s,\eta(s),x_1(s)) - b_k(s,\eta(s),x_2(s))\right) dw_k(s)\right\|^2\right]\}$$

$$\leq (1+r)\gamma^2(T)\{tE\left[\int_{t_o}^{t} \|x_1(s) - x_2(s)\|^2 ds\right] + \sum_{k=1}^{r} E\left[\int_{t_o}^{t} \|x_1(s) - x_2(s)\|^2 ds\right].\}$$

(9) $$E\left(\|x_1(t) - x_2(t)\|^2\right) \leq \gamma^2(T)(1+r)(T+r)\int_{t_o}^{t} E\|x_1(s) - x_2(s)\|^2 ds,$$

for all $t_0 \leq t \leq T$. From Gronwall's lemma it follows that $E\left(\|x_1(t) - x_2(t)\|^2\right) = 0$. Hence $E\left(\|x_1(t) - x_2(t)\|\right) = 0$ for all $t \in [t_0,T]$. The uniqueness of the solution is proved.

For the existence part we shall use the method of successive approximations [5]. Let $x_0(t) = \xi$ and

$$x_{m+1}(t) = \xi + \int_{t_o}^{t} a(s,\eta(s), x_m(s))ds$$

(10)

$$+\sum_{k=1}^{r}\int_{t_o}^{t} b_k(s,\eta(s), x_m(s))dw_k(s),$$

for all $m \in \mathbf{N}$. Arguing exactly as in [5], we can prove by induction on $m \in \mathbf{N}^*$ that $x_m \in L^2_{\eta,w}([t_0,T],H)$ and there is a positive constant *M* depending only on $\gamma(T)$ and $E\|\xi\|^2$, such that

(11) $$\|x_{p+1}(t) - x_p(t)\| \le \frac{(Mt)^p}{(p+1)!}$$

for all $p \le m-1, m \in \mathbf{N}^*$.

Replacing x_1 and x_2 in (6) with x_{m+1} and x_m, respectively, arguing as in (8) and taking the supremum and the mean we get

$$E\left[\sup_{t_0\le t\le T}\|x_{m+1}(t) - x_m(t)\|^2\right] \le (1+r)\{TE\left[\sup_{t_0\le t\le T}\int_{t_o}^{t}\|a(s,\eta(s),x_1(s)) - a(s,\eta(s),x_1(s))\|^2 ds\right] +$$

$$\sum_{k=1}^{r} E\left[\sup_{t_0\le t\le T}\left\|\int_{t_o}^{t} b_k(s,\eta(s),x_1(s)) - b_k(s,\eta(s),x_2(s))dw_k(s)\right\|^2\right]\}.$$

Let us denote f_m (T) = $E\left[\sup_{t_0\le t\le T}\|x_{m+1}(t) - x_m(t)\|^2\right]$. An infinite dimensional version of Theorem 4.36 from [5] ensures that $f_m(T) \le (1+r)\gamma(T)^2(T+4r)\int_{t_o}^{T} f_m(s)ds.$ From (11) we obtain

$$f_m(T) \le \gamma(T)^2(1+r)(T+4r)\int_{t_o}^{T}\frac{(Ms)^m}{(m+1)!}ds \le \gamma(T)^2(1+r)T(T+4r)\frac{(MT)^m}{(m+1)!}.$$

Hence

$$E\left[\sup_{t_0\le t\le T}\|x_{m+1}(t) - x_m(t)\|^2\right] \le \gamma^2(T)(1+r)(T^2+4Tr)\frac{(MT)^m}{(m+1)!}.$$

Denoting $\alpha(T) = \gamma^2(T)(1+r)(T^2+4Tr)$ and using the Chebyshev's Inequality we get

$$P\left(\sup_{t_0\le t\le T}\|x_{m+1}(t) - x_m(t)\| > \frac{1}{2^m}\right) \le \alpha(T)\frac{(4MT)^m}{(m+1)!}.$$

Obviously the series $\sum_{m=1}^{\infty}\frac{(4MT)^m}{(m+1)!}$ converges and the Borel-Cantelli's Lemma ensures that

$$P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\left(\sup_{t_0\le t\le T}\|x_{m+1}(t) - x_m(t)\| > \frac{1}{2^m}\right)\right) = 0.$$

Arguing as in [5], we conclude that the sequence $x_m(t) = x_0(t) + \sum_{p=0}^{m-1} (x_{p+1}(t) - x_p(t))$ converges uniformly in $t \in [t_0, T]$ to a function $x(t)$ which is a continuous and nonanticipative process from $L^2_{\eta,w}([t_0,T], H)$. Passing to the limit for $m \to \infty$ in (10) we see that $x(t)$ is a solution of (2)-(3). The existence part is proved.
Now let us prove (5). From (4) we have

$$\|x(t)\|^2 \le (2+r)\Big(\|\xi\|^2 + T\int_{t_o}^{t} \|a(s,\eta(s),x(s))\|^2 ds$$

$$+\sum_{k=1}^{r}\Big\|\int_{t_o}^{t} b_k(s,\eta(s),x(s))\,dw_k(s)\Big\|^2\Big),$$

Taking the conditional mean we obtain

$$E[\|x(t)\|^2 \mid \eta(t_0)=i] \le (2+r)\{E[\|\xi\|^2 \mid \eta(t_0)=i] + \gamma^2(T)[T\int_{t_o}^{t} (1+E[\|x(s)\|^2 \mid \eta(t_0)=i])ds$$

$$+\sum_{k=1}^{r}\int_{t_o}^{t} (1+E[\|x(s)\|^2 \mid \eta(t_0)=i])ds]\}$$

$$\le (2+r)\{E[\|\xi\|^2 \mid \eta(t_0)=i] + \gamma^2(T)(T+r)\Big(T+\int_{t_o}^{t} E[\|x(s)\|^2 \mid \eta(t_0)=i]ds\Big)\}.$$

Taking $K_1 = \max\{2+r, \gamma^2(T)(T+r)T\}$ we see that

$$E[\|x(t)\|^2 \mid \eta(t_0)=i] \le K_1(E[\|\xi\|^2 \mid \eta(t_0)=i]+1) + \gamma^2(T)(T+r)\int_{t_o}^{t} E[\|x(s)\|^2 \mid \eta(t_0)=i]ds.$$

Another appeal to Gronwall's Lemma ensures that

$$E[\|x(t)\|^2 \mid \eta(t_0)=i] \le K_1(E[\|\xi\|^2 \mid \eta(t_0)=i]+1)e^{\gamma^2(T)(T+r)\int_{t_o}^{T} ds}.$$

Now it is clear that (5) follows. The proof is complete.

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