VIBRATION OF ELASTICALLY SUPPORTED TIMOSHENKO BEAM

Associate Professor PhD. eng. Traian MAZILU
Department of Railway Vehicles, University Politehnica of Bucharest
trmazilu@yahoo.com

Abstract This paper deals with the response of a Timoshenko beam elastically supported on Winkler foundation due to two harmonic forces. Such problem could be interesting to study the dynamic of the sleepers of the track. The Green’s function method has been applied to determine the frequency-response of the beam considering both symmetrical and anti-symmetrical excitation modes. The influence of the foundation on the vibration behavior of the beam is pointed out.

Keywords: Timoshenko beam, Winkler foundation, Green’s function, sleeper, track

1. INTRODUCTION

Nowadays the beam models are an important theoretical tool to investigate both static and dynamic behaviors of technical systems. The most common beam models are the so-called Euler-Bernoulli beam, which is the simplest one, and the Timoshenko beam, which takes into account the shear and the rotary inertia of the cross-section.

In the railway engineering, the beam theory is applied to model the rails [1], sleepers [2] and bridges [3]. Also, the structural vibration behavior of the carbody or the wheelset is studied using either the Euler-Bernoulli beam model [4] or the Timoshenko beam model [5].

Regarding the problems related by the track sleeper vibration, it has to underline a particular difficulty which is represented by the fact that the mono-bloc concrete sleepers have variable cross section. Actually, the sleepers are substantially more robust in the railseat aria than in the centre. Despite of this, many authors used the uniform Timoshenko beam to model the track sleepers [6, 7]. In fact, Grassie has demonstrated that it is possible to model a wide variety of mono-bloc concrete sleepers as Timoshenko beams of constant flexural rigidity and proposed a means of calculating the effective flexural rigidity of such a uniform beam [7].

In this paper, the Timoshenko beam model on a Winkler foundation is used to study the basic features of the vibration behavior of the concrete sleeper. To this end, the Green’s functions method is applied [8].

2. MECHANICAL MODEL.

It considers a uniform Timoshenko beam of 2l length, supported on a viscoelastic Winkler foundation, subjected to the action of two forces \( Q_1 \) and \( Q_2 \) located at the distance \( e \) on both sides of the beam center, as it can see in figure 1. Such model can be used to analyze the vibration behavior of a sleeper on ballast excited by the wheel/rail forces transmitted via rails.

The beam motion is studied in respect to the reference frame \( O_{xz} \). Considering a cross section of the beam at \( x \) distance from the \( O \) point, the beam displacement is \( w(x, t) \) and the rotation is \( \psi(x, t) \). The parameters for the Timoshenko beam are: the Young’s modulus \( E \), the shear modulus \( G \), the density \( \rho \), the mass per length unit \( m \), the cross-section area \( S \), the area moment of inertia \( I \) and the shear coefficient \( \kappa \). The parameters for the Winkler foundation
are: the elastic constant per length unit \( k \) and the damping constant per length unit \( c \).  

\[
m \frac{ \partial^2 w}{ \partial t^2 } + c \frac{ \partial w}{ \partial t } + kw - GSK \left( \frac{ \partial^2 w}{ \partial x^2 } - \frac{ \partial \varphi}{ \partial x } \right) = q(x,t),
\]

\[
\rho l \frac{ \partial^2 \varphi}{ \partial t^2 } + GSK \left( \varphi - \frac{ \partial w}{ \partial x } \right) - EI \frac{ \partial^2 \varphi}{ \partial x^2 } = 0.
\]

where \( q(x,t) \) is the force per length unit applied on the beam

\[
q(x,t) = Q_1(t) \delta(x+e) + Q_2(t) \delta(x-e),
\]

with \( \delta(\cdot) \) is the delta Dirac function.

The boundary conditions at the beam ends (free-free) require that the shear force and the moment be zero

\[
\frac{ \partial w}{ \partial x } (+l,t) - q(+l,t) = 0, \quad \frac{ \partial \varphi}{ \partial x } (+l,t) = 0.
\]

For the steady-state harmonic behavior of \( \omega \) angular frequency, the complex variables have to be introduced

- for beam displacement

\[
\bar{w}(x,t) = \bar{w}(x)e^{i\omega t}, \quad \bar{\varphi}(x,t) = \bar{\varphi}(x)e^{i\omega t};
\]

- for the forces \( Q_{1,2} \) and the force per unit length \( q(x,t) \)
\[
\begin{align*}
\bar{Q}_{1,2}(t) &= \bar{Q}_{1,2}e^{j\omega t}, \quad \bar{q}(x,t) = \bar{q}(x)e^{j\omega t}, \\
\end{align*}
\]

with
\[
\bar{q}(x) = \bar{Q}_0\delta(x+e) + \bar{Q}_2\delta(x-e)
\]
and \(i^2 = -1\).

The equations of motion become
\[
\begin{align*}
(-m\omega^2 + c\omega i + k)\bar{w} - G\kappa \left( \frac{d^2 \bar{w}}{dx^2} - \frac{d\bar{q}}{dx} \right) &= \bar{q}; \\
-\rho\kappa \frac{d^2 \bar{q}}{dx^2} + G\kappa \left( \bar{q} - \frac{d\bar{w}}{dx} \right) - EI \frac{d^2 \bar{q}}{dx^2} &= 0.
\end{align*}
\]

Also, the boundary condition can be rewritten
\[
\frac{d\bar{w}}{dx} (\pm l) - \bar{\phi}(\pm l) = 0; \quad \frac{d\bar{\phi}}{dx} (\pm l) = 0
\]

or better, rewritten in terms of the displacement
\[
\begin{align*}
\frac{d^2 \bar{w}}{dx^2} (\pm l) - \frac{m\omega^2}{G\kappa} \bar{w}(\pm l) &= 0, \quad \frac{d^3 \bar{w}}{dx^3} (\pm l) + \left[ \frac{\rho\omega^2}{E} - \frac{m\omega^2}{G\kappa} \right] \frac{d\bar{w}}{dx} (\pm l) = 0
\end{align*}
\]

In fact, it is more interesting to work with the displacement \(w\) and, because of that, the following equation can be extracted from (8) and (9)
\[
\begin{align*}
EI \frac{d^4 \bar{w}}{dx^4} + \omega^2 \rho I \left[ 1 + \frac{E}{G\kappa} \left( 1 - \frac{c}{\omega m} - \frac{k}{\omega^2 m} \right) \right] \frac{d^2 \bar{w}}{dx^2} + \omega^2 \rho S \left[ 1 - \frac{c}{\omega m} - \frac{k}{\omega^2 m} \right] \left( \frac{\omega^2 \rho I}{G\kappa} - 1 \right) \bar{w} &= 0 \quad (12) \\
= \left[ 1 - \frac{\omega^2 \rho I}{G\kappa} \right] \bar{q}(x) - \frac{EI}{G\kappa} \frac{d^2 \bar{q}}{dx^2}
\end{align*}
\]

Solution of this equation can be given in terms of the Green’s function
\[
\bar{w}(x) = \int_{-l}^{l} g(x, \xi) \left[ 1 - \frac{\omega^2 \rho I}{G\kappa} \right] \bar{q}(\xi) - \frac{EI}{G\kappa} \frac{d^2 \bar{q}}{dx^2} d\xi,
\]

where \(g(x, \xi)\) is the Green’s function of the differential operator of the equation (12).
Inserting equation (7) in (13), the beam displacement can be calculated

\[ \bar{w}(x) = \overline{Q}_1 g^* (x, -e) + \overline{Q}_2 g^* (x, e), \]  

where

\[ g^*(x, \pm e) = \left[ 1 - \frac{\omega^2 \rho l}{GS\kappa} \right] g(x, \pm e) - \frac{EI}{GS\kappa} \frac{d^2 g(x, \pm e)}{dx^2}, \]  

(14)

taking the form

\[ g(x, \xi) = A_1 \sinh \sigma x + A_2 \cosh \sigma x + A_3 \sin \tau x + A_4 \cos \tau x + \frac{H(x - \xi)}{EI (\sigma^2 - \tau^2)} \left[ \frac{\sinh (x - \xi)}{\sigma} - \frac{\sinh (x - \xi)}{\tau} \right], \]  

(19)

where \( H(.) \) is the Heaviside function and \( \pm \sigma \) and \( \pm \tau \) are the solutions of the characteristic equation.

To calculate the constants \( A_i \), \( i = 1 \) to \( 4 \), the Green’s function (19) has to verify the boundary conditions (17) and (18).

3. NUMERICAL APPLICATION

In this section, the numerical results derived for a particular sleeper and ballast bed using the previous model are presented. Parameters for the model are as follows: \( m = 129 \) kg/m, \( E = 52.9 \) GPa, \( G = 22.78 \) GPa, \( I = 2.61 \times 10^{-4} \) m\(^4\), \( \rho = 2729 \) kg/m\(^3\), \( S = 0.04727 \) m\(^2\), \( l = 1 \) m, \( e = 0.75 \) m, \( k = 180 \) MN/m\(^2\). The amplitude of the excitation forces is 1 N.

It has to be observed that the structure under study is symmetrical and due to that, there are two kind of excitation and also two kind of vibration: symmetrical and anti-symmetrical.
The symmetrical mode of excitation implies $Q_1 = Q_2$ – the two forces are in phase, and the anti-symmetrical mode of excitation requires $Q_1 = -Q_2$ – the two forces are in anti-phase.

Figure 2 presents the beam displacement at the force point when the two forces are in phase (symmetrical excitation) and the beam has no foundation ($k = 0$ and $c = 0$) for frequencies between 10 and 3000 Hz. The beam response exhibits many frequencies of resonance and anti-resonance. Indeed, the first resonance frequency takes place at 275 Hz, the second at 1242 and the third at 2506 Hz.

Figure 3 shows the beam displacement at the force point for the anti-symmetrical mode of excitation. This time, the resonance frequencies are 697 and 1855 Hz, and the anti-resonance frequencies are 1377 and 2546 Hz. Both figures show the inertial character of the beam response particularly at low frequency where the response decreases as long the frequency increases.

![Figure 2. Beam response to symmetrical excitation (without foundation).](image1)

![Figure 3. Beam response to anti-symmetrical excitation (without foundation).](image2)

![Figure 4. Beam response to symmetrical excitation (with foundation).](image3)

![Figure 5. Beam response to anti-symmetrical excitation (with foundation).](image4)
Figure 6. Beam response to symmetrical excitation (with damped foundation).

Figure 7. Beam response to anti-symmetrical excitation (with damped foundation).

Figure 4 and 5 present the influence of the foundation stiffness (ballast) for both excitation modes, symmetrical and anti-symmetrical, respectively. The vibration behavior changes especially at low frequency, the beam response has elastic character (constant) and then, the resonance due to the Winkler foundation appears at 132 Hz. The resonances due to the bending modes of the beam are unchanged excepting the first resonance which is slightly higher.

The influence of the foundation damping is exhibited in figures 6 and 7, also for both symmetrical and anti-symmetrical excitation modes. The main influence of the foundation damping consists in the fact that the resonance peaks and anti-resonances deeps are leveled.

4. CONCLUSIONS

In this paper, the model of a Timoshenko beam on viscoelastic foundation has been used to study the main features of the dynamic response of a sleeper from the ballasted track. To this end, the Green’s function method has been applied.

At low frequencies, the frequency – response of the beam is practically constant due to the elastic force of the foundation which is higher than the inertial and damping forces. At middle and high frequency, the dynamic behavior is dominated by the many resonances due to the Winkler foundation and bending modes of the beam. Foundation damping levels the peaks of resonance and the deeps of anti-resonance.

REFERENCES


