THE STATISTICAL METHOD FOR FINDING THE RESPONSE FOR N RANDOM EXCITATION OF THE INVERTED PENDULUM

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Abstract: We present a method for estimating the power spectral density of the stationary response of oscillator with a nonlinear restoring force under external stochastic wide-band excitation. An equivalent linear system is derived, from which the power spectral density is deduced. We consider an inverted pendulum suspended subjected to white noise excitation with a random number n random excitations simultaneously applied. The method will be briefly discussed in the following sections.

Keywords: inverted pendulum, the power spectral density, response statistics, random vibration.

1. SYSTEM MODEL

We consider an inverted pendulum suspended, which has mass m, length l rod has spring constant k, the angle of inclination to the position of equilibrium \( \theta \), subjected to white noise excitation with a random number n random excitations simultaneously applied. It is assumed that the rod has negligible mass.

The differential equation of the motion [1, 8, 9] can be written as:

\[
ml^2 \ddot{\theta} + kd^2 \cos \theta + cd^2 \sin \theta \cos \theta - mgl \sin \theta = W(t)l, \quad (1)
\]

Equation of motion, while neglecting very small terms is

\[
\dot{\theta} + \frac{cd^2}{ml^2} (1 - \lambda \theta^2) \dot{\theta} + \left( \frac{kd^2}{ml^2} - \frac{g}{l} \right) \theta + \alpha \left( \frac{g}{6l} \frac{2kd}{3ml^2} \right) \theta^3 = W(t), \quad (2)
\]

where \( W(t) = \sum_{j=1}^{n} \frac{W_j(t)}{ml} \), \( \lambda \) and \( \alpha \) are the nonlinear factors to control the type and degree of nonlinearity in the system [3,4].

The autocorrelation of the process [1, 4, 5] \( W(t) \) can be evaluated as

\[
R_W(\tau) = \lim_{T \to \infty} \frac{1}{T} \left[ \sum_{j=1}^{n} W_j(t) \right] \left[ \sum_{j=1}^{n} W_j(t+\tau) \right] dt = \lim_{T \to \infty} \frac{1}{T} \left[ \sum_{j=1}^{n} W_j(t) W_j(t+\tau) + W_1(t) W_2(t+\tau) + \ldots + W_n(t) W_n(t+\tau) \right] dt = \sum_{j=1}^{n} R_{W_1}(\tau) + \sum_{r=1}^{n} \sum_{s=1}^{n} R_{W_r W_s}(\tau). \quad (3)
\]

where the cross spectral density is
The real part of the cross-spectrum, \( \text{Re}\left( \sum_{r=1}^{n} S_{W_r W_s}(\omega) \right) \), which contains the in-phase components of the time-functions \( W_1(t), W_2(t), \ldots, W_n(t) \) is called the coincidence-spectrum. The spectral density function of the response can also be determined from the direct superposition of the loads and in the time domain \( W(t) = \sum_{j=1}^{n} W_j(t) \).

For completely uncorrelated processes we have

\[
S_{W_r W_s}(\omega) = S_{W_s W_r}(\omega) = 0, \quad r \neq s
\]

and the power spectral density of the response is

\[
S_p(\omega) = \sum_{j=1}^{n} |H_j(\omega)|^2 S_{W_j}(\omega).
\]

The equations of motion for the equivalent linear system will be written

\[
\ddot{\theta}(t) + \beta_{ech} \dot{\theta}(t) + \gamma_{ech} \theta(t) = w(t).
\]

If we denote

\[
h(\theta(t), \dot{\theta}(t)) = \frac{cd^2}{ml^2} (1 - \lambda \dot{\theta}^2) \dot{\theta} + \left( \frac{kd^2}{ml^2} - \frac{g}{l} \right) \theta + \alpha \left( \frac{g}{6l} - \frac{2kd}{3ml^2} \right) \dot{\theta}^3
\]

the error obtained by linearization is given by

\[
e = h(\theta(t), \dot{\theta}(t)) - \beta_{ech} \dot{\theta}(t) - \gamma_{ech} \theta(t).
\]

If we take into account the error minimization conditions achieved by linearization of nonlinear equations of motion, we obtain the solutions in simplified form

\[
\beta_{ech} = \frac{E\{\theta h\}}{E\{\theta^2\}}, \quad \gamma_{ech} = \frac{E\{\eta h\}}{E\{\eta^2\}}.
\]

Obtain the linearization equation [3,5,6]
\[
\theta(t) + \frac{E(\theta h)}{E(\theta^2)} \dot{\theta}(t) + \frac{E(\theta h)}{E(\theta^2)} \theta(t) = u(t),
\]
where
\[
\frac{E(\theta h)}{E(\theta^2)} = \frac{cd^2}{ml^2} \left( \frac{E(\theta^2)}{E(\theta^2)} - \lambda \frac{E(\Theta^2)}{E(\Theta^2)} \right) + \left( \frac{k \frac{d^2}{ml^2} - \frac{g}{l} \right) \frac{E(\Theta)}{E(\Theta^2)} + \alpha \left( \frac{g}{6l} - \frac{2kd^2}{3ml^2} \right) \frac{E(\Theta^3)}{E(\Theta^2)},
\]
\[
\frac{E(\theta h)}{E(\theta^2)} = \frac{cd^2}{ml^2} \left( \frac{E(\Theta)}{E(\Theta^2)} - \lambda \frac{E(\Theta^2)}{E(\Theta^2)} \right) + \left( \frac{k \frac{d^2}{ml^2} - \frac{g}{l} \right) \frac{E(\Theta)}{E(\Theta^2)} + \alpha \left( \frac{g}{6l} - \frac{2kd^2}{3ml^2} \right) \frac{E(\Theta^3)}{E(\Theta^2)}.
\]

By transforming the above relations, we obtain the relations of motion
\[
\ddot{\theta} + \frac{cd^2}{ml^2} \left( 1 - 2\lambda \sigma^2 \right) \dot{\theta} + \left( \frac{k \frac{d^2}{ml^2} - \frac{g}{l} \right) \frac{E(\Theta)}{E(\Theta^2)} + \alpha \left( \frac{g}{6l} - \frac{2kd^2}{3ml^2} \right) \sigma^2 \theta = u(t).
\]

Using the Fourier transform \[5,8,9\] of equation (15) we obtain for the frequency response function
\[
H(\omega) = \frac{1}{-m\omega^2 + i\omega \frac{cd^2}{l^2} \left( 1 - 2\lambda \sigma^2 \right) + \left( \frac{k \frac{d^2}{ml^2} - \frac{g}{l} \right) + 3\alpha \left( \frac{g}{6l} - \frac{2kd^2}{3ml^2} \right) \sigma^2}.
\]

The response spectrum \[4,5,8\] is given by
\[
S_\eta(\omega) = \left| H(\omega) \right|^2 \left[ \sum_{j=1}^{n} S_{W_j}(\omega) + \sum_{r=1, r \neq s}^{n} S_{W_rW_s}(\omega) \right].
\]

where the cross spectral density function \[2,6,7\] is
\[
S_{W_rW_s}(\omega) = 2\int_{-\infty}^{\infty} R_{W_rW_s}(\tau) \cos \omega \tau d\tau + 2i \int_{-\infty}^{\infty} R_{W_rW_s}(\tau) \sin \omega \tau d\tau,
\]
\[
S_{W_r}(\omega) = 2\int_{-\infty}^{\infty} R_{W_rW_s}(\tau) \cos \omega \tau d\tau + 2i \int_{-\infty}^{\infty} R_{W_rW_s}(\tau) \sin \omega \tau d\tau, r \neq s.
\]

We have
\[
S^*_{W_rW_s}(\omega) = S_{W_rW_s}(\omega), r \neq s.
\]

The response spectrum is
\[
S_\eta(\omega) = \left| H(\omega) \right|^2 \left[ S_{W_1}(\omega) + S_{W_2}(\omega) + \ldots + S_{W_n}(\omega) + 2 \operatorname{Re}(S_{W_1W_2}(\omega) + S_{W_2W_3}(\omega) + \ldots + S_{W_{n-1}W_n}(\omega)) \right]
= \left| H(\omega) \right|^2 \left[ \sum_{j=1}^{n} S_{W_j}(\omega) + 2 \operatorname{Re} \left( \sum_{r=1, r \neq s}^{n} S_{W_rW_s}(\omega) \right) \right].
\]

The mean square value for the displacement of the system \[9\] is
\[
\sigma^2 = \int_{-\infty}^{\infty} \left[ \sum_{j=1}^{n} H_j(\omega)^2 S_{W_j}(\omega) + 2 \sum_{r=1, r \neq s}^{n} H^*_r(\omega) H_s(\omega) S_{W_rW_s}(\omega) \right] d\omega.
\]
The variance [5,7,8] of the process $W(t)$ can be written as

$$\sigma_W^2 = R_W(0) = \sum_{j=1}^{n} R_{W_j}(0) + \sum_{r=1}^{n} \sum_{s=1}^{n} R_{W_rW_s}(0) = \sigma_{W_1}^2 + \sigma_{W_2}^2 + \ldots \sigma_{W_n}^2$$ 

(23)

2. NUMERICAL RESULTS

The pendulum formed from a length bar $l = 0.5m$, with insignificant mass, caught in point O, with the elasticity constant $k = 36 \frac{N}{m}$, in which hangs the body with the $m=1kg$ mass and $c = \frac{N \cdot s}{m}$, $d = 0.4m$, $\alpha = 5rad^{-2}$, $\lambda = 0.34rad^{-2}$.

In figure 1, we consider the power spectral density for two random excitations, with $S_{W_1} = S_{W_2} = 1N^2 \cdot s$. Figure 1 describes the spectral density of the response spectrum for two random excitations applied equal.

The mean square value for the displacement of the system is $\sigma_{\theta}^2 = 0.36rad^2$. We observe that the maximum value of 0.06 $rad^2 \cdot s$ are obtained for values of the frequency of 4.8 Hz although, in the frequency band of values until 2 Hz the spectral power density is almost constant $0.005 rad^2 \cdot s$.

Figure 2 shows the spectral density graphic for three random excitations simultaneously applied $S_{W_1} = 1N^2 \cdot s$, $S_{W_2} = 1N^2 \cdot s$, $S_{W_3} = 1N^2 \cdot s$.
3. REMARKS

In figure 1 we observe that the maximum value of 0.06 rad²·s are obtained for values of the frequency of 4.8 Hz although, in the frequency band of values until 2 Hz the spectral power density is almost constant 0.005 rad²·s. Then follows a sharp decline in the frequency (5.1; 6.40) Decreases in the frequency band (6.3; 9) power spectral density response is very slow and they tend to 0. Unlike the former case, in figure 2 we observe that the maximum value of 0.091 rad²·s are obtained for values of the frequency of 5 Hz although, in the frequency band of values until 1.25 Hz the spectral power density is almost constant 0.006 rad²·s.

4. REFERENCES