SOME PROPERTIES OF \((\alpha, \beta)\)-CUTS FOR INTUITIONISTIC FUZZY SETS

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Abstract. In this article, we give some basic definitions from Intuitionistic fuzzy set theory, we introduced the notion of a complete system of events in intuitionistic fuzzy sets theory with examples and characterizations.

Keywords: Intuitionistic fuzzy sets, \((\alpha, \beta)\)-cut, decomposition theorems, and complete system of events of a intuitionistic fuzzy sets.

1. INTRODUCTION
Zadeh introduced the notion of the \(\alpha\) cut of fuzzy set (FS) and decomposition theorems [7],[8],[11]. The concept of intuitionistic fuzzy sets (IFS) was introduced by K.T. Atanassov [1] as a generalization of the notion of a fuzzy set. Starting from the decomposition theorems from Intuitionistic fuzzy sets described in [7],[8],[11], I will try to develop this theory by showing that IFS not forming a complete system of events, similar to the classical theory of probabilities.

2. PRELIMINARIES

Definition 1.[11] Let \(X\) the non-empty set. A fuzzy subset \(A\) of \(X\) is

\[ A = \left\{ (x, \mu_A(x)) | x \in X \right\} \] (1)

Definition 2.[1] An intuitionistic fuzzy set (IFS) \(A\) in \(X\) is given by

\[ A = \left\{ (x, \mu_A(x), \nu_A(x)) | x \in X \right\} \] (2)

where \(\mu_A : X \rightarrow [0,1]\) is called the membership function, \(\mu_A(x)\) is called degree of membership, \(\nu_A : X \rightarrow [0,1]\) is called non-membership, \(\nu_A(x)\) is called degree of non-membership, with the condition

\[ 0 \leq \mu_A(x) + \nu_A(x) \leq 1, \quad \forall x \in X \] (3)

Definition 3.[11] In the case of fuzzy set theory a \(\alpha\)-cut or a set of level \(\alpha\), \(\alpha \in [0,1]\). of fuzzy set \(A\) is:

\[ [A]^\alpha = \begin{cases} \{ x \in X | \mu_A(x) \geq \alpha \} & \text{if } 0 < \alpha \leq 1 \\ \{ x \in X | \mu_A(x) > 0 \} & \text{if } \alpha = 0 \end{cases} \] (4)

and

\[ [A]^\ast\alpha = \begin{cases} \{ x \in X | \mu_A(x) > \alpha \} & \text{if } 0 < \alpha \leq 1 \\ \{ x \in X | \mu_A(x) > 0 \} & \text{if } \alpha = 0 \end{cases} \] (5)

is called strong \(\alpha\)-cut as the IFS.
Theorem 1. [7] First Decomposition Theorem of FS. For any fuzzy sets A,
\[ A = \bigcup_{\alpha \in [0,1]} [A]_{\alpha} \]

Theorem 2. [7] Second Decomposition Theorem of FS. For any fuzzy sets A,
\[ A = \bigcup_{\alpha \in [0,1]} [A]_{\alpha^+} \]

3. \((\alpha,\beta)\)-CUT AS THE INTUITIONISTIC FUZZY SETS

In the case of IFS we have:

Definition 4. [3] We consider \( \alpha, \beta \in [0,1] \) then, for any IFS set \( A \)
\[ [A]^{(\alpha,\beta)} = \begin{cases} \{ x \in X : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta, 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \} & \text{if } 0 < \alpha, \beta \leq 1 \\ \{ x \in X : \mu_A(x) > 0, \nu_A(x) < 0, 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \} & \text{if } \alpha = 0, \beta = 0 \end{cases} \] (6)

is called \((\alpha,\beta)\)-cut as the IFS A.

\[ [A]^{+(\alpha,\beta)} = \begin{cases} \{ x \in X : \mu_A(x) > \alpha, \nu_A(x) < \beta, 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \} & \text{if } 0 < \alpha, \beta \leq 1 \\ \{ x \in X : \mu_A(x) > 0, \nu_A(x) < 0, 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \} & \text{if } \alpha = 0, \beta = 0 \end{cases} \] (7)

is called strong \((\alpha,\beta)\)-cut as the IFS.

Definition 5. [6] For \( \alpha, \beta \in [0,1] \) with \( \alpha + \beta \leq 1 \) through \([A]^{(\alpha,\beta)}\) and \([A]^{+(\alpha,\beta)}\) we understand
\[ [A]_{(\alpha,\beta)} = \begin{cases} (\alpha, \beta) & \text{if } x \in [A]^{(\alpha,\beta)} \\ (0,1) & \text{if } x \notin [A]^{(\alpha,\beta)} \end{cases} \] (8)

Respectively
\[ [A]^{+(\alpha,\beta)} = \begin{cases} (\alpha, \beta) & \text{if } x \in [A]^{+(\alpha,\beta)} \\ (0,1) & \text{if } x \notin [A]^{+(\alpha,\beta)} \end{cases} \] (9)

Definition 6. [3] Given two IFSs A and B over an universe of discourse X, one can define the following relations:
\( A \subset B \) iff \( \forall x \in X \), \( \mu_A(x) \leq \mu_B(x) \) and \( \nu_A(x) \geq \nu_B(x) \)
\( A = B \) iff \( A \subset B \) and \( B \subset A \)
as well as the following operations [1]:
\( \overline{A} = \{ (x, \nu_A(x), \mu_A(x)) | x \in X \} \)
\( A \cap B = \{ (x, \mu_{A\cap B}(x), \nu_{A\cap B}(x)) | x \in X \} \), where
\[ \mu_{A \cap B} = \min \{ \mu_A(x), \mu_B(x) \} \] and 
\[ \nu_{A \cap B} = \max \{ \nu_A(x), \nu_B(x) \} \]

\[ A \cup B = \{ (x, \mu_{A \cap B}(x), \nu_{A \cap B}(x)) | x \in X \} \], where 
\[ \mu_{A \cap B} = \max \{ \mu_A(x), \mu_B(x) \} \] and 
\[ \nu_{A \cap B} = \min \{ \nu_A(x), \nu_B(x) \} \]

**Theorem 3.** [3] Let A and B two IFS and one \( \alpha, \beta, \gamma, \delta \in [0.1] \). Then following are true:

1) \([A]^{\alpha\beta} \subseteq [A]^{\gamma\delta}\)
2) If \( \alpha \leq \gamma \) and \( \beta \leq \delta \) then \([A]^{\gamma\delta} \subseteq [A]^{\alpha\beta}\)
3) \([A \cup B]^{\alpha\beta} = [A]^{\alpha\beta} \cup [B]^{\alpha\beta} \cup [D_{A \cup B}]^{\alpha\beta} \cup [D_{B \cup A}]^{\alpha\beta}\)
4) \([A \cap B]^{\alpha\beta} = [A]^{\alpha\beta} \cap [B]^{\alpha\beta}\)
5) \([A \cup B]^{(\alpha\beta)} = [A]^{(\alpha\beta)} \cup [B]^{(\alpha\beta)}\)
6) \([A \cap B]^{(\alpha\beta)} = [A]^{(\alpha\beta)} \cap [B]^{(\alpha\beta)}\)
7) \([A \cap B]^{\alpha\beta} \neq ( [A]^{(1-\alpha,1-\beta)} )^{\alpha\beta}\)

Where

\[
\begin{align*}
[D_{A \cup B}]^{(\alpha\beta)} &= \left\{ \begin{array}{ll}
\{ x \in X | \mu_A(x) \geq \alpha, \nu_B(x) \leq \beta, 0 \leq \mu_A(x) + \nu_B(x) \leq 1 \} & \text{if } 0 < \alpha, \beta \leq 1 \\
\{ x \in X | \mu_A(x) > 0, \nu_B(x) > 0, 0 \leq \mu_A(x) + \nu_B(x) \leq 1 \} & \text{if } \alpha = 0, \beta = 0
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
[D_{B \cup A}]^{(\alpha\beta)} &= \left\{ \begin{array}{ll}
\{ x \in X | \mu_B(x) \geq \alpha, \nu_A(x) \leq \beta, 0 \leq \mu_B(x) + \nu_A(x) \leq 1 \} & \text{if } 0 < \alpha, \beta \leq 1 \\
\{ x \in X | \mu_B(x) > 0, \nu_A(x) > 0, 0 \leq \mu_B(x) + \nu_A(x) \leq 1 \} & \text{if } \alpha = 0, \beta = 0
\end{array} \right.
\end{align*}
\]

**Theorem 4.** [3] Let A and B two IFS and one \( \alpha, \beta, \gamma, \delta \in [0.1] \). Then:

1) \( A \subseteq B \) if and only if \([A]^{(\alpha\beta)} \subseteq [B]^{(\alpha\beta)}\)
2) \( A \subseteq B \) if and only if \([A]^{(\gamma\delta)} \subseteq [B]^{(\gamma\delta)}\)
3) \( A = B \) if and only if \([A]^{(\alpha\beta)} = [B]^{(\alpha\beta)}\)

**4. THE DECOMPOSITION THEOREMS OF INTUITIONISTIC FUZZY SETS**

**Theorem 5.** [6] First Decomposition Theorem of IFS. Let \( X \) the non- empty set. For any intuitionistic fuzzy subset \( A \) in \( X \),

\[
A = \bigcup_{\alpha, \beta \in [0,1]} [A]^{(\alpha\beta)}
\]

Where \( \bigcup \) denotes union given in Definition 8 and \([A]^{(\alpha\beta)}\) given in Definition.

**Theorem 6.** [6] Second Decomposition Theorem of IFS. Let \( X \) the non- empty set. For any intuitionistic fuzzy subset \( A \) in \( X \),
\[ A = \bigcup_{\alpha, \beta \in [0.1]} [A]_{[\alpha, \beta]} \]

Where \( \cup \) denotes union given in Definition 8 and \([A]_{[\alpha, \beta]} \) given in Definition.

**Theorem 7.** Let \( X \) the non-empty set, \( \alpha, \beta \in [0.1] \) with \( \alpha + \beta \leq 1 \). For any intuitionistic fuzzy subset \( A \) in \( X \), \([A]_{[\alpha, \beta]} \) not form a complete system of events of \( A \).

**Proof:** The fact that \( A = \bigcup_{\alpha, \beta \in [0.1]} [A]_{[\alpha, \beta]} \) is demonstrated by [6]. Still need to show that \([A]_{[\alpha, \beta]} \cap [A]_{(\gamma, \delta)} \neq \emptyset \), \( \forall (\alpha, \beta) \neq (\gamma, \delta), \alpha, \beta, \gamma, \delta \in [0.1], \alpha + \beta \leq 1, \gamma + \delta \leq 1 \) (10)

If the absurd relationship were true, then \( \exists (\alpha, \beta) \neq (\gamma, \delta), \alpha, \beta, \gamma, \delta \in [0.1], \alpha + \beta \leq 1, \gamma + \delta \leq 1 \) so that \([A]_{[\alpha, \beta]} \cap [A]_{(\gamma, \delta)} = \emptyset \). But \([A]_{[\alpha, \beta]} \cap [A]_{(\gamma, \delta)} = \emptyset \).

**Example 1.** Let \( X = \{a, b, c, d, e, f\} \) and

\[ A = \{(a, 0.2, 0.4), (b, 0.8, 0.2), (c, 0.6, 0.3), (d, 0.4, 0.5), (e, 0.1), (f, 1, 0)\} \]

Let us denote \( A \) for convenience as

\[ A = \left(\begin{array}{c}
0.2, 0.4 \\
0.8, 0.2 \\
0.6, 0.3 \\
0.4, 0.5 \\
0.1, 0.1 \\
1, 0
\end{array}\right) \]

Then

\[ [A]_{(0.2, 0.4)} = \left(\begin{array}{c}
0.2, 0.4 \\
0.8, 0.2 \\
0.6, 0.3 \\
0.4, 0.5 \\
0.1, 0.1 \\
1, 0
\end{array}\right) \]

And, by Definition 6

\[ [A]_{(0.2, 0.4)} = \left(\begin{array}{c}
0.2, 0.4 \\
0.8, 0.2 \\
0.6, 0.3 \\
0.4, 0.5 \\
0.1, 0.1 \\
1, 0
\end{array}\right) \]

Similarly

\[ [A]_{(0.6, 0.2)} = \left(\begin{array}{c}
0.1 \\
0.8, 0.2 \\
0.6, 0.3 \\
0.4, 0.5 \\
0.1, 0.1 \\
0.8, 0.2
\end{array}\right) \]

\[ [A]_{(0.6, 0.3)} = \left(\begin{array}{c}
0.1 \\
0.6, 0.3 \\
0.6, 0.3 \\
0.4, 0.5 \\
0.1, 0.1 \\
0.6, 0.3
\end{array}\right) \]
Theorem 8. Let $X$ the non-empty set, $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. For any intuitionistic fuzzy subset $A$ in $X$, $[A]_{(\alpha, \beta)}$ does not form a complete system of events of $A$.

Proof of Theorem 8 are exactly similar to those of Theorem 7. Only difference is instead of inequality here we will have $> \ inequality$.

5. CONCLUSIONS

Starting from the definition of a complete system of events in probability theory, I tried to show that the set of parts of $(\alpha, \beta)$-cuts and strong $(\alpha, \beta)$-cuts does not form a complete system of events.

REFERENCES