A CONSTRUCTION OF A SYSTEM OF MEASURES SUPPORTED ON (r,d)-FIBRES OF A GROUPOID

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Abstract. The purpose of this paper is to transfer the construction of the Haar measure of a locally compact group to the groupoid settings. The groupoid is endowed with a family of subsets containing the unit space.

Keywords: groupoid; uniform structure; (r,d)-fibre; system of measures.

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1. INTRODUCTION

We use the same definition, notation and terminology concerning groupoids as in [2] and [7]. For developing an algebraic theory of functions on a locally compact groupoid (more precisely, for defining the convolution product), one needs an analogue of Haar measure on locally compact groups. Several generalizations of the Haar measure to the setting of groupoids were taken into considerations in the literature (see for instance [6] or [7]). This analogue is a system of measures, called Haar system, subject to suitable invariance and smoothness conditions. However unlike the case of locally compact group, Haar system on groupoid need not exists, and if it does, it will not usually be unique. Both versions of Haar systems on a groupoid G used in [6] and [7] admit a further decompositions consisting in Haar systems on principal groupoid associated to G and an additional system of measures $\{\beta_v^u, u\sim v\}$ satisfying the following property

$$\beta_v^{r(x)}(f) = \beta_v^{d(x)}(f_x)$$

for all Borel function f≥0 on the (r,d)-fibre $G_v^{r(x)}$ (see [6], [8]).

In this paper we transfer the construction of the Haar measure of a locally compact group to the groupoid settings in order to obtain a system of measures $\{\beta_v^u, u\sim v\}$ with the above property. The only information that we use for characterize the groupoid G is a family $\mathcal{W}(G^{(0)})$ of subsets of G containing the unit space $G^{(0)}$. This collection of subsets of G mimics the properties of a neighborhood basis of the unit space (of a topological groupoid with paracompact unit space) (see [5]).

Though a system of measures $\{\beta_v^u, u\sim v\}$ can be easily obtain from a Haar system on the group bundle of the groupoid, we prefer to reconstruct $\{\beta_v^u, u\sim v\}$ in terms of "neighborhoods of the unit space" order to establish connections with $\mathcal{W}(G^{(0)})$.

2. GROUPOID FRAMEWORK

Let us start with a groupoid G and a collection $\mathcal{W}(G^{(0)})$ of subsets of G satisfying the following conditions:

1. $G^{(0)}\subset W\subset G$ for all $W\in \mathcal{W}(G^{(0)})$.

2. If $W_1, W_2\in \mathcal{W}(G^{(0)})$, then there is $W_3\subset W_1\cap W_2$ such that $W_3\in\mathcal{W}(G^{(0)})$.

3. $W=W^{-1}$ for all $W\in \mathcal{W}(G^{(0)})$.

4. For every $W_1\in \mathcal{W}(G^{(0)})$ there is $W_2\in \mathcal{W}(G^{(0)})$ such that $W_2W_2\subset W_1$.

5. For every $W_1\in \mathcal{W}(G^{(0)})$ and $x\in G$ there is $W_2\in \mathcal{W}(G^{(0)})$ such $xW_2x^{-1}\subset W_1$.

6, For every $x\notin G^{(0)}$ there is $W\in \mathcal{W}(G^{(0)})$ such that $x\notin W$.

7. For every $W\in\mathcal{W}(G^{(0)})$ and $(u,v)\in R$ (the principal groupoid associated to G), $W\cap G_v^u$ is pre-compact (i.e. $W\cap G_v^u$ is contained in a compact set) with respect to the topology induced by $\tau^r(\mathcal{W}(G^{(0)}))$ on G_v^u, where a neighborhood basis for x with respect to $\tau^r(\mathcal{W}(G^{(0)}))$ is given by

$$\{V\subset G: \text{there is } W\in\mathcal{W}(G^{(0)}) \text{ such that } xW\subset V\}.$$

(see [5])

In [1] and [3-5] we introduce various uniform structures on a groupoid. The notion that we propose below is a applied to (r,d)-fibres of the groupoid.

Definition 1. A function $f: G\to\mathbf{C}$ is said to be right uniformly continuous on (r,d)-fibres if and only if for each $\varepsilon>0$ there is $W_\varepsilon\in \mathcal{W}(G^{(0)})$ such that:

$$|f(x)-f(y)| < \varepsilon \text{ for all } x,y \text{ satisfying } (r,d)(x)= (r,d)(y) \text{ and } x^{-1}y\in W_\varepsilon.$$

If $f: G\to\mathbf{C}$ is uniformly continuous on (r,d)-fibres, then f is continuous with respect to the topology induced by $\tau^r(\mathcal{W}(G^{(0)}))$ on each (r,d)-fibre G_v^u. If $f,g:G\to\mathbf{C}$ are right uniformly continuous on (r,d)-fibres, then |f|, f, f+g are right uniformly continuous on (r,d)-fibres. If $f,g:G\to C$ are right uniformly continuous on (r,d)-fibres bounded functions, then fg is a right uniformly continuous on (r,d)-fibres bounded function.

Definition 2. Let G be a groupoid endowed with a family of subsets $\mathcal{W}(G^{(0)})$. Let us denote by $\mathcal{K}(G, \mathcal{W}(G^{(0)}))$ the family of subsets K of G with the property that there is $W\in\mathcal{W}(G^{(0)})$ such that for all $(u,v)\in R$ (the principal groupoid associated to G), the set

$$WKW\cap G_v^u \text{ is pre-compact}$$

(with respect to the topology induced by $\tau^r(\mathcal{W}(G^{(0)}))$ on G_v^u).

Let us denote by $\mathcal{UF}_c(\mathcal{W}(G^{(0)}),G)$ the space of right uniformly continuous on (r,d)-fibres functions $f:G\to\mathbf{C}$ which vanish outside a set $K\in\mathcal{K}(G, \mathcal{W}(G^{(0)}))$.

3. CONSTRUCTION OF SYSTEM OF MEASURES

In this section we shall denote by G a groupoid and by $\mathcal{W}(G^{(0)})$ a family of subsets of G as in Section 2.

Lemma 3. Let G be a groupoid endowed with a family of subsets $\mathcal{W}(G^{(0)})$. Let $f,g\in \mathcal{UF}_c(\mathcal{W}(G^{(0)}),G)$ be two nonnegative functions such that $g\neq 0$ and let $x_g\in G$ satisfying $g(x_g)\neq 0$. If $v=d(x_g)$, then for every $u\in [v]$ there are $n\in\mathbf{N}$, $y_1,\dots,y_n\in G$ and $c_1,\dots,c_n\geq 0$ such that

$$f(x)\leq \sum_{i=1}^{n} c_i g\left(x_g y_i^{-1} x\right) \text{ for all } x\in G_v^u .$$

Proof. Since g is continuous on $G_v^{r(x_g)}$ and $g(x_g)\neq 0$, there is $W\in\mathcal{W}(G^{(0)})$ such that $g(y)>g(x_g)/2$ for all $y\in(x_g W)\cap G_v^{r(x_g)}$. Let $K_f\in \mathcal{K}(G, \mathcal{W}(G^{(0)}))$ be a set such that f vanishes outside K_f. Since $K_f\cap G_v^u$ is pre-compact, there is a finite family

$$\{y_1,y_2,\dots,y_n\}\subset G_v^u$$

such that $\{y_i W\}_{i=1,n}$ covers $K_f\cap G_v^u$. Thus for every $x\in K_f\cap G_v^u$ there is i(x) such that $x\in y_{i(x)}W$ and consequently,

$$f(x)\leq \frac{2M}{g\left(x_g\right)}\ g\left(x_g y_{i(x)}^{-1} x\right) \leq \sum_{i=1}^{n}\frac{2M}{g\left(x_g\right)} g\left(x_g y_i^{-1} x\right),$$

where $M=\sup\{f(x),\ x\in G_v^u\}$.

Definition 4. Let G be a groupoid endowed with a family of subsets $\mathcal{W}(G^{(0)})$. Let $f,g\in\mathcal{UF}_c(\mathcal{W}(G^{(0)}),G)$ be two nonnegative functions and let let u,v,s be three equivalent units of G such that $g\neq 0$ on G_v^s. Let us denote by

$C(f,g)(u,v,s)=\{ \sum_{i=1}^{n(u,v)} c_i\left(u,v,s\right)$, where $c_1(u,v,s),\dots, c_{n(u,v,s)}(u,v,s)\geq 0$ and there are

$y_1(u,v,s),\dots, y_{n(u,v,s)}(u,v,s)\in G_s^u$ with the property

$$f(x)\leq \sum_{i=1}^{n(u,v,s)} c_i\left(u,v,s\right) g\left(y_i\left(u,v,s\right)^{-1} x\right) \text{ for all } x\in G_v^u \ \}$$

Let us denote (f,g)(u,v,s)=inf C(f,g)(u,v,s).

Proposition 5. Let G be a groupoid endowed with a family of subsets $\mathcal{W}(G^{(0)})$. Let $f,g,h\in\mathcal{UF}_c(\mathcal{W}(G^{(0)}),G)$ be three nonnegative functions, and let u,v,s be three equivalent units of G such that $h\neq 0$ on G_v^s and $g\neq 0$ on G_v^t. Then

$$(f,h)(u,v,s)\leq (f,g)(u,v,t)\ (g,h)(t,v,s).$$

Proof. Let $\sum_{i=1}^{n(u,v,t)} c_i\left(u,v,t\right)\in$ C(f,g)(u,v,t). Then there are $y_1(u,v,t),\dots, y_{n(u,v,s)}(u,v,t)\in G_t^u$ with the property

$$f(x)\leq \sum_{i=1}^{n(u,v,t)} c_i\left(u,v,t\right) g\left(y_i\left(u,v,t\right)^{-1} x\right) \text{ for all } x\in G_v^u .$$

If $\sum_{j=1}^{n(u,v,t)} d_j(t,v,s) \in$ C(g,h)(t,v,s), then there are $z_1(u,v,t),\ldots, z_{n(u,v,s)}(u,v,t) \in G_s^t$ with the property

$$g(x) \le \sum_{j=1}^{n(t,v,s)} d_j(t,v,s) h\left(z_j(t,v,s)^{-1} x\right) \text{ for all } x \in G_v^t .$$

Thus

$$f(x) \le \sum_{i=1}^{n(u,v,t)} c_i(u,v,t) g\left(y_i(u,v,t)^{-1} x\right)$$

$$\le \sum_{i=1}^{n(u,v,t)} c_i(u,v,t) \sum_{j=1}^{n(t,v,s)} d_j(t,v,s) h\left(z_j(t,v,s)^{-1} y_i(u,v,t)^{-1} x\right) \text{ for all } x \in G_v^u$$

Therefore $\sum_{i=1}^{n(u,v,t)} c_i(u,v,t) \sum_{j=1}^{n(t,v,s)} d_j(t,v,s) \in$ C(f,h)(u,v,s).

Definition 6. For every $W \in \mathcal{W}(G^{(0)})$ let us fix a function $a_W \in \mathcal{UF}_c(\mathcal{W}(G^{(0)}),G)$ with the property that $a_W(x)=1$ for all $x \in W$ and $a_W(x)=0$ for all $x \notin W^3$. For every $f \in \mathcal{UF}_c(\mathcal{W}(G^{(0)}),G)$, let us denote

$$(W,f)(u,v,s)=(a_W,f)(u,v,s)$$
$$(f,W)(u,v,s)=(f,a_W)(u,v,s)$$

Let us fix $W_0 \in \mathcal{W}(G^{(0)})$ and let us define:

$$I(f,W)(u,v)=\frac{(f,W)(u,v,v)}{(W_0,W)(v,v,v)}$$

Proposition 7. For every $W \in \mathcal{W}(G^{(0)})$, $f \in \mathcal{UF}_c(\mathcal{W}(G^{(0)}),G)$ and $(u,v) \in R$ such that $f \neq 0$ on G_v^u, we have

$$0<\frac{1}{(W_0,f)(v,v,u)} \le I(f,W)(u,v) \le (f,W_0)(u,v,v)<\infty$$

Proof. Let $M=\sup\{f(x), x \in G_v^u\}$. According Proposition 5 we have

$$(f,W)(u,v,v) \le (f,W_0)(u,v,v)\,(W_0,W)(v,v,v).$$

Thus

$$I(f,W)(u,v)=\frac{(f,W)(u,v,v)}{(W_0,W)(v,v,v)} \le (f,W_0)(u,v,v).$$

Let $K_f \in K(G, \mathcal{W}(G^{(0)}))$ be a set such that f vanishes outside K_f. Since $K_f \cap G_v^u$ is pre-compact, there is a finite family

$$\{y_1(u,v),y_2(u,v),\ldots,y_{n(u,v)}(u,v)\} \subset G_v^u$$

such that $\{y_i(u,v)W_0\}_{i=1,n(u,v)}$ covers $K_f \cap G_v^u$. Thus for every $x \in K_f \cap G_v^u$ there is i(x) such that $x \in y_{i(x)}(u,v)W_0$ and consequently,

$$f(x)\leq M\, a_{W_0}\left(y_{i(x)}(u,v)^{-1}x\right) \leq \sum_{i=1}^{n(u,v)} M a_{W_0}\left(y_i(u,v)^{-1}x\right),$$

Hence

$$I(f,W)(u,v)\leq(f,W_0)(u,v,v)\leq n(u,v)M<\infty.$$

On the other hand, applying again Proposition 5, we obtain

$$(W_0,W)(v,v,v)\leq(W_0,f)(v,v,u)(f,W)(u,v,v)$$

Hence

$$I(f,W)(u,v)= \frac{(f,W)(u,v,v)}{(W_0,W)(v,v,v)} \geq \frac{1}{(W_0,f)(v,v,u)}$$

Let $\sum_{i=1}^{n(v,v,u)} c_i(v,v,u) \in C(W_0,f)(v,v,u)$. Then there are $y_1,\ldots, y_n\in G_v^u$ with the property

$$a_{W_0}(x)\leq \sum_{i=1}^{n} c_i(v,v,u) f\left(y_i^{-1}x\right)\leq M\sum_{i=1}^{n} c_i(v,v,u) \text{ for all } x \in G_v^v .$$

Thus for x=v we obtain $\sum_{i=1}^{n(v,v,u)} c_i(v,v,u)\geq 1/M$. Therefore $\frac{1}{(W_0,f)(v,v,u)}>0$.

Let us remark that

1) $I(f_1+f_2,W)(u,v)\leq I(f_1,W)(u,v)+ I(f_1,W)(u,v)$
2) $I(\alpha f_1,W)(u,v)=\alpha I(f_1,W)(u,v),\ \alpha\neq 0$
3) $f_1=f_2$ on G_v^u => $I(f_1,W)(u,v)\leq I(f_2,W)(u,v)$
4) $I(f_x,W)(d(x),v)\leq I(f,W)(r(x),v)$, where $f_x(y)=xy$.

Proposition 8. Let $\varepsilon_0>0$, f_1, $f_2\in \mathcal{UF}_c(\mathcal{W}(G^{(0)}),G)$ two nonnegative functions. We can choose W small enough such that

$$I(f_1,W)(u,v)+ I(f_1,W)(u,v)\leq I(f_1+f_2,W)(u,v)+\varepsilon_0.$$

Construction of the measures β_v^u:

Since the net $(I(f,W)(u,v))_W$ is bounded, it has a convergent subnet, also denoted $(I(f,W)(u,v))_W$. If we define

$$\beta_v^u(f)=\lim_W I(f,W)(u,v),$$

then β_v^u can be extended to a positive Radon measure supported on G_v^u. The system of positive Randon measures $\{\beta_v^u, u\sim v\}$ satisfies the following property

$$\beta_v^{r(x)}(f)=\beta_v^{d(x)}(f_x)$$

for all compactly supported continuous function f on $G_v^{r(x)}$, and consequently all Borel f≥0.

The construction allows us to establish additional properties of $\{\beta_v^u, u\sim v\}$. For instance for each $W\in \mathcal{W}(G^{(0)})$ and $K\in\mathcal{K}(G, \mathcal{W}(G^{(0)}))$, let us denote

$(K:W)(u,v)=\min\{n$: there is $y_1,y_2,\ldots,y_n\in G_v^u$ such that $\{y_iW\}_{i=1,n}$ covers $K\cap G_v^u\}$

$(K:W)=\sup\{(K:W)(u,v): (u,v)\in R\}$.

Then

1. $\beta_v^u(f)\leq (f,W_0)(u,v,v)\leq\sup\{f(x),x\in\in G_v^u\}(K_f:W_0)(u,v)$ (f vanishes outside K_f)
2. $\beta_v^u(f)\geq \dfrac{1}{(W_0,f)(v,v,u)} >1/(((f(x_0)-\varepsilon)(W_0:W))$, where $f(x_0)>0$ and $x_0W\subset\{y: f(y)> f(x_0)-\varepsilon\}$
3. $\beta_v^u(a_{W_0})=1$ for all $(u,v)\in R$ such that $a_{W_0}\neq 0$ on G_v^u.
4. $\beta_u^u(a_W)\leq 1$ for all $u\in G^{(0)}$ and all $W\in \mathcal{W}(G^{(0)})$ such that $W^3\subset W_0$.

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