A GROUPOID FRAMEWORK FOR STUDY ASYMPTOTIC BEHAVIOR OF A DISCRETE SYSTEM

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Abstract. *The purpose of this note is to introduce four different groupoids associated to a function* $f:X\times\mathbf{N}\times\mathbf{N}\to X$*, where X is a uniform space. Thesegroupoids allow to study within a unified framework theasymptotic behavior of thediscrete systems of the form* $x_{n+1} = f_{n,n_0}(x_n, x_{n-1}, \ldots, x_{n_0})$ *that do not necessarily satisfy the semigroup property of a process. For these systems the groupoidsare associated with the functionsf defined by* $f(x,n,n_0) := f_{n,n_0}(x_{n-1}, x_{n-2}, \ldots, x_{n_0})$ *with* $x = x_{n_0}$.

Keywords:groupoid; equilibrium point; asymptotically stable equilibrium point.

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1. INTRODUCTION

In this article we associate several groupoidsto abstract discrete systems that generalize nonautonomousdiscrete-time processes. Let us recall that the mathematical formalization for a nonautonomousdiscrete-time process includesa space X and a sequence $(f_n)_n$ of maps $f_n:X\to X$. Then the nonautonomous difference equation

$$x_{n+1} = f_n(x_n)$$

generates a discrete-time process which is defined for all $x\in X$ and $n, n_0\in\mathbb{N}$ with $n \geq n_0$ by:

$\sigma(n_0, n_0, x) := x$,

$\sigma(n, n_0, x) := f_{n-1}\circ f_{n-2}\circ\ldots\circ f_{n_0}(x)$.

The function $\sigma: \mathbb{N}\times\mathbb{N}\times X\to X$ defined above has the following properties:

1.1) $\sigma(n_0, n_0, x) = x$ for all $n_0\in\mathbb{N}$ and $x\in X$.

1.2) $\sigma(n_2, n_0, x) = \sigma(n_2, n_1, \sigma(n_1, n_0, x))$ for all $n_0\leq n_1\leq n_2$ and $x\in X$

Howeverthere are processes (such as fractional order systems [3]) that do not satisfy semigroup condition 1.2.In this article we consider processes that do not necessarily semigroup condition 1.2. More precisely, we take into consideration the processes generated by the difference equations of the form

$$x_{n+1} = f_{n,n_0}(x_n, x_{n-1}, \ldots, x_{n_0}),\ n\geq n_0.$$

We introduce different groupoids associated to an uniform space X and a function

$$f: \mathbb{N}\times\mathbb{N}\times X\to X$$

having the meaning that for all $(n,n_0)\in\mathbb{N}\times\mathbb{N}$ such that $n\geq n_0$

$$f(n,n_0,x) = f_{n-1}(x_{n-1}, x_{n-2}, \ldots, x_{n_0}) \text{ with } x_{n_0} = x.$$

The use of an abstract notion of system allows to study within a unified framework diverse processesand provides a unique language for several different areas of applications.

2. EQUILIBRIUM AND STABILITY

Definition 2.1. Let $f: \mathbb{N}\times\mathbb{N}\times X\to X$ be a function. For each $x\in X$ we write

$$N_e(x)=\{n_0\in\mathbb{N} : f(n,n_0,x)=x \text{ for all } n\in\mathbb{N},\ n\geq n_0\}.$$

An element $x\in X$ is said to be an equilibrium point of the system defined by $f: \mathbb{N}\times\mathbb{N}\times X\to X$ if $N_e(x)\neq\emptyset$, i.e. if there is $n_0\in\mathbb{N}$ such that $f(n,n_0,x)=x$ for all $n\in\mathbb{N}$, $n\geq n_0$.

Remark 2.2. Let $f: \mathbb{N}\times\mathbb{N}\times X\to X$ be a process that satisfies semigroup condition, i.e

$$f(n_2, n_0, x) = f(n_2, n_1, f(n_1, n_0, x)) \text{ for all } n_0\leq n_1\leq n_2 \text{ and } x\in X.$$

If x_e be an equilibrium point of f, then there is $n_0\in\mathbb{N}$ such that

$$N_e(x)=\{n\in\mathbb{N} : n\geq n_0\}.$$

Indeed, let $n_0=\min N_e(x_e)$ and let $m\in\mathbb{N}$ such that $m\geq n_0$. Then for all $n\geq m$ we have

$$f(n,n_0,x_e) = f(n, m, f(m, n_0, x_e)) \text{ (semigroup condition).}$$

Since $n_0\in N_e(x_e)$, it follows that $x_e=f(n, m, x_e)$ for all $n\geq m$, or equivalently, $m\in N_e(x_e)$.

Let us recall that a uniform space is a set X equipped with a nonempty family $\mathfrak{A}$ of subsets of X × X (called uniform structure on X) satisfying the following conditions:

1. $\Delta\subset U$ for all $U\in\mathfrak{A}$, where $\Delta = \{ (x, x) : x \in X \}$.
2. If $U\in\mathfrak{A}$ and $U\subset V\subset X\times X$, then $V\in\mathfrak{A}$.
3. If $U\in\mathfrak{A}$ and $V\in\mathfrak{A}$, then $U\cap V\in\mathfrak{A}$.
4. For each $U\in\mathfrak{A}$ there exists $V\in\mathfrak{A}$ such that $VV\subset U$, where
$$VV=\{(x,z): \text{there is } y \text{ such that } (x,y)\in V \text{ and } (y,z)\in V\}$$
5. If $U\in\mathfrak{A}$, then $U^{-1} = \{ (y, x) : (x, y) \in U \} \in\mathfrak{A}$.

If $\mathfrak{A}$ is a uniform structure on X, then $\mathfrak{A}$ induces a topology on X: $A\subset X$ is openset if and only if for every $x\in A$ there exists $U\in\mathfrak{A}$ such that $\{y: (x,y)\in U\}\subset A$.

Definition 2.3. Let X be a space endowed with a uniform structure $\mathfrak{A}$. Let us consider a system defined by a function $f: \mathbb{N}\times\mathbb{N}\times X\to X$ and let x_e be an equilibrium point of the system. We write

$N_s(x_e)=\{n_0\in N_e(x_e)$: for every $U\in\mathfrak{A}$ there is $V_U\in\mathfrak{A}$ with the property that if $(x_e,x)\in V_U$, then $(x_e, f(n,n_0,x))\in U$ for all $n\in\mathbb{N}$, $n\geq n_0\}$

The equilibrium point x_e is said to be stable if $N_e(x)\neq\emptyset$.

Definition 2.4. Let us consider a system defined by a function $f: \mathbb{N}\times\mathbb{N}\times X\to X$ and let us assume that X is endowed with a uniform structure $\mathfrak{A}$.

- An equilibrium point x_e is said to be attractive if there is $n_0\in N_e(x_e)$ and there is $U\in\mathfrak{A}$ such that for all $(x_e,x)\in U$ we have $\lim_{n\to\infty} f(n,n_0,x) = x_e$.
- An equilibrium point x_e is asymptotically stable if it is stable and attracting.

3. GROUPOIDS ASSOCIATED TO A GENERAL DISCRETE SYSTEM

In this section we use the same notation concerning groupoids as in [1] and [2].

Proposition 3.1. Let $\mathfrak{A}$ be a uniform structure on a set X and let f: $\mathbb{N}\times\mathbb{N}\times X\to X$ be a function. If
$G_1(X,\mathfrak{A},f)$ = {(x,n_1 k, y, n_2) $\in X\times\mathbb{N}\times\mathbb{Z}\times X\times\mathbb{N}$:
for all U$\in\mathfrak{A}$ there is $n_U\in\mathbb{N}$ such that for all $n\geq n_U$ we have
$n\geq n_2$, $n+k\geq n_1$ and $(f(n+k,n_1,x),f(n,n_2,y))\in U$},
Then

1. $G_1(X,\mathfrak{A},f)$ is a subgroupoid of $X\times\mathbb{N}\times\mathbb{Z}\times X\times\mathbb{N}$ seen as a groupoid under the operations

$(x,n_1\, k, y, n_2)(y, n_2, m, z, n_3) = (x, n_1,k+m, z, n_3)$ (product)

$(x, n_1, k, y\, n_2)^{-1} = (y, n_2, -k, x, n_1)$ (inversion).

2. If $G_1(X,\mathfrak{A},f)$ is endowed with the induced topology from $X\times\mathbb{N}\times\mathbb{Z}\times X\times\mathbb{N}$, then $G_1(X,\mathfrak{A},f)$ is a topological groupoid.

Proof. 1. Let $(x, n_1, k, y, n_2)\in G_1(X,\mathfrak{A},f)$ and let us prove that $(y, n_2, -k, x, n_1)\in G_1(X,\mathfrak{A},f)$. Let $U\in\mathfrak{A}$. Then $U^{-1}\in\mathfrak{A}$, hence there is $n_U\in\mathbb{N}$ such that for all $n\geq n_U$ we have

$$n\geq n_2,\ n+k\geq n_1 \text{ and } (f(n+k,n_1,x),f(n,n_2,y))\in U^{1}.$$

Thus for all $n\geq n_U+k$ we have
$n-k\geq n_2$, $n\geq n_1$ and $(f(n,n_1,x),f(n-k,n_2,y))\in U^{1}$ (or equivalently, $(f(n-k,n_2,y),f(n,n_1,x))\in U$).
Consequently, $(y,n_2, -k, x,n_1)\in G_1(X,\mathfrak{A},f)$.

Let (x, n_1, k, y,n_2), $(y,n_2, m, z,n_3)\in G(X,\mathfrak{A},f)$ and let us prove that $(x, n_1, k+m, z,n_3)\in G_1(X,\mathfrak{A},f)$. Let $U\in\mathfrak{A}$. Then $V\in\mathfrak{A}$ such that $VV\subset U$. Since $(x, n_1, k, y,n_2)\in G_1(X,\mathfrak{A},f)$ there is $n_V\in\mathbb{N}$ such that for all $n\geq n_V$, $n\geq n_2$, $n+k\geq n_1$ and $(f(n+k,n_1,x),f(n,n_2,y))\in V$}. Since $(y,n_2, m, z,n_3)\in G_1(X,\mathfrak{A},f)$, there is $n'_V\in\mathbb{N}$ such that for all $n\geq n'_V$, $n\geq n_3$, $n+m\geq n_2$ and $(f(n+m,n_2,y),f(n,n_3,z))\in V$. Thus for all $n\geq\max\{n_V-m, n'_V\}$ we have

$n+m\geq n_2$, $n+m+k\geq n_1$ and $(f(n+m+k,n_1,x),f(n+m,n_2,y))\in V$.

On the other hand $n\geq n_3$, $n+m\geq n_2$ and $(f(n+m,n_2,y),f(n,n_3,z))\in V$. Therefore

$$(f(n+m+k,n_1,x), f(n,n_3,z))\in VV\subset U \text{ for all } n\geq\max\{n_V-m, n'_V\}.$$

Hence $(x, n_1, k+m, z, n_3)\in G_1(X,\mathfrak{A},f)$.

2. Since $X\times\mathbb{N}\times\mathbb{Z}\times X\times\mathbb{N}$ is a topological groupoid, and $G(X,\mathfrak{A},f)$ a subgroupoid, it follows that $G_1(X,\mathfrak{A},f)$ is a topological groupoid,

Proposition 3.2. Let $\mathfrak{A}$ be a uniform structure on a set X and let f: $\mathbb{N}\times\mathbb{N}\times X\to X$ be a function. If

$G_2(X,\mathfrak{A},f)$ = {(x, k, y, n_0) $\in X\times\mathbb{Z}\times X\times\mathbb{N}$:
for all U$\in\mathfrak{A}$ there is $n_U\in\mathbb{N}$ such that for all $n\geq n_U$ we have
$n\geq n_0$, $n+k\geq n_0$ and $(f(n+k,n_0,x),f(n,n_0,y))\in U$},

Then

1. $G_2(X,\mathfrak{A},f)$ is a subgroupoid of $X\times\mathbb{Z}\times X\times\mathbb{N}$ seen as a groupoid under the operations

$$(x, k, y, n_0)(y, m, z, n_0) = (x, k+m, z, n_0) \text{ (product)}$$
$$(x, k, y\ n_0)^{-1} = (y, -k, x, n_0) \text{ (inversion).}$$

2. If $G_2(X,\mathfrak{A},f)$ is endowed with the induced topology from $X\times\mathbb{Z}\times X\times\mathbb{N}$, then $G_2(X,\mathfrak{A},f)$ is a topological groupoid.

Proof. The proof is similar to that of Proposition 3.1.

Remark 3.3. The groupoid $G_2(X,\mathfrak{A},f)$ introduced in Proposition 3.2 is a disjoint union of groupoids defined in[2 ,Proposition 2.1].

Also $G_2(X,\mathfrak{A},f)$ can be seen as a subgroupoid $G_1(X,\mathfrak{A},f)$ introduced in Proposition 3.1 by the identification (x,k, y, n_0) $\mapsto$ (x,n_0, k, y, n_0).

Proposition 3.4. Let $\mathfrak{A}$ be a uniform structure on a set X and let f: $\mathbb{N}\times\mathbb{N}\times X\to X$ be a function. If

$G_3(X,\mathfrak{A},f)$ = {(x,n_1 k, y, n_2) $\in X\times\mathbb{N}\times\mathbb{Z}\times X\times\mathbb{N}$:
for all U$\in\mathfrak{A}$ there is $n_U\in\mathbb{N}$ such that for all $n\geq n_U$ we have $n\geq n_2$, $n+k\geq n_1$ and
$(f(n+k,m_1,x),f(n,m_2,y))\in U$ for all m_1, $m_2\in\mathbb{N}$ such that $n+k\geq m_1\geq n_1$ and $n\geq m_2\geq n_2$}

Then $G_3(X,\mathfrak{A},f)$ is a subgroupoid of $G_1(X,\mathfrak{A},f)$ introduced in Proposition 3.1.

Proof. The proof is similar to that of Proposition 3.1.

Proposition 3.5. Let $\mathfrak{A}$ be a uniform structure on a set X and let f: $\mathbb{N}\times\mathbb{N}\times X\to X$ be a function. If

$G_4(X,\mathfrak{A},f)$ = {(x, k, y) $\in X\times\mathbb{N}\times\mathbb{Z}\times X\times\mathbb{N}$:
there is $n_0\in\mathbb{N}$ such that for all U$\in\mathfrak{A}$ there is $n_U\in\mathbb{N}$ such that for all $n\geq n_U$ we have $n\geq n_0$, $n+k\geq n_0$ and
$(f(n+k,m_1,x),f(n,m_2,y))\in U$ for all m_1, $m_2\in\mathbb{N}$ such that $n+k\geq m_1\geq n_0$ and $n\geq m_2\geq n_0$}

Then

1. $G_4(X,\mathfrak{A},f)$ is a subgroupoid of $X\times\mathbb{Z}\times X$ seen as a groupoid under the operations

$$(x, k, y)(y, m, z) = (x, k+m, z) \text{ (product)}$$
$$(x, k, y)^{-1} = (y, -k, x) \text{ (inversion).}$$

2. If $G_4(X,\mathfrak{A},f)$ is endowed with the induced topology from $X \times \mathbb{Z} \times X$, then $G_4(X,\mathfrak{A},f)$ is a topological groupoid.

Proof.1. Let (x, k, y)$\in G_4(X,\mathfrak{A},f)$ and let us show that (y,-k, x)$\in G_4(X,\mathfrak{A},f)$. Let $U\in\mathfrak{A}$. Since $U^{-1}\in\mathfrak{A}$, it follows that there are $n_0,n_U \in \mathbb{N}$ such that for all $n\geq n_U$ we have $n\geq n_0$, $n+k\geq n_0$ and

$(f(n+k,m_1,x),f(n,m_2,y)) \in U^1$ for all $m_1, m_2 \in \mathbb{N}$ such that $n+k\geq m_1\geq n_0$ and $n\geq m_2\geq n_0$. Thus for all $n \geq n_U+k$ we have $n-k\geq n_0$, $n\geq n_0$ and $(f(n,m_1,x),f(n-k,m_2,y)) \in U^1$ (or equivalently, $(f(n-k,m_2,y),f(n,m_1,x)) \in U$) for all $m_1, m_2 \in \mathbb{N}$ such that $n+k\geq m_1\geq n_0$ and $n\geq m_2\geq n_0$. Consequently, (y,-k, x)$\in G_4(X,\mathfrak{A},f)$.

Let (x, k, y), (y,m, z)$\in G_4(X,\mathfrak{A},f)$ and let us prove that (x, n_1, k+m, z,n_3)$\in G_4(X,\mathfrak{A},f)$. Let $U\in\mathfrak{A}$. Then $V\in\mathfrak{A}$ such that $VV\subset U$. Since (x, k, y)$\in G_4(X,\mathfrak{A},f)$ there are $n_0,n_V \in \mathbb{N}$ such that for all $n\geq n_V$ we have $n\geq n_0$, $n+k\geq n_0$ and $(f(n+k,m_1,x),f(n,m_2,y)) \in V$ for all $m_1, m_2 \in \mathbb{N}$ such that $n+k\geq m_1\geq n_0$ and $n\geq m_2\geq n_0$. Since (y, m, z)$\in G_4(X,\mathfrak{A},f)$, there are n_1, $n'_V \in \mathbb{N}$ such that for all $n\geq n'_V$ we have $n\geq n_1$, $n+m\geq n_1$ and $(f(n+m,m_3,y),f(n,m_4,z)) \in V$ for all $m_3, m_4 \in \mathbb{N}$ such that $n+m\geq m_3\geq n_1$ and $n\geq m_4\geq n_1$. Thus for all $n\geq\max\{n_V-m, n'_V\}$ we have $n+m\geq n_0$, $n+m+k\geq n_0$ and $(f(n+m+k,m_1,x),f(n+m,m_2,y)) \in V$ for all $m_1, m_2 \in \mathbb{N}$ such that $n+m+k\geq m_1\geq n_0$ and $n+m\geq m_2\geq n_0$.On the other hand $n\geq n_1$, $n+m\geq n_1$ and

$(f(n+m,m_3,y),f(n,m_4,z)) \in V$ for all $m_3, m_4 \in \mathbb{N}$ such that $n+m\geq m_3\geq n_1$ and $n\geq m_4\geq n_1$. Therefore

$$(f(n+m+k,m_1,x), f(n,m_4,z)) \in VV\subset U$$

for all $n\geq\max\{n_V-m, n'_V\}$ and all $m_1, m_4 \in \mathbb{N}$ such that $n+m+k\geq m_1\geq \max\{n_0, n_1\}$ and $n\geq m_4\geq \max\{n_0, n_1\}$. Consequently,(x,k+m, z)$\in G_4(X,\mathfrak{A},f)$.

2 . Since $G(X,\mathfrak{A},f)$ is subgroupoid of the topological groupoid $X \times \mathbb{Z} \times X$ is,it follows that $G_4(X,\mathfrak{A},f)$ is a topological groupoid,

Remark 3.6. Let x be an equilibrium point of the system defined by f: $\mathbb{N} \times \mathbb{N} \times X\to X$.

1. If G is the groupoid $G_1(X,\mathfrak{A},f)$ or $G_2(X,\mathfrak{A},f)$, then $G^{(x,n_0)}_{(x,n_0)} =\{(x,n_0)\} \times \mathbb{Z} \times \{(x,n_0)\}\approx\mathbb{Z}$ for all $n_0\in N_e(x)$.
2. If G is the groupoid $G_3(X,\mathfrak{A},f)$ and f has semigroup property, then $G^{(x,n_0)}_{(x,n_0)} =\{(x,n_0)\} \times \mathbb{Z} \times \{(x,n_0)\}\approx\mathbb{Z}$ for all $n_0\in N_e(x)$.
3. If G is the groupoid $G_4(X,\mathfrak{A},f)$ and f has semigroup property, then $G^x_x =\{x\} \times \mathbb{Z} \times \{x\}\approx\mathbb{Z}$.

Lemma 3.7. Let $\mathfrak{A}$ be a uniform structure on X, let f: $\mathbb{N} \times \mathbb{N} \times X\to X$ be a function and let x_e be an equilibrium point of the system associated with f and let $n_0\in N_e(x_e)$.

1. If G is the groupoid $G_1(X,\mathfrak{A},f)$ or $G_2(X,\mathfrak{A},f)$, then (x, n_1) and (x_e, n_0) are equivalent units of G if and only if $\lim_{n\to\infty} f(n,n_1,x)$ = x_e.
2. If (x, n_1) and (x_e, n_0) are equivalent units of $G_3(X,\mathfrak{A},f)$, then $\lim_{n\to\infty} f(n,n_1,x)$ = x_e.

Proof. 1. Let G= $G_1(X,\mathfrak{A},f)$.Then (x, n_1) and (x_e, n_0) are equivalent units of G if and only if there is $k\in\mathbb{Z}$ such that (x_e, n_0, k,x,n_1) $\in$G if and only if for every U$\in\mathfrak{A}$ there is $n_U\in\mathbb{N}$ such that for all $n\geq n_U$, we have $n\geq n_1$, $n+k\geq n_0$ and $(f(n+k,n_0,x_e),f(n,n_1,x))\in$ U. Since x_e is an equilibrium point, f(n+k, n_0, x_e)=x_e for all n and k such that $n+k \geq n_0$. It follows that (x, n_1) and (x_e, n_0) are equivalent units of G if and only if$(x_e,f(n,n_1,x))\in$ U for all $n\geq n_U$ or equivalently, $\lim_{n\to\infty} f(n,n_1,x)$ = x_e. Similarly, we can prove that if G = $G_2(X,\mathfrak{A},f)$, then (x, n_1) and (x_e, n_0) are equivalent units of G if and only if $\lim_{n\to\infty} f(n,n_1,x)$ = x_e.

2. (x, n_1) and (x_e, n_0)are two equivalent units of $G_3(X,\mathfrak{A},f)$if and only if there is $k\in\mathbb{Z}$ such that (x_e, n_0, k,x,n_1) $\in G_3(X,\mathfrak{A},f)$ if and only if for every U$\in\mathfrak{A}$ there is $n_U\in\mathbb{N}$ such that for all $n\geq n_U$ we have $n\geq n_1$, $n+k\geq n_0$and $(f(n+k,m_1,x_e),f(n,m_2,x))\in$ U for all m_1, $m_2\in\mathbb{N}$ such that $n+k\geq m_1\geq n_0$ and $n\geq m_2\geq n_1$. Since x_e is an equilibrium point, f(n+k, n_0, x_e)=x_e for all n and k such that $n+k \geq n_0$. It follows that if (x, n_1) and (x_e, n_0) are two equivalent units of $G_3(X,\mathfrak{A},f)$, then $(x_e,f(n,n_1,x))\in$ U for all $n\geq n_U$ or equivalently, $\lim_{n\to\infty} f(n,n_1,x)$ = x_e.

Corollary 3.8. Let $\mathfrak{A}$ be a uniform structure on X such that

$$\bigcap_{U\in\mathfrak{A}} U = \{ (x, x) : x \in X \}.$$

andf: $\mathbb{N}\times\mathbb{N}\times X\to X$ be a function. Then each orbit of the groupoids G, where G isthe groupoid $G_1(X,\mathfrak{A},f)$, $G_2(X,\mathfrak{A},f)$ or $G_3(X,\mathfrak{A},f)$,contains at most an element (x,n_0) with the property that x an equilibrium point of the system associated with f and $n_0\in N_e(x)$.

Proof. Let be (x,n_1) and (y,n_2) two equivalent units of G. Let us assume that x and y are equilibrium points and that $n_2\in N_e(y)$. Then x= $\lim_{n\to\infty} f(n,n_2,y) = \lim_{n\to\infty} y$ =y (since in this case the topology on X induced by the uniform structure $\mathfrak{A}$ is Hausdorff).

Proposition 3.9. Let $\mathfrak{A}$ be a uniform structure on X, let f: $\mathbb{N}\times\mathbb{N}\times X\to X$ be a function, letx_e be an equilibrium point and $n_0\in N_e(x_e)$. Then x_e is attractive if and only if (x_e,n_0) is in interior of its orbitwith respect to the structure of the groupoid $G_i(X,\mathfrak{A},f)$, i=1,2.

Proof. Let us assume that (x_e, n_0) belongs to the interior of $[(x_e, n_0)]\subset X\times\mathbb{N}$. Then there is $U\in\mathfrak{A}$ such that $\{(x,n_0): (x_e, x) \in U\}\subset[(x_e,n_0)]$. Let $x\in X$ such that $(x_e, x) \in U$. Then $(x,n_0)\in[(x_e,n_0)]$ and according Lemma 3.7, $\lim_{n\to\infty} f(n,n_0,x)$ = x_e. Thus x_e is attractive. Conversely, assume that x_e is attractive.

Then there is $U \in \mathfrak{A}$ such that if $(x_e,x) \in U$, then $\lim_{n\to\infty} f(n,n_0,x) = x_e$. ApplyingLemma 3.7, if $\lim_{n\to\infty} f(n,n_0,x) = x_e$, then $(x,n_0)\in[(x_e,n_0)]$. Consequently,

$$\{(x,n_0): (x_e, x) \in U\}\subset[(x_e,n_0)].$$

Therefore (x_e, n_0) belongs to the interior of $[(x_e, n_0)]\subset X\times\mathbb{N}$.

Corollary 3.10. Let $\mathfrak{A}$ be a uniform structure on a set X, let f: $\mathbb{N}\times\mathbb{N}\times X\to X$ be a function, let x_e be a stable equilibrium point and $n_0\in N_e(x_e)$. Then x_e is asymptotically stable if and only if (x_e,n_0) is in interior of its orbitwith respect to the structure of the groupoidG, where $G=G_1(X,\mathfrak{A},f)$ or $G=G_2(X,\mathfrak{A},f)$.

REFERENCES

[1] M. Buneci, Morphisms of discrete dynamical systems, Discrete and Continuous Dynamical Systems, 29 (1) (2011), 91-107.

[2] M. Buneci, A groupoid associated to discrete-time system that does not satisfies semigroup property, Annals of the “Constantin Brâncuşi” University of Târgu-Jiu. Engineering Series. No. 1(2016), 20-23.

[2] W. Mitkowski, Is a Fractional System a Dynamical System?,Automatyka 15 (2011), 67-69.