CARDINALITY AND ENTROPY FORINTUITIONISTIC FUZZY SETS

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Abstract.*In this article, a non-probabilistic-type entropy measure for intuitionistic fuzzy sets is proposed, starting from theratio of distances between them proposed in Szmidt and Kacprzyk [13] and De Luca and Termini - first axiomatized non-probabilistic entropy [7].*

Keywords: Intuitionistic fuzzy sets; Distance between intuitionistic fuzzy sets; Cardinality of intuitionistic fuzzy set; Entropy of intuitionistic fuzzy set.

1. INTRODUCTION

A measure of fuzziness often used and cited in the literature is an entropy first mentioned in 1965 by Zadeh [15-16]. The name entropy was chosen due to an intrinsic similarity of equations to the ones in the Shannon entropy.De Luca and Termini [7], introduced in 1972 some requirements which capture our intuitive comprehension of the degree of fuzziness.

In this paper, we propose a measure of fuzziness for intuitionistic fuzzy sets introduced by Atanassov [1-5]. The measure of entropy is a result of a geometric interpretation of intuitionistic fuzzy sets and basically uses a ratio of distances between them [13]. It is also shown that the proposed measure can be stated as the ratio of intuitionistic fuzzy cardinalities: that of $F \cap F^C$ and that of $F \cup F^C$, where F^C is the complement of F.

2.PRELIMINARIES

In this section, we will present those aspects of intuitionistic fuzzy sets which will be needed in our next discussion. Basic definitions and properties of these setsare those used in [1-5].

Definition 1.A fuzzy set *A* in $X = \{x\}$ may be given as

$$A = \{(x, \mu_A(x)) | x \in X\} \quad (1)$$

where each element had a degree of membership.

Definition 2. The intuitionistic fuzzy set on a universe *X*, is form

$$A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\} \quad (2)$$

where besides the degree of membership $\mu_A : X \to [0,1]$ of each element $x \in X$ to a set *A* there was considered a degree of non-membership $\nu_A : X \to [0,1]$, but such that

$$0 \le \mu_A(x) + \nu_A(x) \le 1,\ \forall x \in X . \tag{3}$$

Definition 3. We call degrees of indeterminacy of *x* to *A*, for each *A* in *X* the numbers []:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \forall x \in X \tag{4}$$

It is a hesitancy degree of *x* to *A* [1-5]. It is obvious that $0 \le \pi_A(x) \le 1,\ \forall x \in X$. For each fuzzy set $A \in X$, evidently, we have:

$$\pi_A(x) = 1 - \mu_A(x) - \left[1 - \mu_A(x)\right], \forall x \in X \tag{5}$$

A geometric interpretation of intuitionistic fuzzy sets and fuzzy sets is presented in Fig.1 [13]. An intuitionistic fuzzy set *X* is mapped into the triangle ABD [13] in that each element of *X* corresponds to an element of *ABD* - in Fig. 1, [13], as an example, a point $x \in \triangle ABC$ corresponding to $x \in X$ is marked (the values of $\mu_A(x), \nu_A(x), \pi_A(x)$ fulfill Eq. (4)). When $\pi_A(x) = 0$, then $1 = \mu_A(x) - \nu_A(x)$. In Fig. 1, this condition is fulfilled only on the segment AB. Segment AB may be therefore viewed to represent a fuzzy set.

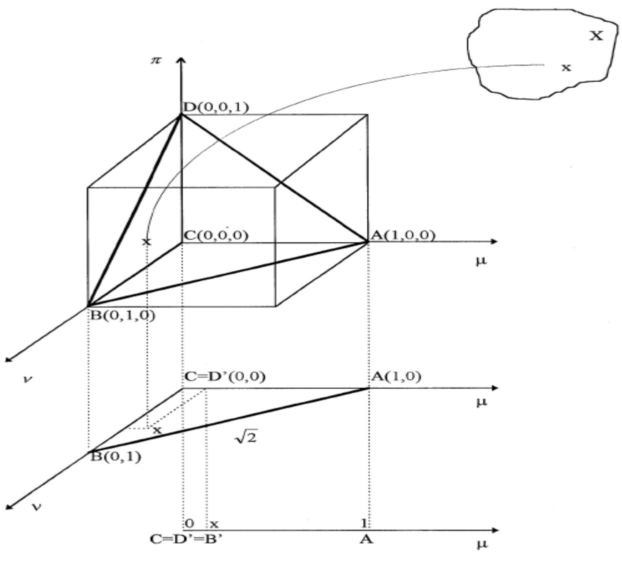

Fig. 1 *[13] A geometrical interpretation of an intuitionistic fuzzy set.*

The orthogonal projection of the triangle ABD gives the representation of an intuitionistic fuzzy set on theplane. (The orthogonal projection transfers $x' \in \triangle ABD$ into $x'' \in \triangle ABC$.) The interior of the triangle *ABC =ABD'* is the area where $\pi>0$. Segment *AB* represents a fuzzy set described by two parameters: μ and ν.The orthogonal projection of the segment AB on the axis μ (the segment [0; 1] is only considered) gives thefuzzy set represented by one parameter μ only.

As it was shown in [13], distances between intuitionistic fuzzy sets should be calculated taking into accountthree parameters describing an intuitionistic fuzzy set.The most popular distances between intuitionistic fuzzy sets A, B in $X = \{x_1, \dots, x_n\}$ are [13]:

- The Euclidian distance between *A*and *B* is defined as follows:

$$e(A,B) = \sqrt{\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2} \quad (6)$$

-The Hamming distance:

$$\text{h}(A,B) = \sum_{i=1}^{n} \left|\mu_A(x) - \mu_A(x)\right| + \left|\nu_A(x) - \nu_A(x)\right| + \left|\pi_A(x) - \pi_A(x)\right| \quad (7)$$

Definition 4.[13]Let *A*be an intuitionistic fuzzy set in *X*. First, we define the following two cardinalities of an intuitionistic fuzzy set:

- the least ("sure") cardinality of *A* is equal to the so-called sigma-count (cf. [23,24]), and is called here the $\min \sum Count$ (min-sigma-count):

$$\min \sum Count(A) = \sum_{i=1}^{n} \mu_A(x_i) \quad (8)$$

- the biggest cardinality of *A*, which is possible due to π_A, is called the $max \sum Count$ (max-sigma-count),and is equal to

$$\max \sum Count(A) = \sum_{i=1}^{n} \left(\mu_A(x_i) + \pi_A(x_i)\right) \quad (9)$$

For A^C we have:

$$\min \sum Count(A^C) = \sum_{i=1}^{n} \nu_A(x_i) \quad (10)$$

$$\max \sum Count(A^C) = \sum_{i=1}^{n} \left(\nu_A(x_i) + \pi_A(x_i)\right) \quad (11)$$

The cardinality of an intuitionistic fuzzy set is defined as the interval

$$card\ A = \left[\min \sum Count(A), \max \sum Count(A)\right] \quad (12)$$

3.ENTROPY

De Luca and Termini [7] first axiomatized non-probabilistic entropy, axioms formulated for intuitionistic fuzzy sets are intuitive and have been widely employed in the fuzzy literature.

E is an entropy measure if it satisfies axioms[7]:

1. *E(A)=0*iff $A \in 2^x$ (A non-fuzzy);
2. *E(A)=1*iff $\mu_A(x_i) = 0.5$ for all *i*;
3. $E(A) \le E(B)$ if*A* is less fuzzy than *B*,i.e., if $\mu_A(x_i) \le \mu_B(x_i)$ when $\mu_B(x_i) \le 0.5$ and $\mu_A(x_i) \ge \mu_B(x_i)$ when $\mu_B(x_i) \ge 0.5$;
4. $E(A) = E(\mathrm{A}^C)$.

Considertriangle ABD (Fig. 2. [13]).

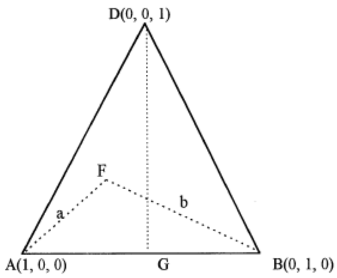

Fig.2

A non-fuzzy set (a crisp set) corresponds to the point *A* (the element fully belongs to it as $(\left(\mu_A(x), \nu_A(x), \pi_A(x)\right)$= (1; 0; 0)) and at point *B* (the element fully does not belong to it as $\left(\mu_A(x), \nu_A(x), \pi_A(x)\right)$=(0; 1; 0)). Points *A* and *B*representing a crisp set have the degree of fuzziness equal to 0.As shown in [13] : “An intuitionistic fuzzy set is represented by the triangle ABD and its interior. All points which are above the segment AB have a hesitancy margin greater than 0. The most undefined is point D. As the hesitancy margin for D is equal to 1, we cannot tell if this point belongs or does not belong to the set. The distance from D to A (full belonging) is equal to the distance to B (full non-belonging). So, the degree of fuzziness for D is equal to 100%. But the same situation occurs for all points xi on the segment DG. For DG we have $\mu_{DG}(x_i) = \nu_{DG}(x_i)$, $\pi_{DG}(x_i) \ge 0$ (equality only for point G), and certainly $\mu_{DG}(x_i) + \nu_{DG}(x_i) + \pi_{DG}(x_i) = 1$. Forevery $x_i \in DG$ we have: *distance (A; x_i)=distance (B; x_i)*. This geometric representation of an intuitionistic fuzzy set motivates a ratio-based measure of fuzziness (a similar approach was proposed in [13] to calculate the entropy of fuzzy sets):

$$E(\mathrm{F}) = \frac{a}{b} \quad (13)$$

where a is a distance *(F; F_{near})* from F to the nearer point F_{near}among A and B, and b is the distance *(F; F_{far})*from F to the farther point Ffar among A and B. The geometric interpretation confirms that Eq. (34) satisfies axioms (1)-(4)."

For n points belonging to an intuitionistic fuzzy set we have

$$E(\mathrm{F}) = \frac{1}{n}\sum_{i=1}^{n} E(F_i) \quad (14)$$

Szmidt andKacprzyk [13] remark the factwhile applying the Hamming distances in Eq. (13), the entropy of intuitionistic fuzzy sets is the ratio of the biggest cardinalities ($max \sum Count$) involving only F and F^c. Their result from [13] generalized form of the entropy measure for fuzzy sets presented in [14].

I have noticed that can also be appliedtheEuclidian distancesin Eq. (13),as long as

$$\mu_F^{\,2}(x_i) + \nu_F^{\,2}(x_i) + \pi_F^{\,2}(x_i) = 1 \quad (15)$$

Theorem 1.A generalized entropy measure of an intuitionstic fuzzy set F of n elements is:

$$E(\mathrm{F}) = \frac{1}{n}\sum_{i=1}^{n}\sqrt{\frac{1-\min \Sigma Count(F_i \cap F_i^C)}{1-\min \Sigma Count(F_i \cup F_i^C)}} \quad (16)$$

where[1-5]

$$F_i \cap F_i^C = \left\langle \min\left(\mu_{F_i}(x), \mu_{F_i^C}(x)\right), \max\left(\nu_{F_i}(x), \nu_{F_i^C}(x)\right)\right\rangle$$

$$F_i \cup F_i^C = \left\langle \max\left(\mu_{F_i}(x), \mu_{F_i^C}(x)\right), \min\left(\nu_{F_i}(x), \nu_{F_i^C}(x)\right)\right\rangle.$$

Proof. Let F a point having coordinates(μ_F, ν_F, π_F), F^ca point having coordinates$(\mu_{F^C}, \nu_{F^C}, \pi_{F^C}) = (\nu_F, \mu_F, \pi_F)$, $\overline{F}$ the nearest non-fuzzy neighbor of F (i.e. point A for Fig. 3a, or point $B$for Fig. 3b), $\underline{F}$ the farthest non-fuzzy neighbor of F (i.e. point B for Fig. 3a, or point A for Fig. 3b).Due to Eq. (34), we have $E(\mathrm{F}) = \frac{a}{b} = \frac{d_{IFS}(F, \overline{F})}{d_{IFS}(F, \underline{F})}$ and for the situation in Fig. 3a we have [using the Euclidiandistance (6)]

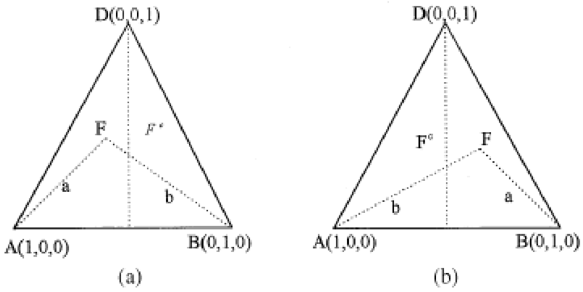

Fig. 3 [13]

$$E(\mathrm{F})=\frac{\sqrt{(1-\mu_F)^2+(0-\nu_F)^2+(0-\pi_F)^2}}{\sqrt{(0-\mu_F)^2+(1-\nu_F)^2+(0-\pi_F)^2}}=\sqrt{\frac{1-2\mu_F+\mu_F{}^2+\nu_F{}^2+\pi_F{}^2}{1-2\nu_F+\mu_F{}^2+\nu_F{}^2+\pi_F{}^2}}=\sqrt{\frac{2-2\mu_F}{2-2\nu_F}}=\sqrt{\frac{1-\mu_F}{1-\nu_F}}$$

$$=\sqrt{\frac{1-\min Count(F)}{1-\min Count(F^C)}}\,. \qquad (17)$$

For multiple elements F_i ($i=1\ldots n$) whose point A is their nearest fuzzy neighbor, Eq. (39) becomes owing to Eqs. (8), (10) and (14)

$$E(\mathrm{F})=\frac{1}{n}\sum_{i=1}^{n}\sqrt{\frac{1-\min Count(F)}{1-\min Count(F^C)}} \qquad (18)$$

For the situation in Fig. 3a we have

$$\min Count(\mathrm{F}\cap F^C)=\min Count\left\langle \min\left(\mu_F,\mu_{F^C}\right),\max\left(\nu_F,\nu_{F^C}\right)\right\rangle=\min Count\left(F^C\right) \qquad (18)$$

$$\min Count(\mathrm{F}\cup F^C)=\min Count\left\langle \max\left(\mu_F,\mu_{F^C}\right),\min\left(\nu_F,\nu_{F^C}\right)\right\rangle=\min Count\left(F\right) \qquad (19)$$

and a similar consideration for the situation in Fig. 3b. Formulas (19), (20) lead to formulas (18) as formula (16).

Example 1. Let us calculate the entropy for an element F_1 with the coordinates

$$F_1=\left(\frac{1}{3}\sqrt{3}+\frac{1}{3},\frac{1}{3}-\frac{1}{3}\sqrt{3},\frac{1}{3}\right).$$

$$d(A,F_1)=\sqrt{(1-\left(\frac{1}{3}\sqrt{3}+\frac{1}{3}\right))^2+(0-\left(\frac{1}{3}-\frac{1}{3}\sqrt{3}\right))^2+(0-\frac{1}{3})^2}=0.42265$$

$$d(\mathrm{B},F_1)=\sqrt{(0-\left(\frac{1}{3}\sqrt{3}+\frac{1}{3}\right))^2+(1-\left(\frac{1}{3}-\frac{1}{3}\sqrt{3}\right))^2+(0-\frac{1}{3})^2}=1.5774$$

$$E(\mathrm{F})=\frac{d(\mathrm{A},F_1)}{d(\mathrm{B},F_1)}=0.2679.$$

We can obtain the same result using formula (16):

$$F_1^C=\left(\frac{1}{3}-\frac{1}{3}\sqrt{3},\frac{1}{3}\sqrt{3}+\frac{1}{3},\frac{1}{3}\right),\ F_1\cap F_1^C=\left(\frac{1}{3}-\frac{1}{3}\sqrt{3},\frac{1}{3}\sqrt{3}+\frac{1}{3},\frac{1}{3}\right)=F_1^C,$$

$$F_1\cup F_1^C=\left(\frac{1}{3}\sqrt{3}+\frac{1}{3},\frac{1}{3}-\frac{1}{3}\sqrt{3},\frac{1}{3}\right)=F_1$$

$$E(\mathrm{F}_1)=\sqrt{\frac{1-(\frac{1}{3}\sqrt{3}+\frac{1}{3})}{1-(\frac{1}{3}-\frac{1}{3}\sqrt{3})}}=0.2679 \text{ i.e. the same value.}$$

5. CONCLUSIONS

Starting from [13] I tried to introduced a measure of entropy for an intuitionistic fuzzy set. This measure is consistent with similar considerations for ordinary fuzzy sets.

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