A FEW REMARKS ON LOCALLY COMPACT TOPOLOGIES AND HAAR SYSTEMS

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Abstract. *We start from the question raised by Williams (Proc. Am. Math. Soc. 2016): Must a second countable, locally compact, transitive groupoid G have open range map? If the answer is positive, the topology of G is in fact locally transitive (in the sense of [Seda, 1976]). We prove that even if the answer is negative, we can replace the original topology of G with a local transitive topology so that the topologies of the r-fibres are not affected. The new topology is locally compact Hausdorff but not necessary second countable. However its full C*-algebra (introduced in [Renault, 1980]) is still isomorphic to $C^*(H)\otimes K(L^2(\mu))$, where H is the isotropy group at a unit u and μ is a positive Radon measure on the unit space.*
We also present a few remarks concerning the Haar systems on locally compact groupoids and for every locally compact groupoid having paracompact unit space and second countable r-fibres, we prove the existence of a pre-Haar system bounded on the compact sets.

Keywords: transitive groupoid; Haar system; pre-Haar system; C*-algebra.

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1. A few remarks on the haar systems on locally compact groupoids

The construction of the full (respectively, reduced) C*-algebra associated to a locally compact Hausdorff groupoid G, introduced by Renault in [21], extends the case of a group: the space $C_c(G)$ of continuous functions with compact support on G is made into a *-algebra (using convolution) and then completed in a C*-norm making all its representations continuous (respectively, the “regular” representation continuous). In the locally compact non-Hausdorff case, then as pointed out by A. Connes [10], one has to modify the choice of $C_c(G)$ (because $C_c(G)$ may be too small to capture the topological structure of G). Usually is this case (assuming that G is locally Hausdorff) $C_c(G)$ is replaced with the space $C_c(G)$ of complex valued functions on G spanned by functions which vanishes outside a compact set K contained in an open Hausdorff subset U of G and being continuous on U. In order to define the convolution on $C_c(G)$ (or on the space $C_c(G)$ that replaces $C_c(G)$ in the non-Hausdorff case), one needs an analogue of Haar measure on locally compact groups, called Haar system (see Definition 2.2 [21]), subject to suitable support, invariance and smoothness conditions called respectively “full support on fibres”, "left invariance" and "continuity".

Unlike the case for locally compact groups, Haar systems on groupoids (in the sense of Definition 2.2 [21]) need not exist. The “continuity condition” entails that the range map (and hence the domain/source map) of the groupoid is an open map (Proposition I.4 [30]). According to a result of Seda [25], the “continuity condition” is essential for construction of the convolution algebra $C_c(G)$ (and consequently, for construction of the C*-algebras introduced in [21]). More precisely, he proved that for a locally compact Hausdorff groupoid G, $C_c(G)$ is closed to convolution if and only if the system of measures used as analogue of the Haar measure satisfies the “continuity condition”. Thus in order to construct the C*-algebras associated to G as in [21], the range map of G should be open. However as Deitmar pointed out in [9], it is possible that, although the range map is open, the “full support on fibres condition” of a Haar system cannot be satisfied (see Proposition 2.2 [9] where Deitmar constructed a compact transitive principal groupoid $G=X\times X$ for which no Haar system exists, though for any positive Radon measure λ on X, $\{\delta_x \times \lambda, x \in X\}$ is a system of measures on G satisfying "left invariance" and "continuity" conditions, but not "full support on fibres condition", since X is chosen such that it not be the support of a Radon measure). Such a space exists. If fact for a compact space X a necessary and sufficient condition to admit a full supported positive Radon measure (Borel regular measure) was established in [13] and for a locally compact Hausdorff space in Theorem 4.6 [26] (based on Theorem 1.6 [13]): a locally compact Hausdorff space X admits a full supported positive Borel finite measure if and only if X satisfies Kelley condition [15] (i.e. the family of its nonempty open subsets is the union of a countable family of positive collections, where a positive collection H is a family of subsets of X with the property that there is $\alpha>0$ such that for each finite sequence (S_1, S_2, …, S_n) in H, there is a subsequence $\left(S_{i_1}, S_{i_2}, ..., S_{i_k}\right)$ such that $\bigcap_{j=1}^{k} S_{i_j} \neq \varnothing$ and $\frac{k}{n} > \alpha$). Obviously, every locally compact Hausdorff second countable space satisfies Kelley condition.

Let us briefly recall some results concerning the existence of Haar systems (we use the notation and definition of a groupoid given by J. Renault in [21] with the exception of the source (domain) map which is denoted d as in [12]; all groupoids are assume topological locally compact, locally Hausdorff in the sense that each point has a Hausdorff compact neighborhood and having Hausdorff unit space):

- locally transitive groupoids (groupoïde microtransitif [11]), i.e. groupoids for which the restriction of the range map to the d-fibre G_u is an open map for each unit u: Seda proved that if a locally compact Hausdorff groupoid G is locally transitive, then the "continuity assumption" of a system of measures follows from the left invariance assumption (Theorem 2 [24]).

- totally intransitive groupoids (group bundles), i.e. groupoids with the property that r (x) = d (x) for all x: Renault proved that if a locally compact (or more generally, a locally conditionally compact), locally Hausdorff group bundle G admits a continuous function F_0:G$\to$[0, 1] with conditionally compact support such that $F_0(u)=1$ for all units $u \in G^{(0)}$, then G has a Haar system if and only if its range map r is open (Lemma 1.3/p. 6 [23]).

- r-discrete groupoids, i.e. groupoids having open unit space (as a subspace of the groupoid): Thomsen proved (Lemma 2.1 [28]) that a groupoid G has open unit space $G^{(0)}$ if and only if its range map is locally injective. This result together with Proposition 2.8 [21] imply that a locally compact groupoid with open unit space has a Haar system if and only if its range map is open.

- equivalent groupoids: Williams proved (Theorem 2.1 [31]) that if G is a second countable, locally compact Hausdorff groupoid having a Haar system, and H is a second countable, locally compact groupoid which is equivalent to G (in the sense of Definition 2.1 [17]), then also H has a Haar system.

Another difference between the Haar systems and the Haar measures on locally compact groups is that even when a Haar system does exist, it need not be unique. However groupoid C*-algebras associated to different Haar systems on the same groupoid are canonically Morita–Rieffel equivalent. This is a particular case of a more general result: equivalent groupoids (in the sense of Definition 2.1 [17]), have Morita-Rieffel equivalent C*-algebras. For full C*-algebras associated with second countable locally compact groupoids the result was proved in [17] (in Hausdorff case) and in [18] (in locally Hausdorff case). The Morita-Rieffel equivalence for reduced C*-algebras associated with second countable locally compact Hausdorff groupoids was proved in [27] (for locally Hausdorff case a similar result is stated in [29]). In [17], [18] and [27], the proof of the Morita-Rieffel equivalence of (full/reduced) C*-algebras associated to equivalent groupoid uses Renault's Disintegration Theorem (Proposition 4.2 [22]) and therefore the second countability hypothesis is essential. A recent result in [8] (Corollary 6.2) establishes the I-boundedness of densely defined $*$-representations of $C_c(G)$, for a locally compact Hausdorff groupoid G, even if G is not second countable. As a consequence it follows that equivalent locally compact Hausdorff groupoids (not necessarily second countable) have Morita-Rieffel equivalent C*-algebras (Corollary 6.3 [8]).

2. Existence of the pre-haar systems and c*-algebras for transitive groupoids

When we try to associate C*-algebras to locally compact groupoids which do not necessarily admit Haar systems in the sense of Definition 2.2 [21], one variant is to replace $C_c(G)$ with a larger space closed to convolution (see [28] for r-discrete (semi-etale) groupoids, [2] for transitive groupoids). Also in [21], [20] and [3] various subspaces of Borel functions was used. However those constructions need a left invariant system of measures which satisfies at least a Borel condition (instead of the continuity condition). The constructions of the convolution algebras in [5] require systems of measures satisfying only "support" and "left invariance" conditions. The same is true for [4], where in addition the system of measures should be bounded on compact sets. Let us prove that for locally compact (locally Hausdorff) topological groupoid having paracompact (Hausdorff) unit space and second countable r-fibres there exist systems of measures satisfying "full support on fibres" and "left invariance" conditions and which are bounded on compact sets.

Let us start with a groupoid G endowed with a topology that satisfies the following conditions

1. The groupoid operations are continuous.
2. Each point in G has a compact Hausdorff neighborhood.
3. The unit space $G^{(0)}$ is paracompact (Hausdorff).

We shall call such a groupoid locally compact groupoid with paracompact unit space.

Definition 2.1. Let G be a locally compact groupoid with paracompact unit space. A (left) pre-Haar system on G is a family of positive measures, $\nu=\{\nu^u, u\in G^{(0)}\}$, with the following properties:

1. ν^u is a positive Radon measure on G^u with $supp(\nu^u) = G^u$ for all $u\in G^{(0)}$.
2. $\int f(y)d\nu^{r(x)}(y)= \int f(xy)d\nu^{d(x)}(y)$ for all $x\in G$ and $f:G\to R$ such that $f|G^{r(x)}$ is continuous and compactly supported.

The pre-Haar system $\nu=\{\nu^u, u\in G^{(0)}\}$ on G is said to be bounded on compact sets if

$\sup\{\nu^u (K\cap G^u), u\in G^{(0)}\} < \infty$

for all compact subsets of G.

In [7] for a groupoid G we introduced the notion of G-uniformity meaning a collection $W(G^{(0)})$ of subsets of G satisfying the following conditions:

1. $G^{(0)}\subset W\subset G$ for all $W\in W(G^{(0)})$.
2. If $W_1, W_2\in W(G^{(0)})$, then there is $W_3\subset W_1\cap W_2$ such that $W_3\in W(G^{(0)})$.
3. $W=W^{-1}$ for all $W\in W(G^{(0)})$.
4. For every $W_1\in W(G^{(0)})$ there is $W_2\in W(G^{(0)})$ such that $W_2W_2\subset W_1$.

If G is a topological groupoid with paracompact unit space, then for each neighborhood W_0 of $G^{(0)}$ there is a symmetric neighborhood W_1 of $G^{(0)}$ such that $W_1W_1 \subset W_0$ (Proposition 3 [6]). A similar result is true for topological paracompact groupoids ([19] p. 362).

A subset K of G is said to be diagonally compact in [18] (p. 10) (or conditionally compact in [23] p. 5) if $K\cap r^{-1}(L)$ and $K\cap d^{-1}(L)$ are compact whenever L is a compact subset of $G^{(0)}$. According to Lemma 2.10 [18], if G is a locally compact groupoid with paracompact unit space, then G has a fundamental system of diagonally compact neighborhoods of $G^{(0)}$. Therefore by Proposition 3 [6] and Lemma 2.10 [18], every locally compact groupoid G with paracompact unit space has a G-uniformity $W(G^{(0)})$ consisting in a fundamental system of diagonally compact neighborhoods of $G^{(0)}$. Moreover since G_v^u is closed in G, it follows that for $u\neq v$, $G \setminus G_v^u$ is a neighborhood of $G^{(0)}$. Thus the G-uniformity $W(G^{(0)})$ consisting in a fundamental system of diagonally compact neighborhoods of $G^{(0)}$ fulfills the condition

$$\bigcap_{W\in W\left(G^{(0)}\right)} (r,d)(W) = \mathrm{diag}\left(G^{(0)}\right).$$

Also the G-uniformity $W(G^{(0)})$ is compatible with the topology of the fibres in the sense of Definition 3.4 [7]. Let us consider the following topology τ_W on G: $A\in\tau_W$ if and only if for every $x \in A$ there is $W_x \in W(G^{(0)})$ such that $W_x x W_x \subset A$ (Definition 3.1 [7]). Then according to Proposition 3.7 and Proposition 3.8 [7]:

1. G endowed with τ_W is a topological locally transitive groupoid.
2. The topology τ_W is usually finer than the original topology of G.
3. The topologies induced by τ_W and the original topology on r-fibres (respectively, on d-fibres) coincide.
4. The topology τ_W and the original topology coincide if and only if G endowed with the original topology is locally transitive.

Proposition 2.2. Let G be a locally compact groupoid having paracompact unit space. If every orbit [u] of G endowed with the quotient topology induced by d_u: $G^u\to[u]$ ($d_u(x)=d(x)$, for all x) satisfies Kelley condition (cf. [15], [26]), then G admits a pre-Haar system bounded on compact sets.

Proof. Since $G^{(0)}$ is paracompact, we showed above that there is a G-uniformity $W(G^{(0)})$ on G consisting of diagonally compact neighborhoods of the unit space such that

$$\bigcap_{W\in W\left(G^{(0)}\right)} (r,d)(W) = \mathrm{diag}\left(G^{(0)}\right).$$

Let us endow G with the topology τ_W presented above. Then (G, τ_W) is a locally transitive and locally compact Hausdorff groupoid. Thus for (G, τ_W) there is a coherent continuous system of measures $\left\{\beta_v^u, (u,v)\in(r,d)(G)\right\}$ constructed as in Section 1 [23]. Let us recall that Renault's construction [23] is done under the assumption of the existence of a continuous function $0\leq F_0\leq 1$ on (G, τ_W) that is 1 at each $u \in G^{(0)}$ and has diagonally compact support. In our case we can take $F_0 = 1-f_{W0}$, where $W_0\in W$ and f_{W0} is the function constructed in Theorem 2.5 [7] for $A=W_0$ and $W=W_0$. The system of measures $\left\{\beta_v^u, (u,v)\in(r,d)(G)\right\}$ are constructed as follows: for each $u\in G^{(0)}$, let us choose a left Haar measure β_u^u on G_u^u so the integral of F_0

with respect to β_u^u is 1; for each (u,v)∈(r,d)(G) with u≠v, let us choose $x\in G_v^u$ and let us define $\beta_v^u = x\beta_v^v$

$$\int f(y)d\beta_v^u(y) = \int f(xy)d\beta_v^v(y).$$

The measure β_v^u is independent of the choice of x, since if x' is another element of G_v^u, then $x^{-1}x'\in$, and since β_v^v is a left Haar measure on G_v^v, it follows that

$$x\beta_v^u = x'\beta_v^u.$$

Applying Lemma 1.3 [23], $\{\beta_u^u, u\in G^{(0)}\}$ is a (continuous) Haar system on the group bundle G'={x∈G: r(x)=d(x)} endowed with the topology coming from **(G, τ_W)**. Consequently, $\{\beta_v^u, (u,v)\in(r,d)(G)\}$ is a continuous system of measures on (r,d)(G) endowed with the quotient topology induced by (r,d):G →(r,d)(G), where G is endowed with τ_W. Since (G, τ_W) is locally transitive, the quotient topology on (r,d)(G) coincides with the product topology coming from $G^{(0)}\times G^{(0)}$, where the topology of $G^{(0)}$ coming from (G, τ_W). This topology is the disjoint union topology of $\bigcup_{u\in G^{(0)}}[u]$, where [u] is endowed with quotient topology induced by $d_u:G^u \to[u]$ ($d_u(x)=d(x)$ for all x). As in [23] we prove that $\{\beta_v^u, (u,v)\in(r,d)(G)\}$ is bounded on compact sets (we recall here the reasoning because, unlike [23], we work with two different topologies: we take into consideration compact sets with respect to the original topology, but the system of measure $\{\beta_v^u, (u,v)\in(r,d)(G)\}$ was constructed using the new topology τ_W). Since

$$1 = \int F_0(y)d\beta_u^u(y) \geq \int_{W_0} F_0(y)d\beta_u^u(y) = \int_{W_0} 1d\beta_u^u(y) = \beta_u^u(W_0)$$

it follows that $\beta_u^u(W_0)\leq 1$. Let K be a compact subset of G with respect to the original topology. For each (u,v)∈(r,d)(K), there is $x_v^u \in G_v^u\bigcap K$ and thus

$$\beta_v^u(K) = \beta_v^v\left(\left(x_v^u\right)^{-1}K\right) \leq \beta_v^v\left(K^{-1}K\right).$$

The compact set $K^{-1}K$ can be covered by finitely many open sets $\{V_i$, i=1,2,..,n$\}$ such that $V_i^{-1}V_i \subset W_0$ for all i∈{1,2, …, n}. For each i with the property that $G_v^v\bigcap V_i \neq \emptyset$, let us choose $x_i \in G_v^v\bigcap V_i$. Then

$$\beta_v^v(V_i) = \beta_v^v\left(x_i x_i^{-1}V_i\right) = \beta_v^v\left(x_i^{-1}V_i\right) \leq \beta_v^v\left(V_i^{-1}V_i\right) \leq \beta_v^v(W_0) \leq 1.$$

Hence

$$\beta_v^u(K) \leq \beta_v^v\left(K^{-1}K\right) \leq \sum_{i=1}^{n}\beta_v^v\left(V_i^{-1}V_i\right) \leq n$$

for all (u,v)∈(r,d)(G).

Let us remark that topology of G^u coming from (G, τ_W) as well as from din original topology of G coincide. Now to construct a pre-Haar system on G we take a finite positive Borel measure $\mu^{[u]}$ on [u] of full support (according to Theorem 4.6 [26] there is such a measure on [u]). Without loss of generality, we may assume that $\mu^{[u]}([u]) = 1$.

For every $u \in G^{(0)}$ and every $f \in C_c(G^u)$ let us define

$$\nu^u(f) = \int\int f(y)\, d\beta_v^u(y)\, d\mu^{[u]}(v).$$

Then for every $u \in G^{(0)}$, ν^u is a positive Radon measure on G^u with $supp(\nu^u)=G^u$. Also for all compact subsets K of G (with the respect to the original topology) we have
$\sup\{\nu^u(K), u \in G^{(0)}\} < \infty$.

Corollary 2.3. Let G be a locally compact groupoid having paracompact unit space and second countable r-fibres. Then G admits a pre-Haar system bounded on compact sets.

Proof. First let us remark that [u] endowed with the quotient topology induced by
$d_u: G^u \to [u]$ ($d_u(x) = d(x)$ for all x),
has a locally compact Hausdorff topology which is second countable. Since every second countable space satisfies Kelley condition (see [15] and [26]), the conclusion follows from Proposition 2.2.

Remark 2.3. The existence of a pre-Haar system bounded on compact sets was also proved in Theorem 1 [1] for locally compact Hausdorff second countable groupoids. For transitive groupoids the system of measures $\{\nu^u, u \in G^{(0)}\}$ constructed in Theorem 1 [1] is Borel ($u \to \int f(y) d\nu^u(y)$ is a Borel real extended map for every Borel nonnegative function f on G) but not necessarily continuous (there is gap in the proof of Proposition 2 (3) [1], where Mackey's lemma (Lemma 1.1 [16]) was applied for a possible non open map between locally compact second countable spaces; this invalidates Theorem 2 [1] and Corollary 1 [1]).

Remark 2.4. In [31] the following question was raised: Must a second countable, locally compact, transitive groupoid G have open range map? If the answer is positive, then G is also locally transitive (Theorems 2.2A and 2.2B [17] or Theorem 2.1 [19]) and according to Theorem 3.1 [17], the C*-algebra associated to G endowed with a Haar system is isomorphic to $C^*(G_u^u) \otimes K(L^2(G^{(0)}, \mu))$. If the answer is negative, then there is a second countable, locally compact, transitive groupoid G that has no Haar system. However even in this case G can be endowed with a pre-Haar system which becomes a Haar system on (G, τ_W), since (G, τ_W) is locally transitive and Seda result [24] can be applied (τ_W is the topology described in beginning of this section). Therefore we can associated to (G, τ_W) the classical full C*-algebra introduced in [21]. Since the topology τ_W is not necessarily second countable we cannot apply Theorem 3.1 [17] but we show below that the result stated in 3.1 [17] remain true.

Proposition 2.5. Let G be a locally compact transitive groupoid having second countable Hausdorff fibres and paracompact unit space. If $\{\nu^u, u \in G^{(0)}\}$ is a pre-Haar system on G and τ_W is the topology described in beginning of this section, then the classical full C*-algebra associated to (G, τ_W) and $\{\nu^u, u \in G^{(0)}\}$ is isomorphic to $C^*(G_u^u) \otimes K(L^2(G^{(0)}, \mu))$, where μ is a positive Radon measure on $G^{(0)}$ and $K(L^2(G^{(0)}, \mu))$ is the algebra of compact operators on the Hilbert space $L^2(G^{(0)}, \mu)$ (Theorem 3.1 [17] without global second countability assumption).

Proof. Let us denote by $C^*_W(G, \nu)$ the full C*-algebra associated to (G, τ_W) and its

Haar system $\{\nu^u, u\in G^{(0)}\}$. Let us fix $u\in G^{(0)}$. The topological groupoid (G, τ_W) is a locally transitive and locally compact (Hausdorff). Thus G^u is a $\left(G_u^u, G\right)$-equivalence in the sense of Definition 2.1 [17].

There are only two points in the proof of Theorem 3.1 [17] where it is needed that the groupoid involved is second countable. One is the Morita-Rieffel equivalence of $C^*_W(G, \nu)$ and $C^*\left(G_u^u\right)$ which was proved using Theorem 2.8 [17] based on Renault's Disintegration Theorem (Proposition 4.2 [22]). However according Corollary 6.3 [8], $C^*_W(G, \nu)$ and $C^*\left(G_u^u\right)$ are Morita-Rieffel equivalent (without second countability assumption). The other point where the second countability is needed is for choosing a regular Borel cross section of $d_u:G^u\to G^{(0)}$ via Mackey's lemma (Lemma 1.1 [16]). Since for (G, τ_W), $d_u:G^u\to G^{(0)}$ is open and since the topology induced by τ_W on G^u coincides with the original topology (hence is second countable) it follows that we can apply Lemma 1.1 [16]. Thus exactly as in the proof of Theorem 3.1 [17], using Lemma 9 [14], $C^*_W(G, \nu)$ is isomorphic to $C^*\left(G_u^u\right)\otimes K\left(L^2\left(G^{(0)},\mu\right)\right)$.

References

[1] M. Buneci, Haar systems and homomorphisms on groupoids. Operator algebras and mathematical physics (Constanta, 2001), (eds. J-M. Combes, J. Cuntz, G. Elliott, Gh. Nenciu, H. Siedentop, Ş. Strătilă), Theta, Bucharest, 2003.

[2] M. Buneci, Groupoid algebras for transitive groupoids. Math. Rep. (Bucur.) 5(55), No. 1 (2003) 9-26.

[3] M. Buneci, Isomorphic groupoid C*-algebras associated with different Haar systems. New York J. Math. 11 (2005), 225-245.

[4] M. Buneci, Uniformly continuous functions on non-Hausdorff groupoids, Surveys in Mathematics and its Applications, 5 (2010), 229-245.

[5] M. Buneci, Convolution algebras for topological groupoids with locally compact fibres, Opuscula Mathematica 31 (2011), no. 2, 159-172.

[6] M. Buneci, Various notions of amenability for not necessarily locally compact groupoids, Surveys in Mathematics and its Applications, 9 (2014), 55-78.

[7] M. Buneci, A Urysohn type lemma for groupoids, Theory and Applications of Categories 32 (28) (2017), 970-994.

[8] A. Buss, R. D. Holkar and R. Meyer, A universal property for groupoid C*-algebras. I, Proceedings of the London Mathematical Society. (online April 2018). arXiv:1612.04963v2.

[9] A. Deitmar, On Haar systems for groupoids. Z. Anal. Anwend. (2018), arXiv:1605.08580.

[10] A. Connes, Sur la theorie noncommutative de l'integration, Lecture Notes in Math. Springer-Verlag, Berlin 725 (1979) 19-143.

[11] C. Ehresmann, Catégories topologiques. I, II, III, (French) Nederl. Akad. Wetensch. Proc. Ser. A 69=Indag. Math. 28 1966.

[12] P. Hahn, Haar measure for measure groupoids, Trans. Amer. Math. Soc. 242(1978)1-33

[13] D. J. Hebert and H. E. Lacey, On supports of regular Borel measures. Pacif. J. Math. 27 (1968), 101–118.

[14] G. G. Kasparov, Hilbert C*-modules: Theorems of Stinespring and Voiculescu, J. Oper. Theory 4, 133-150 (1980).

[15] J. L. Kelley, Measures on Boolean algebras, Pacific J. Math. 9 (1959), 1165-1176.

[16] G. Mackey, Induced representations of locally compact groups. I, Ann. of Math., 55 (1952)
101-139.
[17] P. Muhly, J. Renault and D. Williams, Equivalence and isomorphism for groupoid C*-algebras, J. Operator Theory 17 (1987), 3–22.
[18] P. Muhly and D. Williams, Renault's Equivalence Theorem for groupoid crossed products, NYJM Monographs 3, 2008.
[19] A. Ramsay, The Mackey-Glimm dichotomy for foliations and other Polish groupoids, J. Funct. Anal. 94 (1990), 358-374.
[20] A. Ramsay and M. E. Walter, Fourier-Stieltjes algebras of locally compact groupoids, J. Funct. Anal. 148 (1997), 314–367
[21] J. Renault, A groupoid approach to C∗-algebras, Lecture Notes in Math., 793, Springer, Berlin, 1980.
[22] J. Renault, Représentation des produits croisés d'algèbres de groupoïdes. J. Oper. Theory 18, No. 1, 67-97 (1987).
[23] J. Renault, The ideal structure of groupoid crossed product C∗-algebras, J. Operator Theory, 25 (1991), 3–36.
[24] A.K. Seda, Haar measures for groupoids, Proc. Royal Irish Acad. 76, Sec. A, no. 5, (1976), 25-36.
[25] A. K. Seda, On the continuity of Haar measure on topological groupoids, Proc. Amer. Math. Soc., 96 (1986), 115-120.
[26] W. Seidel, Supports of Borel measures. Fundam. Math. 133, No.1, 67-80 (1989).
[27] A. Sims and D. P. Williams, Renault's equivalence theorem for reduced groupoid C*-algebras. J. Operator Theory 68 (2012), no. 1, 223-239. arXiv:1002.3093v2
[28] K. Thomsen, Semi etale groupoids and applications, Annales de l'institut Fourier, 60 no. 3 (2010), 759-800. arXiv:0901.2221v3
[29] J. L. Tu, Non-Hausdorff groupoids, proper actions and K-theory, Doc. Math. 9 (2004), 565–597.
[30] J. Westman, Nontransitive groupoid algebras, preprint, Univ. of California at Irvine, 1967.
[31] D. P. Williams, Haar systems on equivalent groupoids, Proc. Am. Math. Soc., Ser. B 3, 1-8 (2016).