BOOLEN NORMED ALGEBRAS

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ABSTRACT: In this section we begin a systematic study of algebras given by algebraic measures. Knowing that the measure is an essential positive total additive function. The most important lesson is made up of probability measures. Each algebra with measures of probability can be interpreted as a system of events, with the measure itself as the probability of this system. The bulk of it in the next chapters allow the translation in the language of probability theory.

ALGEBRE BOOLEAN NORMATE
1. standardized Algebra 1.1 Definition and its properties of Boolean algebra normed topological definition. A Boolean algebra (abbreviated normata ABN) is a pair \( \{ \xi, \mu \} \), where \( \xi \) is complete and \( \mu \) is the measure of \( \xi \). Through the routine of the language, we'll talk often about a "ABN \( \xi \)". If \( \mu \) is a measure of the probability then sometimes use the term algebra to probability. As is always the outer measure. Therefore, all statements for Boolean algebra complement proven with external measures applied to an ABN. In particular, each ABN \( \{ \xi, \mu \} \) turns into a metric space with metric

\[
\rho_{\mu} \equiv \rho : \quad \rho(x, y) = \mu(\vert x - y \vert) . \quad (1)
\]

Considering that the measure is totally additive, \( x_n \downarrow \Phi \) involve \( p(x_n, \varnothing) = \mu x_n \to 0 \). Indeed,

\[
x_n = \bigvee_{k=n}^{\infty} (x_k \land Cx_{k+1} ),
\]

\[
\mu x_n = \sum_{k=n}^{\infty} \mu(x_k \land Cx_{k+1} ) . \quad (2)
\]

Theorem 1. The Topology the metrical of a \( \rho_{\mu} \) on \( \xi \) coinciding with the topology (o).

This theorem is a special case. It is enough to recall the fact that the metric topology satisfies the condition (os).

Corollary 1. Either \( \mu \) si \( \nu \) two measures on a \( AB \) complete. Then for each \( \varepsilon > 0 \) real, there are \( \delta > 0 \) so that the inequality \( \mu x < \delta \) involve \( \nu x < \varepsilon \).
To prove it is sufficient to observe that the relationship $\mu x_n \to 0$ and $\nu x_n \to 0$, in virtue of the theorem 1, expressing the same thing: convergence in topology from scratch \((o)\). This convergence can be called "convergence to the extent". It is obvious that each measure is a function. Corollary 1 looks like this concept of absolute continuity, while important for a set of functions, it becomes useless in our case. For each $\mathbb{A} \mathbb{B} \xi$ complete all the steps, admits metric spaces $\{\xi, \mu\}$ are homeomorphic to each other.

**Corollary 2.**

Because the relationship $\mu x_n \to 0$ keep it off, it is necessary and sufficient that each absolute increase of indices $\{n_k\}$ There is a joint $\{n_{k_i}\} (k_1 < k_2 < ...)$ so $x_{n_{k_i}} \to \emptyset$. In fact, the coincidence of topology \((o)\) si \((os)\) It is sufficient to refer to the corollary of Theorem 1 \(2\) shows us that each sequence $\{n_k\}$ anulandu-it contains a joint to the extent \((o)\) convergence $\{x_{n_k}\}$. This statement can be made more precise method of choice highlighting such consequential: a convergence \((o)\) It is enough that the series $\sum_{k=1}^{\infty} \mu x_{n_k}$ converge. We demonstrate that.

Lemma 1. If $\sum_{i=1}^{\infty} \mu y_i < +\infty$ then $y_i \to \emptyset$.

Demonstratie. Se $m$ and si $k$, We have $\mu \left( \bigvee_{i=m}^{m+k} y_i \right) \leq \sum_{i=m}^{m+k} \mu y_i \leq \sum_{i=m}^{\infty} \mu y_i$. Considering that so far as is \((o)\) - continue:

$$\mu \left( \bigvee_{i=m}^{\infty} y_i \right) \leq \sum_{i=m}^{\infty} \mu y_i \quad (3)$$

The right side of this inequalities disappears: $0 \leq \mu \left( \varliminf y_i \right) = \mu \left( \bigwedge_{m=i}^{\infty} \bigvee y_i \right) \leq \inf_{m} \sum_{i=m}^{\infty} \mu y_i = 0 \quad (4)$

The right side of this inequalities disappears: $0 \leq \mu \left( \varliminf y_i \right) = \mu \left( \bigwedge_{m=i}^{\infty} \bigvee y_i \right) \leq \inf_{m} \sum_{i=m}^{\infty} \mu y_i = 0$

Considering that the measure is essentially positive, we see that $\varliminf y_i = \emptyset$ or \((o)\) - $\lim y = \emptyset$

**Lemma is proved.** Using this Lemma, we consider the problem of complete metrics of a Boolean algebra.
Teorema 2

A \( \xi \) AB complete with metric \( \rho_{\mu} \). It is a complete metric space.

Demonstrație

Fie \( \{x_n\}_{n=1}^{\infty} \) a Cauchy sequence which comply with \( \rho_{\mu} \). We can choose a joint \( \{x_n\}_{k=1}^{\infty} (n_1 < n_2 < \ldots) \) so series \( \sum_{k=1}^{\infty} p(x_{n_k}, x_{n_{k+1}}) = p(\mid x_{n_k} - x_{n_{k+1}} \mid, \Theta) \) converged.

Therefore, according to Lemma 1, \( \mid x_{n_k} - x_{n_{k+1}} \mid \rightarrow x \). So, we have

\[
\Theta \leq \lim_{n_k} - \lim_{n_{k+1}} = \lim_{n_k} \mid x_{n_k} - x_{n_{k+1}} \mid = \Theta.
\]

Therefore, there is an element \( x \) so \( x_{n_k} \rightarrow x \). Now, the inequality \( \rho(x_n, x) \leq \rho(x_n, x) + \rho(x_n, x_n) \) We can easily notice that \( \rho(x_n, x) \rightarrow 0 \) taking into account that \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence. It has been demonstrated that \( \{x, \mu\} \) is complete.

Observație. Theorem 2 remain valid if \( \mu \) is a foreign measure, to demonstrate, it is sufficient to slightly change the proof Theorem 2.

1.2 its properties of a converged \( (o) \)

Starting from Lemma 1, it is easy to demonstrate another important theorem expressed by m. Frechet.

Teorema 3

Every ABN is regular. I actually set a fact even more generally: regularity of the algebra. Thus, each can be considered a ABN space metric. What else can we say about the properties in this space? For example, will be connected? The answer is negative, in general, since some lie algebras are normed listed all finite AB, which are clearly disconnected.

Teorema 4

An ABN is a metric space continue connected in the shape of an arc. This theorem implies, in particular, that the set of values in a measure coincide (in accordance with the terms of the theorem) with the range \([0, \mu]\). But this has been proven previously: base is a Theorem Demonstration 4 follow now. So either two elements \( x_0, x_1, x \) database. We will build a path between these elements. This is done in a few steps. 1. Show that for all \( u, v \in \xi, u \neq v \)

There are \( w \in \xi \) so \( \rho(u, w) = \rho(v, w) = \frac{1}{2} \rho(u, v) \). Suppose \( u' \equiv u \land Cv \) şi \( v' \equiv v \land Cv \).

These elements are disjoint, but not equal to zero at the same time and \( \rho(u', v') = \mu|u' - v'| = \mu|u - v| = \rho(u, v) = \mu u' + \mu v' \) (5)

Let’s assume that for the determination of \( \mu u' > 0 \) si \( \mu u' \geq \mu v' \) (6)
Moreover, we can consider the $\pi_o$ with all quasimasurele $\pi$ regulated by condition of $\pi(1)=1$. It is clear that $K_0 \subset \pi_o$. These categories (and their subclasses) are listed in mathematical statistics "statistical structures." We will build some elements $u^*$ si $v^*$ with the following properties: $u^* \leq u', u^* \leq v', \mu u^* = \frac{1}{2} \mu u', \mu v^* = \frac{1}{2} \mu v'$.

We Place $w = u^* + v^* + u \land v$. Then $\rho(w,u) = \mu(u'-u^*) + \mu v^*$ = $\frac{1}{2} (\mu u' + \mu v') = \frac{1}{2} \rho(u,v)$. Similarly $\rho(w,v) = \frac{1}{2} \rho(u,v)$.

2. Now, knowing that $u = x_0$ si $v = x_1$ as we build the item above $w = x_{01}$ a which distance between $x_0$ and $x_1$ is $\frac{1}{2} \rho(x_0,x_1)$ In addition, we define $x_{001}$ and $x_{011}$ also, continuing this process we will apply our crowd of AB has all the numbers $r \rho(x_0,x_1)$, where $r \in [0,1]$ dyadic rational-it. Moreover, such a built function $\gamma_0$ is uniformly continuous: $p\left(\gamma_0(t'), \gamma_0(t^*)\right) \leq |t'-t^*|$.

3. We remain to extend this application continuity through the whole range $\Delta = [0, \rho(x_0,x_1)]$.

Get some continuous functions $\gamma$ that apply $\Delta$ in AB such that $\gamma(0) = x_0$ and $\gamma(\rho(x_0,x_1)) = x_1$. It's a good trajectory between points $x_0$ and $x_1$.

The demonstration is complete. We will add to this very important Observation. In an ABN form each lot continues $M_{\alpha} = \{x | \mu x = \alpha\}$

It is connected in the shape of an arc, $\alpha \in [0, \mu 1]$. The Crowd $M_{\alpha}$ is closed. That's why each trajectory $\gamma$ What unites $x_0, x_1 \in M_{\alpha}$ lies entirely in $M_{\alpha}$.

This observation will allow us later to demonstrate his famous Theorem of Lyapunov vector measures about. Definition: a complete AB is called normata if it admits a few steps. Each normata holds many measures AB in general; but are equivalent in some places such as the corollary of the theorem 1 shows 1: define the same topology, topological order metric. Theorems 3 and 4 are obviously valid for Lie algebras normed. In essence, a normata is a AB AB $\xi$ analyzed together with complete system of measure $\xi$. Moreover, it is considered that the system is not empty. From now on we will recover through $M(\xi) = M$.

Also consider the set K of all total additive on quasimasurele $\xi$. Each of these is a continuous function quasimasuuri. If $v \in K$ then the crowd $\Omega = \{x | \nu x = 0\}$ is an ideal of the form $\xi_u$.

Quasimasurelor restriction $v$ the complementary area $\xi_{Cu}$. It is a measure. Usually we can consider probability measures satisfying the condition $\mu 1 = 1$. We note the probability measures with $M_\nu(\xi)$. Similarly we will put Moreover, we can consider the $\pi_o$ with all quasimasurele $\pi$ regulated by condition of $\pi(1)=1$. It is clear that $K_0 \subset \pi_o$. These categories (and their subclasses) are listed in mathematical statistics "statistical structures."
1.3 The ABN Izomorfisme

Either \( \{ \xi, \mu \} \) and \( \{ \mathfrak{I}, \nu \} \) the two ABN. An application \( f \) on poset \( E \subset \xi \) (for example, a subalgebra or area) \( \mathfrak{I} \) It is called as constanta \( \nu(f(x)) = \mu x \) for all \( x \in E \). First of all we are interested in homomorfism and isomorphism. Talking about a homomorfism or homorfism of the \( \{ \xi, \mu \} \) on \( \{ \mathfrak{I}, \nu \} \). We always refer to it as homomorfismul (izomorfismul) keeps his \( \xi \) on \( \mathfrak{I} \). The method is often useful in izomorfismelor.

The following theorem relies on this method.

**Theorem 5.**

If \( \xi_0 \subset \xi \) and \( \mathfrak{I}_0 \subset \mathfrak{I} \) there are two subalgebre of everywhere \( \xi \) and \( \mathfrak{I} \) (o) - dense then each measure constant monomorfismului \( \varphi_0 \) of \( \xi_0 \) on \( \mathfrak{I}_0 \) extends to a measure of constant izomorfismului \( \varphi_0 \) of \( \xi_0 \) on \( \mathfrak{I}_0 \). Such an extension is unique.

Demonstration.

Lie Algebras \( \xi \) and I’m a regular. Constant-mean that measures homomorfismele \( \varphi_0 \) si \( \varphi_0^{-1} \) satisfying the condition \( (E_{\sigma}) \). Indeed, either: \( x = \bigvee_{n=1}^{\infty} x_n \), \( x, x_n \in \xi_0 \).

If: \( x_1 = x_1 \), \( x_2 = x_2 \wedge Cx_1 \), \( x_3 = x_3 \wedge C(x_1 \vee x_2) \),.... Then

\[
\nu \left( \bigvee_n \varphi_0(x_n) \right) \geq \nu \left( \bigvee_n \varphi_0(x'_n) \right) = \sum_n \nu \varphi_0(x'_n)
\]

\[
= \sum_n \mu x'_n = \mu x = \nu \varphi_0(x).
\]

Hence, \( \varphi_0(x) = \lim_{n \to \infty} \nu \varphi_0(x_n) \).

The last equality says that that condition \( (E_{\sigma}) \) is fulfilled. In our case it involves \( (E_{\sigma}^+) \).

Thus, homomorfismul \( \varphi_0 \) satisfies the condition \( (E_{\sigma}^+) \). By analogy, we will check if this condition is satisfied for the \( \varphi_0^{-1} \). AB \( \xi \) and \( \mathfrak{I} \) meet the condition; It is easy to see, for such terms of algebras \( (E_{\sigma}) \) and \( (E_{\sigma}^+) \) are equivalent to \( (E) \) and \( (E^+) \).

In this case, the extension \( \varphi \) of monomorfismului \( \varphi_0 \) is an isomorphism \( \xi \) on \( \mathfrak{I} \). Show that it is able. For each \( x \in \xi \) there is a sequence \( \{ x_n \} \) in \( \xi_0 \) converging at \( x \). Passing to the limit in the relationship \( \nu \varphi(x_n) = \nu \varphi_0(x_n) = \mu x_n \). (7)
We arrived at the desired equality \( \nu \varphi(x) = \mu \).

The uniqueness of an extension obtained thus is obvious. The demonstration is complete. We will cite two examples in which the theorem applies to \( S \). In these examples \( \xi \) and \( S \). There are two normed algebras. And \( \mu \) and \( \nu \). There are two measures of the probability of these algebras.

I. Be \( \xi \) and \( S \) containing compact and autonomous subalgebras everywhere \( \xi_0 \) and that \( S_0 \). Let’s say these subalgebra include not only of systems alternators, but also (considering some measures \( \mu \) and \( \nu \) ) with the same power. Either \( E_{\xi_0} \) an independent system of generators \( \mu \) for \( \xi_0 \) and either \( E_{S_0} \) an independent system of generators \( \nu \) for \( S_0 \).

Finally, we assume for simplicity that
\[
\mu x = \nu y = \frac{1}{2}
\]
for all \( x \in E_{\xi_0} \) and \( S \in E_{S_0} \). In this case, each of which applied to each \( \phi \) from \( E_{\xi_0} \) on \( E_{S_0} \) an isomorphism is extensive \( \varphi_0 \). None of his \( \xi_0 \) on \( S_0 \). It is easily checked that this measure is isomorphism. In fact, if
\[
x = x_1 \land x_2 \land \ldots \land x_k ,
\]
where \( x_i \in E_{\xi_0} \cup CE_{\xi_0} (i = 1, 2, \ldots, k) \) and \( x_i \neq x_j \) for \( i \neq j \) then \( \mu x = \frac{1}{2^k} \).
At the same time \( \varphi_0(x) = \phi(x_1) \land \phi(x_2) \land \ldots \land \phi(x_k) \). \( \varphi_0(x) = \phi(x_1) \land \phi(x_2) \land \ldots \land \phi(x_k) \).

Asadar
\[
\nu \varphi_0(x) = \frac{1}{2^k} = \mu x.
\]

We must take into account the fact that each element of \( \xi_0 \). It is a finite sum of elements of disjoint form (1). Thus, by Theorem 5, there is a constant measure of izomorphismului \( \xi \) on \( S \).

II. Now let’s consider a typical situation when there are two pairs of regular subalgebras \( \tilde{\xi}, \tilde{\xi}' \subset \xi \) and \( \tilde{y}, \tilde{y}' \subset S \) with the following properties:

1) \( \mu(\tilde{x} \land \tilde{x}') = \mu_{\tilde{x} \land \tilde{x}'} \) for any \( \tilde{x} \in \tilde{\xi} \) and \( \tilde{x}' \in \tilde{\xi}' \); in a similar way \( \nu(\tilde{y} \land \tilde{y}') = \nu_{\tilde{y} \land \tilde{y}'} \) for any \( \tilde{y} \in \tilde{S} \) and \( \tilde{y}' \in \tilde{S}' \);

2) \( \tilde{\xi} \langle \tilde{\xi}', \tilde{\xi} \rangle = \tilde{\xi} \) and \( \tilde{S} \langle \tilde{S}', \tilde{S} \rangle = \tilde{S} \).
Suppose there are isomorphisms with constant measures $\phi$ and $\phi'$ many of his $\xi$ on $\mathcal{A}$ and that of $\xi'$ on $\mathcal{A}'$. We show that in this case there is an isomorphism with the constant measure $\phi$. None of his $\xi$ on $\mathcal{A}$ by extension of $\phi$ and $\phi'$.

To this end, we introduce subalgebrele $\xi_0 = \xi(\tilde{x}, \tilde{x'})$ and $\mathcal{A}_0 = \mathcal{A}(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}')$. To this end, we introduce subalgebrele $\xi_0 = \xi(\tilde{x}, \tilde{x'})$ and $\mathcal{A}_0 = \mathcal{A}(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}')$. If $\xi_0 = \xi$ and $\mathcal{A}_0 = \mathcal{A}$, it is sufficient to construct an isomorphism with the constant measure of $\xi_0$ pe $\mathcal{A}_0$.

**CONCLUSIONS**

The main issue is topical because it approaches the fuzzy systems underlying the artificial intelligence that is implemented in the economic and industrial machines.

**REFERENCES**

[1] Balbes si Dwinger, The curators of the University of Missouri, 1974