VOLTAGE BASED SOLUTION OF TRANSIENTS IN LINEAR ELECTRICAL CIRCUITS USING THE LAPLACE TRANSFORMATION

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ABSTRACT: The paper inhere presents the implementation of voltage based solutions of transients using the Laplace transformation in the study course of “Basic principles of electrical engineering II”. The proposed approach is intended to expand and improve the knowledge, the skills, and the understanding of students regarding finding the solutions of transients in linear electrical circuits in the Laplace domain.

KEY WORDS: transients, linear circuit, Laplace transform.

INTRODUCTION

In linear circuits with energy storage elements (inductors and capacitors), voltages and currents are calculated as solutions of linear differential equations with constant coefficients, found by application of Kirchhoff’s laws for the circuit after the commutation. Engineers almost never solve the differential equations directly.

An easier approach is to use the primary Laplace transformation which actually transforms the system of differential equations regarding the voltages and the currents in the time domain into a completely algebraic system of linear equations which consists same magnitudes, but regarded as functions of the complex operator $p = a + jb$. In this way the solution of the system regarding a certain current or voltage within the circuit is easily calculated in the Laplace domain. To get a time domain solution, the reverse Laplace transform is used.

According to a number of literature sources [1, 2, 3] the application of the Operator method engages current based solutions applying mostly the Kirchhoff’s laws and the Loop current method in the “$p$” domain. The Syllabus for university students in the course “Basic principles of electrical engineering II” also discusses only the application of the Operator method for finding of current based solutions regarding transients in linear electrical circuits.

The current paper offers an approach for introducing of voltage based solutions using the Operator method and the Laplace transform that will help students to better understand the nature of the transient processes in linear electrical circuits.

VOLTAGE DROPS AND EQUIVALENT OPERATOR CIRCUITS OF IDEAL PASSIVE ELEMENTS. TRAVIAL CASES

According to the theory the following primary Laplace transformations of simple functions are used in regard with the application in electrical circuits [3, 4]:

- resistor - $R$

It is obvious that the resistor in the time domain has the same resistance as in the Laplace domain (fig.1):

$$u_R(t) = R \cdot i_R(t) \tag{1}$$
\[
U_R (p) = R \cdot i_R (p)
\]

Time domain

\[
u_R(t)
\]

\[
i_R(t)
\]

\[
I_R(p)
\]

Laplace domain

Figure 1. Resistor in the time domain and its operator equivalent circuit

• inductor – L (fig. 2)

The voltage of an inductor in the time domain is calculated as:

\[
u_L(t) = L \cdot \frac{di_L(t)}{dt}
\]

(2)

Using the Laplace transform, the inductor’s voltage in the Laplace domain is:

\[
U_L (p) = pL \cdot I_L (p) - L \cdot i_L (0)
\]

(3)

Laplace domain

Figure 2. Inductor in the time domain and its operator equivalent circuit

\[
u_L(t)
\]

\[
i_L(t)
\]

\[
U_L(p)
\]

\[
i_L(0)
\]

\[
L \cdot i_L(0)
\]


• capacitor – C (fig. 3)

The voltage of a capacitor in the time domain is calculated as:

\[
u_C(t) = \frac{1}{C} \int_{0}^{t} i_C (t) \, dt + u_C (0)
\]

(4)

Using the Laplace transform, the capacitor’s voltage in the Laplace domain is:

\[
U_C (p) = \frac{1}{pC} \cdot I_C (p) + \frac{u_C (0)}{p}
\]

(5)

Laplace domain

Figure 3. Capacitor in the time domain and its operator equivalent circuit

\[
u_C(t)
\]

\[
i_C(t)
\]

\[
U_C(p)
\]

\[
i_C(0)
\]

\[
1\frac{1}{pC}
\]

\[
-u_C(0)
\]

\[
-u_C(0)
\]

\[
p
\]


The above equivalents of ideal elements in the Laplace domain are widely used for creation of overall equivalent circuits, implemented in current based solutions of transients using Kirchhoff’s laws or the Method of loop currents.

In order to improve the knowledge, the understanding and skills of students in the discipline “Basic principles of electrical engineering II” it is a good approach to derive and solve the system linear equations in the Laplace domain not only regarding the transient currents in the branches of a circuit, but also to find voltage based solutions in the Laplace domain.

In order to find a voltage based solution of the transient process, the following rearrangements are made regarding the momentary values of the electrical magnitudes in the time domain and their operator equivalents:

• resistor – R – no change. In some cases it is better to work with resistor’s conductance - G
\[ i_R(t) = \frac{u_R(t)}{R} \]  
\[ i_R(p) = \frac{1}{R} \cdot U_R(p) = G \cdot U_R(p) \]  

- inductor – \( L \) (fig. 4) 
If the current in the time domain is expressed as a function of the voltage upon the inductor:
\[ i_L(t) = \frac{1}{L} \int_0^t u_L(t) \, dt + i_L(0) \]  

The inductor’s current in the Laplace domain therefore could be obtained by rearrangement of (3):
\[ U_L(p) = pL \cdot I_L(p) - L \cdot i_L(0) \]
\[ \Downarrow \]
\[ pL \cdot I_L(p) = U_L(p) + L \cdot i_L(0) / : pL \]

\[ I_L(p) = \frac{U_L(p)}{pL} + \frac{i_L(0)}{p} \]  
\[ I_L(p) = \frac{U_L(p)}{pL} + \frac{i_L(0)}{p} \]  
(9) 

The obtained expression represents the Kirchhoff’s 1-st law regarding the current of an inductor in the Laplace domain. The term \( \frac{i_L(0)}{p} \) then could be substituted in the equivalent operator circuit with an impulsive current source in parallel (fig. 4) 

\[ U_c(p) = \frac{1}{pC} \cdot I_c(p) + \frac{u_c(0)}{p} \]
\[ \Downarrow \]
\[ \frac{1}{pC} \cdot I_c(p) = U_c(p) - \frac{u_c(0)}{p} / : pC \]  

\[ I_c(p) = \frac{pC \cdot U_c(p) - \frac{R}{C} \cdot u_c(0)}{R} \]

\[ I_c(p) = \frac{pC \cdot U_c(p) - C \cdot u_c(0)}{} \]

The obtained expression represents the Kirchhoff’s 1-st law regarding the current of a capacitor in the Laplace domain. The term \(- C \cdot u_c(0)\) then could be substituted in the equivalent operator circuit with an impulsive current source in parallel (fig. 5) 

\[ \text{Figure 4. Capacitor in the time domain and its equivalent circuit according to the Kirchhoff’s I-st law} \]

\[ \text{APPLICATION IN REAL ELECTRICAL CIRCUITS} \]

The expressions derived in the previous chapter and the operator equivalent circuits of ideal elements will be demonstrated in two simple instances of transient processes solved using the Operator method regarding the transient voltage of an inductor \( u_L(t) = ? \) and the transient voltage of a capacitor \( u_c(t) = ? \). 

\[ \text{Example 1} \]
Find the transient voltage $u_L(t) = ?$ upon the inductor if: $E_1 = 200 \text{ V}$, $E_2 = 300 \text{ V}$, $R_1 = 10 \text{ Ω}$, $R_2 = 20 \text{ Ω}$, $R_3 = 5 \text{ Ω}$, $L = 400 \text{ mH}$

**Solution:**

1) Independent initial conditions for the circuit prior the commutation:

$$i(0) = \frac{E_1}{R_1 + R_2} = \frac{200}{10 + 15} = 13.3 \text{ A}$$

2) Equivalent operator circuit after the commutation:

3) Two equations for the respective operator equivalent:

- Kirchhoff's 1\textsuperscript{st} law for node 1:

$$I(p) = \frac{U_L(p) + i(0)}{pL}$$

- Kirchhoff's 2\textsuperscript{nd} law for loop $\Sigma$:

$$(R_2 + R_3)I(p) + U_L(p) = -\frac{E_2}{p}$$

4) Both equations are combined in a system:

$$\begin{cases} I(p) = \frac{U_L(p) + i(0)}{pL} \\ (R_2 + R_3)I(p) + U_L(p) = -\frac{E_2}{p} \end{cases}$$

5) $I(p)$ from the first eq. is substituted in the second eq.
Find the transient voltage \( u_c(t) \) upon the capacitor if:

\[ E_1 = 200 \text{ V}, \quad E_2 = 300 \text{ V}, \quad R_1 = 10 \Omega, \quad R_2 = 20 \Omega, \quad R_3 = 5 \Omega, \quad C = 250 \mu F \]

**Solution:**

1) Independent initial conditions for the circuit prior the commutation:

\[ u_c(0) = E_1 = 200 \text{ V} \]

2) Equivalent operator circuit after the commutation:

3) Two equations for the respective operator equivalent:

- Kirchhoff’s 1\textsuperscript{st} law for node 1:

\[ I(p) = \frac{U_c(p)}{pC} - C \cdot u_c(0) \]

- Kirchhoff’s 2\textsuperscript{nd} law for loop \( \Sigma \):

\[ (R_2 + R_3)I(p) + U_c(p) = -\frac{E_2}{p} \]

4) Both equations are combined in a system:

\[
\begin{align*}
I(p) &= pC \cdot U_c(p) - C \cdot u_c(0) \\
(R_2 + R_3)I(p) + U_c(p) &= -\frac{E_2}{p}
\end{align*}
\]

5) \( I(p) \) from the first eq. is substituted in the second eq.:

\[
(R_2 + R_3)[pC \cdot U_c(p) - C \cdot u_c(0)] + U_c(p) = -\frac{E_2}{p}
\]

\[
(R_2 + R_3) \cdot pC \cdot U_c(p) - (R_2 + R_3)C \cdot u_c(0) + U_c(p) = -\frac{E_2}{p}
\]

6) An expression is derived regarding \( u_c(p) \):

\[
U_c(p)[(R_2 + R_3) \cdot pC + 1] = -\frac{E_2 + pC(R_2 + R_3) \cdot u_c(0)}{p}
\]

\[
U_c(p) = \frac{-E_2 + pC(R_2 + R_3) \cdot u_c(0)}{p \cdot [(R_2 + R_3) \cdot pC + 1]} = \frac{G(p)}{H(p)}
\]

7) Laplace to time domain conversion is made:

- The roots of \( H(p) = 0 \) are found:

\[ p \cdot \left[ (R_2 + R_3) \cdot pC + 1 \right] = 0 \]

\[ \frac{p_1}{p} = 0; \quad \frac{p_2}{p} = -\frac{1}{(R_2 + R_3) \cdot C} \]

\[ p_2 = -\frac{1}{(20 + 5) \cdot 250 \cdot 10^{-6}} = -160 \]

- The first derivative \( H'(p) \) is found:

\[ H'(p) = \left\{ p \cdot \left[ (R_2 + R_3) \cdot pC + 1 \right] \right\}' = 2p(R_2 + R_3) + 1 \]

- The time domain solution about \( u_c(t) \) is:

\[ u_c(t) = \frac{G(p_1)}{H'(p_1)} \cdot e^{p_1 \cdot t} + \frac{G(p_2)}{H'(p_2)} \cdot e^{p_2 \cdot t} \]

\[ u_c(t) = \frac{G(0)}{H'(0)} + \frac{G(-160)}{H'(-160)} \cdot e^{-160 \cdot t} = \]

\[ = -\frac{E_2}{p} + \frac{-E_2 + p_2 \cdot (R_2 + R_3)C \cdot u_c(0)}{1} \cdot e^{p_2 \cdot t} = \]

\[ = -\frac{300}{1} - \frac{300 + (-160)(20 + 5) \cdot 250 \cdot 10^{-6} \cdot 200 \cdot e^{-160 \cdot t}}{2(-160)(20 + 5) \cdot 250 \cdot 10^{-6} + 1} \cdot e^{160 \cdot t} \]

\[ u_c(t) = -300 + 500 \cdot e^{-160 \cdot t} \text{ V} \]

**CONCLUSIONS**

The derived equivalent operator circuits and their implementation in the Operator method could be successfully included and
applied in the syllabus of the discipline “Basic principles of electrical engineering II”. That will improve the knowledge and the understanding of students about transient processes in linear electrical circuits.

As a result, voltage based solutions of transient processes in complex linear circuits with energy storage elements in the Laplace domain could be solved by using not only current based methods, but also by using of voltage base methods such as the method of node potentials etc.

REFERENCES