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EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS TO A QUASILINEAR ELLIPTIC PROBLEM IN $\mathbb{R}^N$

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Abstract. We prove the existence of a unique positive solution to the problem

$$-\Delta_p u = a(x)f(u)$$

in $\mathbb{R}^N$, $N > 2$. Our result extended previous works by Cirstea-Radulescu and Dinu, while the proofs are based on two theorems on bounded domains, due to Diaz-Saà and Goncalves-Santos.

1. Introduction

Our purpose in this paper is to study the problem

$$\begin{gathered} -\Delta_p u = a(x)f(u) \quad \text{in } \mathbb{R}^N \\ u > 0 \quad \text{in } \mathbb{R}^N \\ u(x) \to 0 \quad \text{as } |x| \to \infty\,, \end{gathered} \tag{1.1}$$

where $N > 2$, $\Delta_p u$, $(1 < p < \infty)$ is the p-Laplacian operator and the function $a(x)$ satisfies the following hypotheses:

- (A1) $a(x) \in C^{0,\alpha}_{\rm loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$;
- (A2) $a(x) > 0$ in $\mathbb{R}^N$;
- (A3) For $\Phi(r) = \max_{|x|=r} a(x)$ and $p < N$,

$$\int_0^\infty r^{1/(p-1)}\Phi^{1/(p-1)}(r)dr < \infty \quad \text{if } 1 < p \le 2$$

$$\int_0^\infty r^{\frac{(p-2)N+1}{p-1}}\Phi(r)dr < \infty \quad \text{if } 2 \le p < \infty\,.$$

This problem has been studied extensively in the case $p = 2$ and $f(u) = u^{-\gamma}$, with $\gamma > 0$. Lazer and McKenna [12] studied the special case when $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ is a bounded domain with smooth boundary. They proved the existence and the uniqueness of a positive solution $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ with homogeneous Dirichlet boundary condition, provided that $a(x) \in C^\alpha(\overline{\Omega})$ and $a(x) > 0$ for all $x \in \overline{\Omega}$.

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The existence of entire positive solutions on $\mathbb{R}^N$ for $\gamma \in (0,1)$ and under certain additional hypotheses has been established by Edelson [7] and Kusano-Swanson [10].

Kusano-Swanson proved that the problem (1.1) has an entire positive solution in $\mathbb{R}^2$ with logarithmic growth at ∞ if $a(x) > 0$, $x > 0$, $a(x) \in C(0,\infty)$ and

$$\int_e^\infty t(Logt)^{-\gamma}\big(\max_{|x|=t} a(x)\big)dt < \infty.$$

Edelson proved the existence of a solution provided that

$$\int_1^\infty r^{N-1+\gamma(N-2)}(\max_{|x|=t} a(x))dt < \infty,$$

for some $\gamma \in (0,1)$. This result is generalized for any $\gamma > 0$ via the sub- and super solutions method in Shaker [13] and by other methods by Dalmasso [4].

Shaker proved that problem (1.1) with $p = 2$ and $f(u) = u^{-\gamma}$, $\gamma > 0$ has an entire positive solution $u(x)$ such that $c_1 \le u(x)|x|^{q|N-2|} \le c_2$ for some c_1, c_2 and $0 < q < 1$ as $x \to \infty$ if

(1) $a(x) \in C^\alpha_{\rm loc}(\mathbb{R}^N)$, $a(x) > 0$ for $x \in \mathbb{R}^N\setminus\{0\}$;
(2) There exists $0 < c < 1$ such that $c\Phi(|x|) \le a(x) \le \Phi(|x|)$ where $\Phi(r) := \max_{|x|=r} a(x)$, $r \in [0,\infty)$;
(3) $\int_1^\infty r^{N-1+\gamma(N-2)}(\max_{|x|=t} a(x))dt < \infty$.

Lair and Shaker continued in [11] the study of (1.1) for $p = 2$ and $f(u) = u^{-\gamma}$, $\gamma > 0$. Under the above conditions the authors proved the existence of a unique positive solution $u \in C^{2,\alpha}_{\rm loc}(\mathbb{R}^N)$ vanishing at infinity to this special problem.

Zhang [14], imposed the following condition to guarantee the existence of positive solutions to problem (1.1):

(A4) $f \in C^1((0,\infty),(0,\infty))$, $\lim_{s\searrow 0+} \lim f(s) = \infty$, and $f'(s) < 0$, for all $s \in (0,\infty)$, namely, f is strictly decreasing in $(0,\infty)$.

Under the above condition Zhang's proved that problem (1.1) has a unique positive solution, $u \in C^{2+\alpha}_{\rm loc}(\mathbb{R}^N)$, vanishing at infinity.

Cirstea-Radulescu [2] and Dinu [6] extended the results of Lair, Shaker and Zhang for the case of a nonlinearity that is not necessarily decreasing on $(0,\infty)$.

Our aim is to extend the results of Cirstea-Radulescu and Dinu in the sense that $1 < p < \infty$. More exactly, let $f : (0,\infty) \to (0,\infty)$ be a C^1 function that satisfies the following assumptions:

(F1) There exists $\beta > 0$ such that the mapping $u \mapsto f(u)/(u+\beta)^{p-1}$ is decreasing on $(0,\infty)$
(F2) $\lim_{u\searrow 0} f(u)/u^{p-1} = +\infty$ and f is bounded in a neighbourhood of $+\infty$.

Our main results are the following:

Theorem 1.1. *Under hypotheses (F1), (F2), (A1), (A2), (A3), problem* (1.1) *has a unique positive global solution vanishing at infinity.*

Theorem 1.2. *Suppose $a(r)$ is a positive radial function which is continuous on $\mathbb{R}^N$ and fulfills*

$$\int_0^\infty r^{1/(p-1)}a^{1/(p-1)}(r)dr = \infty \quad \text{if } 2 \le p < \infty$$

Then (1.1) *has no positive radial solution.*

Theorem 1.3. *Problem* (1.1) *has no positive radial solution if* $p \geq N$.

Theorem 1.4. *Suppose* $a(r)$ *is a positive radial function which is continuous on* $\mathbb{R}^N$ *and*

$$\int_0^\infty r^{\frac{(p-2)N+1}{p-1}} a(r)dr = \infty \quad \text{if } 1 < p \leq 2.$$

Then (1.1) *has no positive radial solution.*

2. Uniqueness

Suppose u and v are arbitrary solutions of problem (1.1). Let us show that $u \leq v$ or, equivalently, $\ln(u(x)+\beta) \leq \ln(v(x)+\beta)$, for any $x \in \mathbb{R}^N$. Assume the contrary. Since, we have

$$\lim_{|x|\to\infty} (\ln(u(x)+\beta) - \ln(v(x)+\beta)) = 0,$$

we deduce that

$$\max_{\mathbb{R}^N} (\ln(u(x)+\beta) - \ln(v(x)+\beta))$$

exists and is positive. At that point, say x_0, we have

$$\nabla(\ln(u(x_0)+\beta) - \ln(v(x_0)+\beta)) = 0,$$

so

$$\frac{1}{u(x_0)+\beta} \cdot \nabla u(x_0) = \frac{1}{v(x_0)+\beta} \cdot \nabla v(x_0),$$

and

$$\frac{1}{(u(x_0)+\beta)^{p-2}} \cdot |\nabla u(x_0)|^{p-2} = \frac{1}{(v(x_0)+\beta)^{p-2}} \cdot |\nabla v(x_0)|^{p-2}. \tag{2.1}$$

By (f1) we obtain

$$\frac{f(u(x_0))}{(u(x_0)+\beta)^{p-1}} < \frac{f(v(x_0))}{(v(x_0)+\beta)^{p-1}}. \tag{2.2}$$

Since $0 \geq \Delta(\ln(u(x_0)+\beta) - \ln(v(x_0)+\beta))$, it follows that

$$\frac{\Delta u(x_0)}{u(x_0)+\beta} \leq \frac{\Delta v(x_0)}{v(x_0)+\beta},$$

so

$$\frac{1}{(u(x_0)+\beta)^{p-1}} \cdot |\nabla u(x_0)|^{p-2} \Delta u(x_0) \leq \frac{1}{(v(x_0)+\beta)^{p-1}} \cdot |\nabla v(x_0)|^{p-2} \Delta v(x_0)\,. \tag{2.3}$$

Since

$$|\nabla \ln(u(x_0)+\beta)|^{p-2} = \frac{1}{(u(x_0)+\beta)^{p-2}} \cdot |\nabla u(x_0)|^{p-2},$$

it follows that

$$\begin{aligned}
&\nabla(|\nabla \ln(u(x_0)+\beta)|^{p-2}) \\
&= -(p-2)\frac{|\nabla u(x_0)|^{p-2}(u(x_0)+\beta)^{p-3}}{(u(x_0)+\beta)^{2(p-2)}} \cdot \nabla u(x_0) + \frac{\nabla(|\nabla u(x_0)|^{p-2})}{(u(x_0)+\beta)^{p-2}}.
\end{aligned}$$

Then

$$\begin{aligned}
&\nabla(|\nabla \ln(u(x_0)+\beta)|^{p-2}) \cdot \nabla(\ln(u(x_0)+\beta)) \\
&= -(p-2)\frac{|\nabla u(x_0)|^{p-2}|\nabla u(x_0)|^2}{(u(x_0)+\beta)^p} + \frac{\nabla(|\nabla u(x_0)|^{p-2}) \cdot \nabla u(x_0)}{(u(x_0)+\beta)^{p-1}}
\end{aligned} \tag{2.4}$$

and

$$|\nabla \ln(u(x_0)+\beta)|^{p-2}\Delta(\ln(u(x_0)+\beta)) = \frac{|\nabla u(x_0)|^{p-2}\Delta u(x_0)}{(u(x_0)+\beta)^{p-1}} - \frac{|\nabla u(x_0)|^p}{(u(x_0)+\beta)^p}.$$

So, by (2.1), (2.2), (2.3) and (2.4) we have

$$\begin{aligned}
0 &\geq \Delta_p(\ln(u(x_0)+\beta)) - \Delta_p(\ln(v(x_0)+\beta))\\
&= \frac{\Delta_p u(x_0)}{(u(x_0)+\beta)^{p-1}} - (p-1)\frac{|\nabla u(x_0)|^p}{(u(x_0)+\beta)^p} - \frac{\Delta_p v(x_0)}{(v(x_0)+\beta)^{p-1}}\\
&\quad + (p-1)\frac{|\nabla v(x_0)|^p}{(v(x_0)+\beta)^p}\\
&= \frac{\Delta_p u(x_0)}{(u(x_0)+\beta)^{p-1}} - \frac{\Delta_p v(x_0)}{(v(x_0)+\beta)^{p-1}}\\
&= -a(x_0)\Big(\frac{f(u(x_0))}{(u(x_0)+\beta)^{p-1}} - \frac{f(v(x_0))}{(v(x_0)+\beta)^{p-1}}\Big) > 0
\end{aligned}$$

which is a contradiction. Hence $u \leq v$. By symmetry we also have $v \leq u$, and the proof is complete.

3. Existence of a solution

We first show that our hypothesis (F1) implies $\lim_{u\searrow 0} f(u)$ exists, finite or $+\infty$. Indeed, since $\frac{f(u)}{(u+\beta)^{p-1}}$ is decreasing, there exists $L := \lim_{u\searrow 0}\frac{f(u)}{(u+\beta)^{p-1}} \in (0,+\infty]$. It follows that $\lim_{u\searrow 0} f(u) = L\beta^{p-1}$.

To prove the existence of a solution to Problem (1.1), we need to employ a corresponding result by Diaz-Saà [5] for bounded domains. They considered the problem

$$\begin{gathered}
-\Delta_p u = g(x,u) \quad \text{in } \Omega\\
u \geq 0 \quad \text{in } \Omega\\
u(x) = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $g(x,u) : \Omega \times [0,\infty) \to \mathbb{R}$.

Assume that

-for a.e. $x \in \Omega$ the function $u \to g(x,u)$ is continuous on $[0,\infty)$

and the function $u \to g(x,u)/u^{p-1}$ is decreasing on $(0,\infty)$; (3.2)

-for each $u \geq 0$ the function $x \to g(x,u)$ belongs to $L^\infty(\Omega)$; (3.3)

-there exists $C > 0$ such that $g(x,u) \leq C(u^{p-1}+1)$ a.e. $x \in \Omega$, for all $u \geq 0$. (3.4)

Under these hypotheses on g, Diaz-Saà [5] proved that there is at most one solution of (1.1).

Let us consider the problem

$$\begin{gathered}
-\Delta_p u_k = a(x)f(u_k), \quad \text{if } |x| < k,\\
u_k(x) = 0, \quad \text{if } |x| = k.
\end{gathered} \tag{3.5}$$

The following two distinct situations may occur:

Case 1: f is bounded on $(0,+\infty)$. In this case, as we have initially observed, $\lim_{u\searrow 0} f(u)$ exists and is finite, so f can be extended by continuity at the origin.

To obtain a solution to (3.5), it is sufficinet to verify that the hypotheses of the Diaz-Saà theorem are fulfilled.

* Since $f \in C^1((0,\infty),(0,\infty))$ it follows that the mapping $u \to a(x)f(u)$ is continuous in $[0,\infty)$.

* From $a(x)\frac{f(u)}{u^{p-1}} = a(x)\frac{f(u)}{(u+\beta)^{p-1}} \cdot \frac{(u+\beta)^{p-1}}{u^{p-1}}$, using positivity of a and (F1) we deduce that the function $u \to a(x)\frac{f(u)}{u^{p-1}}$ is decreasing on $(0,\infty)$.

* For all $u \geq 0$, since $a(x) \in C^{0,\alpha}_{\rm loc}(\mathbb{R}^N)$, we obtain $x \to a(x)f(u)$ belongs to $L^\infty(\Omega)$.

* By $\lim_{u\to\infty} \frac{f(u)}{u^{p-1}+1} = \lim_{u\to\infty} \frac{f(u)}{u^{p-1}} \cdot \frac{u^{p-1}}{u^{p-1}+1} = 0$ and $f \in C^1((0,\infty),(0,\infty))$, there exists $C > 0$ such that $f(u) \leq C(u^{p-1}+1)$ for all $u \geq 0$. Therefore, $a(x)f(u) \leq C(u^{p-1}+1)$ for all $u \geq 0$.

* Observe that $a_0(x) = \lim_{u\searrow 0} \frac{p(x)f(u)}{u^{p-1}} = +\infty$ and $a_\infty(x) = \lim_{u\to+\infty} \frac{p(x)f(u)}{u^{p-1}} = 0$. Thus by Diaz-Saa, problem (3.5) has a unique solution u_k which, by the maximum principle, is positive in $|x| < k$.

Case 2. $\lim_{u\searrow 0} f(u) = +\infty$. We will apply the method of sub- and supersolutions in order to find a solution to the problem (3.5). We first observe that 0 is a subsolution for this problem.

We construct in what follows a positive supersolution. By the boundedness of f in a neighbourhood of $+\infty$, there exists $A > 0$ such that $f(u) \leq A$, for any $u \in (1,+\infty)$. Let $f_0 : (0,1] \to (0,+\infty)$ be a continuous nonincreasing function such that $f_0 \geq f$ on $(0,1]$. We can assume without loss of generality that $f_0(1) = A$. Set

$$g(u) = \begin{cases} f_0(u), & \text{if } 0 < u \leq 1,\\ A, & \text{if } u > 1. \end{cases}$$

Then g is a continuous nonincreasing function on $(0,+\infty)$. Let $h : (0,+\infty) \to (0,+\infty)$ be a C^1 nonincreasing function such that $h \geq g$. Thus by in [8, Theorem 1.3] the problem

$$\begin{gathered} -\Delta_p U = p(x)h(U), \quad \text{if } |x| < k,\\ U = 0, \quad \text{if } |x| = k. \end{gathered}$$

has a positive solution. Now, since $h \geq f$ on $(0,+\infty)$, it follows that U is supersolution of (3.5).

In both cases studied above we define $u_k = 0$ for $|x| > k$. Using a comparision principle argument as already done above for proving the uniqueness, we can show that $u_k \leq u_{k+1}$ on $\mathbb{R}^N$.

We now justify the existence of a continuous function $v : \mathbb{R}^N \to \mathbb{R}$ such that $u_k \leq v$ in $\mathbb{R}^N$. We first construct a positive radially symmetric function w such that $-\Delta_p w = \Phi(r)$, $(r = |x|)$ on $\mathbb{R}^N$ and $\lim_{r\to\infty} w(r) = 0$. A straightforward computation shows that

$$w(r) := K - \int_0^r \Big[\xi^{1-N}\int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)} d\xi,$$

where

$$K = \int_0^\infty \Big[\xi^{1-N}\int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)} d\xi.$$

We first show that (A3) implies that

$$\int_0^{+\infty}\Big[\xi^{1-N}\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)}d\xi,$$

is finite.

Theorem 3.1. *If $j : I \subseteq R \to R$ is a locally integrable nonnegative function, then*

$$\Big(\frac{1}{b-a}\int_a^b j(x)dx\Big)^h \le \text{ (resp. } \ge) \ \frac{1}{b-a}\int_a^b j^h(x)dx$$

for all $a, b \in I$, $a < b$ and $1 \le h < +\infty$ (resp $0 < h \le 1$)

Case 1: Let $1 < p \le 2$, so $0 < p-1 \le 1$, follows that $1 \le \frac{1}{p-1} < +\infty$. By Theorem 3.1 for any $r > 0$, we have

$$\begin{aligned}
&\int_0^r \xi^{\frac{1-N}{p-1}}\Big[\frac{\xi}{\xi}\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)}d\xi\\
&= \int_0^r \xi^{\frac{1-N}{p-1}}\xi^{1/(p-1)}\Big[\frac{1}{\xi}\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)}d\xi\\
&\le \int_0^r \xi^{\frac{2-N}{p-1}}\frac{1}{\xi}\int_0^{\xi}\sigma^{\frac{N-1}{p-1}}\Phi^{1/(p-1)}(\sigma)d\sigma d\xi\\
&= \int_0^r \xi^{\frac{2-N}{p-1}-1}\int_0^{\xi}\sigma^{\frac{N-1}{p-1}}\Phi^{1/(p-1)}(\sigma)d\sigma d\xi\\
&= -\frac{p-1}{N-2}\int_0^r \frac{d}{d\xi}\xi^{\frac{2-N}{p-1}}\int_0^{\xi}\sigma^{\frac{N-1}{p-1}}\Phi^{1/(p-1)}(\sigma)d\sigma d\xi\\
&= \frac{p-1}{N-2}\Big[-r^{\frac{2-N}{p-1}}\int_0^r \sigma^{\frac{N-1}{p-1}}\Phi^{1/(p-1)}(\sigma)d\sigma + \int_0^r \xi^{1/(p-1)}\Phi^{1/(p-1)}(\xi)d\xi\Big].
\end{aligned}$$

Now, by L'Hôpital's rule, we have

$$\begin{aligned}
&\lim_{r\to\infty}\Big[-r^{\frac{2-N}{p-1}}\int_0^r \sigma^{\frac{N-1}{p-1}}\Phi^{1/(p-1)}(\sigma)d\sigma + \int_0^r \xi^{1/(p-1)}\Phi^{1/(p-1)}(\xi)d\xi\Big]\\
&= \lim_{r\to\infty}\frac{-\int_0^r \sigma^{\frac{N-1}{p-1}}\Phi^{1/(p-1)}(\sigma)d\sigma + r^{\frac{N-2}{p-1}}\int_0^r \xi^{1/(p-1)}\Phi^{1/(p-1)}(\xi)d\xi}{r^{\frac{N-2}{p-1}}}\\
&= \lim_{r\to\infty}\int_0^r \xi^{\frac{1}{p-1}}\Phi^{1/(p-1)}(\xi)d\xi\\
&= \int_0^{\infty}\xi^{1/(p-1)}\Phi^{1/(p-1)}(\xi)d\xi < \infty,
\end{aligned}$$

Case 2: Let $2 \le p < +\infty$, so $1 \le p-1$, it follows that $1 \ge \frac{1}{p-1} > 0$. Set

$$\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma \le 1 \quad \text{for } \xi > 0, \quad \text{or}$$

$$\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma > 1 \quad \text{for } \xi > 0,$$

In the first case

$$\Big[\int_0^{\xi}\sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)} \le 1,$$

so

$$\int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)} d\xi \le \int_0^r \xi^{\frac{1-N}{p-1}} d\xi$$

is finite as $r \to \infty$ and $N > p$. In the second case,

$$\Big[\int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)} \le \int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma$$

for $\xi \ge 0$, so

$$\int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma\Big]^{1/(p-1)} d\xi \le \int_0^r \xi^{\frac{1-N}{p-1}} \int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma d\xi\,.$$

Integration by parts gives

$$\begin{aligned}
&\int_0^r \xi^{\frac{1-N}{p-1}} \int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma d\xi \\
&= -\frac{p-1}{N-p}\int_0^r \frac{d}{d\xi}\xi^{\frac{p-N}{p-1}} \int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma d\xi \\
&= \frac{p-1}{N-p}(-r^{\frac{p-N}{p-1}} \int_0^r \sigma^{N-1}\Phi(\sigma)d\sigma + \int_0^r \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi)\,.
\end{aligned}$$

Now, by L' Hôpital's rule, we have

$$\begin{aligned}
&\lim_{r\to\infty}\Big[-r^{\frac{p-N}{p-1}}\int_0^r \sigma^{N-1}\Phi(\sigma)d\sigma + \int_0^r \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi\Big] \\
&= \lim_{r\to\infty} \frac{-\int_0^r \sigma^{N-1}\Phi(\sigma)d\sigma + r^{\frac{N-p}{p-1}}\int_0^r \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi}{r^{\frac{N-p}{p-1}}} \\
&= \lim_{r\to\infty}\int_0^r \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi \\
&= \int_0^\infty \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi < \infty,
\end{aligned}$$

From cases 1 and 2 above, it follows that

$$\begin{gathered}
K = \frac{p-1}{N-2}\cdot\int_0^\infty \xi^{\frac{1}{p-1}}\Phi^{1/(p-1)}(\xi)d\xi \text{ if } 1<p<2, \text{ or} \\
K = \frac{p-1}{N-p}\cdot\int_0^\infty \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi \quad \text{if } 2 \le p < +\infty.
\end{gathered}$$

Clearly, for all $r > 0$,

$$\begin{gathered}
w(r) < \frac{p-1}{N-2}\cdot\int_0^\infty \xi^{1/(p-1)}\Phi^{1/(p-1)}(\xi)d\xi \quad \text{if } 1<p\le 2, \text{ or} \\
w(r) < \frac{p-1}{N-p}\cdot\int_0^\infty \xi^{\frac{(p-2)N+1}{p-1}}\Phi(\xi)d\xi \quad \text{if } 2 \le p < +\infty,
\end{gathered}$$

An upper-solution to (1.1) will be constructed. Consider the function $\overline{f}(u) = (f(u)+1)^{1/(p-1)}$, for $u > 0$.

Note that the hypothesis $u \to f(u)/(u+\beta)^{p-1}$ is a decreasing function on $(0,\infty)$ implies that $u \to f(u)/u^{p-1}$ is a decreasing function on $(0,\infty)$, because $\frac{v+\beta}{v} \le \frac{u+\beta}{u} \Leftrightarrow vu + \beta u \le vu + v\beta \Leftrightarrow \beta(u-v) \le 0$, is true $\forall u \le v$ and $\beta > 0$. We have

(F1') $\overline{f}(u) \ge f(u)^{1/(p-1)}$

(F2') $\lim_{u\searrow 0}\overline{f}(u)/u=\infty$ and $u\mapsto \overline{f}(u))/u^{p-1}$ is decreasing on $(0,\infty)$.

Let v be a positive function such that $w(r)=\frac{1}{C}\int_0^{v(r)}t^{p-1}/\overline{f}(t)\,dt$, where C is a positive constant such that $KC\leq\int_0^{C^{1/(p-1)}}t^{p-1}/\overline{f}(t)\,dt$. We prove that we can find $C>0$ with this property. From our hypothesis (F2') we obtain that $\lim_{x\to+\infty}\int_0^x t^{p-1}/\overline{f}(t)\,dt=+\infty$. Now using L'Hôpital's rule we have

$$\lim_{x\to\infty}\frac{1}{x^{p-1}}\int_0^x\frac{t^{p-1}}{\overline{f}(t)}dt=\lim_{x\to\infty}\frac{x}{(p-1)\overline{f}(x)}=+\infty.$$

This means that there exists $x_1>0$ such that $\int_0^x t^{p-1}/\overline{f}(t)\,dt\geq Kx^{p-1}$, for all $x\geq x_1$. It follows that for any $C\geq x_1$,

$$KC\leq\int_0^{C^{1/(p-1)}}\frac{t^{p-1}}{\overline{f}(t)}dt.$$

But w is a decreasing function, and this implies that v is a decreasing function too. Then

$$\int_0^{v(r)}\frac{t^{p-1}}{\overline{f}(t)}dt\leq\int_0^{v(0)}\frac{t^{p-1}}{\overline{f}(t)}dt=C\cdot w(0)=C\cdot K\leq\int_0^{C^{1/(p-1)}}\frac{t^{p-1}}{\overline{f}(t)}dt.$$

It follows that $v(r)\leq C^{1/(p-1)}$ for all $r>0$. From $w(r)\to 0$ as $r\to+\infty$ we deduce $v(r)\to 0$ as $r\to+\infty$. By the choice of v we have

$$\nabla w=\frac{1}{C}\cdot\frac{v^{p-1}}{\overline{f}(v)}\nabla v\quad\text{and}\quad\Delta w=\frac{1}{C}\cdot\frac{v^{p-1}}{\overline{f}(v)}\Delta v+\frac{1}{C}\big(\frac{v^{p-1}}{\overline{f}(v)}\big)'|\nabla v|^2.$$

so

$$|\nabla w|^{p-2}=\frac{1}{C^{p-2}}\big(\frac{v^{p-1}}{\overline{f}(v)}\big)^{p-2}|\nabla v|^{p-2}.$$

It follows that

$$\begin{aligned}|\nabla w|^{p-2}\Delta w&=\frac{1}{C^{p-2}}\big(\frac{v^{p-1}}{\overline{f}(v)}\big)^{p-2}|\nabla v|^{p-2}\Big(\frac{1}{C}\frac{v^{p-1}}{\overline{f}(v)}\Delta v+\frac{1}{C}\big(\frac{v^{p-1}}{\overline{f}(v)}\big)'|\nabla v|^2\Big)\\&=\frac{1}{C^{p-1}}\big(\frac{v^{p-1}}{\overline{f}(v)}\big)^{p-1}|\nabla v|^{p-2}\Delta v+\frac{1}{C^{p-1}}\big(\frac{v^{p-1}}{\overline{f}(v)}\big)^{p-2}\big(\frac{v^{p-1}}{\overline{f}(v)}\big)'|\nabla v|^p,\end{aligned}$$

so

$$\begin{aligned}&\nabla(|\nabla w|^{p-2})\cdot\nabla w\\&=\Big\{\frac{1}{C^{p-2}}\big(\frac{v^{p-1}}{\overline{f}(v)}\big)^{p-2}\nabla(|\nabla v|^{p-2})+\frac{1}{C^{p-2}}\Big[\big(\frac{v^{p-1}}{\overline{f}(v)}\big)^{p-2}\Big]'|\nabla v|^{p-2}\nabla v\Big\}\cdot\big[\frac{1}{C}\cdot\frac{v^{p-1}}{\overline{f}(v)}\nabla v\big]\\&=\frac{1}{C^{p-1}}\big(\frac{v^{p-1}}{\overline{f}(v)}\big)^{p-2}\nabla(|\nabla v|^{p-2})\cdot\nabla v+\frac{1}{C^{p-1}}\Big[\big(\frac{v^{p-1}}{\overline{f}(v)}\big)^{p-2}\Big]'\frac{v^{p-1}}{\overline{f}(v)}|\nabla v|^p,\end{aligned}$$

so that

$$\begin{aligned}\Delta_p w&=\frac{1}{C^{p-1}}\big(\frac{v^{p-1}}{\overline{f}(v)}\big)^{p-1}\Big[|\nabla v|^{p-2}\Delta v+\nabla(|\nabla v|^{p-2})\cdot\nabla v\Big]\\&\quad+\frac{1}{C^{p-1}}\big(\frac{v^{p-1}}{\overline{f}(v)}\big)^{p-2}|\nabla v|^p\big(\frac{v^{p-1}}{\overline{f}(v)}\big)'+\frac{1}{C^{p-1}}\Big[\big(\frac{v^{p-1}}{\overline{f}(v)}\big)^{p-2}\Big]'\frac{v^{p-1}}{\overline{f}(v)}|\nabla v|^p\\&=\frac{1}{C^{p-1}}\big(\frac{v^{p-1}}{\overline{f}(v)}\big)^{p-1}\Delta_p v+(p-1)\frac{1}{C^{p-1}}|\nabla v|^p\big(\frac{v^{p-1}}{\overline{f}(v)}\big)^{p-2}\big(\frac{v^{p-1}}{\overline{f}(v)}\big)'.\end{aligned}\tag{3.6}$$

From (3.6) we deduce that

$$\Delta_p w = \frac{1}{C^{p-1}}(\frac{v^{p-1}}{\overline{f}(v)})^{p-1}\Delta_p v + (p-1)\frac{1}{C^{p-1}}|\nabla v|^p(\frac{v^{p-1}}{\overline{f}(v)})^{p-2}(\frac{v^{p-1}}{\overline{f}(v)})'. \tag{3.7}$$

From (3.7) and the fact that $u \to \frac{\overline{f}(u)}{u^{p-1}}$ is a decreasing function on $(0,+\infty)$, we deduce that

$$\Delta_p v \le C^{p-1}(\frac{\overline{f}(v)}{v^{p-1}})^{p-1}\Delta_p w = -C^{p-1}(\frac{\overline{f}(v)}{v^{p-1}})^{p-1}\Phi(r) \le -f(v)\Phi(r). \tag{3.8}$$

By (3.7) and (3.8) and using in an essential manner the hypothesis (F1), as already done for proving the uniqueness, we obtain that $u_k \le v$ for $|x| \le k$ and, hence, for all $\mathbb{R}^N$. Now we have a bounded increasing sequence $u_1 \le u_2 \le \dots \le u_k \le dots \le v$ with v vanishing at infinity. Thus there exists a function, say $u \le v$ such that $u_k \to u$ pointwise in $\mathbb{R}^N$. Using

$$u'(r) = \Big[r^{1-N}\int_0^r \sigma^{N-1}p(\sigma)f(u(\sigma))d\sigma\Big]^{1/(p-1)},$$

$$u''(r) = -\frac{p(r)f(u(r)) + (1-N)r^{-N}\int_0^r \sigma^{N-1}p(\sigma)f(u(\sigma))d\sigma}{p-1} \times\Big[r^{1-N}\int_0^r \sigma^{N-1}p(\sigma)f(u(\sigma))d\sigma\Big]^{\frac{2-p}{p-1}},$$

$$\frac{2-p}{p-1} \ge 0 \iff 1 < p \le 2$$

$$\lim_{r\to 0}\frac{\int_0^r \sigma^{N-1}p(\sigma)f(u(\sigma))d\sigma}{r^N} = 0$$

$$\lim_{r\to 0}\frac{\int_0^r \sigma^{N-1}p(\sigma)f(u(\sigma))d\sigma}{r^{N-1}} = 0$$

it is easy to prove that $u(r) \in C^2(\mathbb{R}^N)$ if $1 < p \le 2$ because $\lim_{r\to\infty} u''(r)$ is finite and $u(r) \in C^1(\mathbb{R}^N)$ if $2 < p < \infty$ because $\lim_{r\to\infty} u'(r)$ is finite.

4. Proof of Theorem 1.2

Suppose (1.1) has a solution $u(r)$, then

$$(r^{N-1}|u'(r)|^{p-2}u'(r))' = -r^{N-1}f(u(r))a(r),$$

integrating from 0 to r, we have

$$|u'(r)|^{p-2}u'(r) = -r^{1-N}\int_0^r \sigma^{N-1}f(u(\sigma))a(\sigma)d\sigma,$$

hence $u'(r) < 0$. We put $\ln(u(r)+1) := \overline{u}(r) > 0$ for all $r > 0$. Then we have

$$\Delta_p\overline{u}(r) = \frac{\Delta_p u(r)}{(u(r)+1)^{p-1}} - (p-1)\frac{|\nabla u(r)|^p}{(u(r)+1)^p}.$$

Then $\overline{u}(r)$ satisfies

$$\frac{1}{r^{N-1}}\Big(r^{N-1}(-\overline{u}'(r))^{p-2}\overline{u}'(r)\Big)' + (p-1)\frac{|\nabla u(r)|^p}{(u(r)+1)^p} = -\frac{f(u(r))a(r)}{(u(r)+1)^{p-1}}. \tag{4.1}$$

Multiplying (4.1) by r^{N-1} and integrating on $(0,\xi)$ yield

$$\int_0^\xi \Big((-\overline{u}'(\sigma))^{p-1}\sigma^{N-1}\Big)' d\sigma - (p-1)\int_0^\xi \frac{\sigma^{N-1}|\nabla u(\sigma)|^p}{(u(\sigma)+1)^p} d\sigma \\ = \int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma,$$

equivalently

$$(-\overline{u}'(\xi))^{p-1}\xi^{N-1} - \int_0^\xi (p-1)\frac{\sigma^{N-1}|\nabla u(\sigma)|^p}{(u(\sigma)+1)^p} d\sigma = \int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma\,. \tag{4.2}$$

Multiplying equation (4.2) by ξ^{1-N}, we deduce

$$(-\overline{u}'(\xi))^{p-1} - \xi^{1-N}(p-1)\int_0^\xi \frac{\sigma^{N-1}|\nabla u(\sigma)|^p}{(u(\sigma)+1)^p} d\sigma = \xi^{1-N}\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma. \tag{4.3}$$

From (4.3), we have

$$\big(-\overline{u}'(\xi)\big)^{p-1} \geq \xi^{1-N}\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma,$$

so

$$-\overline{u}'(\xi) \geq \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma\Big]^{1/(p-1)}, \tag{4.4}$$

integrating (4.4) on $(0,r)$, we have

$$\overline{u}(0) - \overline{u}(r) \geq \int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma\Big]^{1/(p-1)} d\xi.$$

We observe that $\overline{u}(r) < \overline{u}(0)$, for all $r > 0$ implies $u(r) < u(0)$, for all $r > 0$.

If $\beta \geq 1$, then the function $u \mapsto \frac{f(u)}{(u+\beta)^{p-1}}$ is decreasing on $(0,+\infty)$. This implies

$$\frac{f(u(\sigma))}{(u(\sigma)+1)^{p-1}} > \frac{f(u(0))}{(u(0)+1)^{p-1}}. \tag{4.5}$$

Since $\overline{u}$ is positive, we have

$$\int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma\Big]^{1/(p-1)} d\xi \leq \overline{u}(0), \quad \forall r > 0,$$

substituting (4.5) into this expression, we obtain

$$\int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi a(\sigma)\sigma^{N-1} d\sigma\Big]^{1/(p-1)} d\xi \leq \frac{u(0)+1}{f(u(0))^{\frac{1}{p-1}}}\overline{u}(0) < \infty.$$

Let $2 \le p < +\infty$, so $1 \le p-1$, follows that $1 \ge \frac{1}{p-1} > 0$. We have

$$\begin{aligned}
&\int_0^r \xi^{\frac{1-N}{p-1}} \Big[\frac{\xi}{\xi}\int_0^\xi \sigma^{N-1} a(\sigma)d\sigma\Big]^{1/(p-1)} d\xi \\
&= \int_0^r \xi^{\frac{1-N}{p-1}} \xi^{1/(p-1)} \Big[\frac{1}{\xi}\int_0^\xi \sigma^{N-1} a(\sigma)d\sigma\Big]^{1/(p-1)} d\xi \\
&\ge \int_0^r \xi^{\frac{2-N}{p-1}} \frac{1}{\xi}\int_0^\xi \sigma^{\frac{N-1}{p-1}} a^{1/(p-1)}(\sigma)d\sigma d\xi \\
&= \int_0^r \xi^{\frac{2-N}{p-1}-1} \int_0^\xi \sigma^{\frac{N-1}{p-1}} a^{1/(p-1)}(\sigma)d\sigma d\xi \\
&= -\frac{p-1}{N-2}\int_0^r \frac{d}{d\xi}\xi^{\frac{2-N}{p-1}} \int_0^\xi \sigma^{\frac{N-1}{p-1}} a^{1/(p-1)}(\sigma)d\sigma d\xi \\
&= \frac{p-1}{N-2}(-r^{\frac{2-N}{p-1}}\int_0^r \sigma^{\frac{N-1}{p-1}} a^{1/(p-1)}(\sigma)d\sigma + \int_0^r \xi^{1/(p-1)} a(\xi)^{1/(p-1)}d\xi) \\
&\ge \frac{p-1}{N-2}\frac{1}{r^{\frac{N-2}{p-1}}}\int_0^r \Big[r^{\frac{N-2}{p-1}} - (t)^{\frac{N-2}{p-1}}\Big] t^{1/(p-1)} a^{1/(p-1)}(t)dt \\
&\ge \frac{p-1}{N-2}\frac{1}{r^{\frac{N-2}{p-1}}}\Big(r^{\frac{N-2}{p-1}} - (\frac{r}{2})^{\frac{N-2}{p-1}}\Big)\int_0^{r/2} t^{1/(p-1)} a^{1/(p-1)}(t)dt \\
&= \frac{p-1}{N-2}(1-(\frac{1}{2})^{\frac{N-2}{p-1}})\int_0^{r/2} t^{1/(p-1)} a^{1/(p-1)}(t)dt \to \infty \quad \text{as } r\to\infty.
\end{aligned}$$

So

$$\infty > \frac{u(0)+1}{f(u(0))^{1/(p-1)}}\overline{u}(0) \ge \infty,$$

which is a contradiction.

If $\beta < 1$ then the function $u \mapsto \frac{(u+\beta)^{p-1}}{(u+1)^{p-1}}$ is increasing on $(0,+\infty)$. In this case

$$\begin{aligned}
\overline{u}(0) &\ge \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}}d\sigma\Big]^{1/(p-1)} d\xi \\
&= \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \frac{f(u(\sigma))a(\sigma)(u(\sigma)+\beta)^{p-1}\sigma^{N-1}}{(u(\sigma)+\beta)^{p-1}(u(\sigma)+1)^{p-1}}d\sigma\Big]^{1/(p-1)} d\xi \\
&\ge \frac{f(u(0))^{1/(p-1)}}{u(0)+\beta}\beta\int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\Big]^{1/(p-1)} d\xi,
\end{aligned}$$

which implies

$$\infty > \frac{u(0)+\beta}{f(u(0))^{1/(p-1)}\beta}\overline{u}(0) \ge \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\Big]^{1/(p-1)} d\xi \ge \infty,$$

which is a contradiction.

5. Proof of Theorem 1.3

Assume u is positive for $r > 0$ and satisfies

$$(r^{N-1}|u'(r)|^{p-2}u'(r))' = -r^{N-1}f(u(r))a(r).$$

Since $f(u(r))a(r)$ is positive for $r > 0$, follows that

$$(r^{N-1}|u'(r)|^{p-2}u'(r))' < 0, \quad \text{for } r > 0,$$

and that $r^{N-1}|u'(r)|^{p-2}u'(r)$ is a decreasing function. Because this function is decreasing and $u' < 0$,

$$r^{N-1}|u'(r)|^{p-2}u'(r) \leq -C, \quad \text{for } r \geq R,$$

where C is positive constant. As a consequence

$$-u'(r) \geq C_1 r^{-\frac{1-N}{p-1}}, \quad \text{with } C_1 > 0.$$

Integrating this inequality from R to r we have

$$u(R) - u(r) \geq C_1 \int_R^r r^{-\frac{1-N}{p-1}} dr, \quad \text{for } r \geq R.$$

Letting $r \to \infty$, we arrive at a contradiction.

6. Proof of Theorem 1.4

As in proof of Theorem 1.2, we have

$$\overline{u}(0) - \overline{u}(r) \geq \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma \Big]^{1/(p-1)} d\xi,$$

We observe that $\overline{u}(r) < \overline{u}(0)$, for all $r > 0$ implies $u(r) < u(0)$, for all $r > 0$. If $\beta \geq 1$ then the function $u \mapsto \frac{f(u)}{(u+\beta)^{p-1}}$ is decreasing on $(0, +\infty)$. This implies

$$\frac{f(u(\sigma))}{(u(\sigma)+1)^{p-1}} > \frac{f(u(0))}{(u(0)+1)^{p-1}}, \tag{6.1}$$

Since $\overline{u}$ is positive, we have

$$\int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \frac{f(u(\sigma))a(\sigma)\sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma \Big]^{1/(p-1)} d\xi \leq \overline{u}(0), \quad \forall r > 0$$

substituting 6.1 into this expression we obtain

$$\int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi a(\sigma)\sigma^{N-1} d\sigma \Big]^{1/(p-1)} d\xi \leq \frac{u(0)+1}{f(u(0))^{1/(p-1)}} \overline{u}(0) < \infty.$$

Let $1 < p < 2$, so $0 < p-1 < 1$, it follows that $1 < \frac{1}{p-1} < +\infty$. Set

$$\int_0^\xi r^{N-1}a(r)dr < 1 \quad \text{for } \xi > 0, \quad \text{or}$$

$$\int_0^\xi r^{N-1}a(r)dr \geq 1 \quad \text{for } \xi > 0,$$

In the second case, we have

$$\Big[\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma \Big]^{1/(p-1)} \geq \int_0^\xi \sigma^{N-1}a(\sigma)d\sigma,$$

so

$$\int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma \Big]^{1/(p-1)} d\xi \geq \int_0^r \xi^{\frac{1-N}{p-1}} \int_0^\xi \sigma^{N-1}a(\sigma)d\sigma d\xi\,.$$

Integration by parts gives

$$\begin{aligned}
&\int_0^r \xi^{\frac{1-N}{p-1}} \int_0^\xi \sigma^{N-1} a(\sigma) d\sigma d\xi \\
&= -\frac{p-1}{N-p} \int_0^r \frac{d}{d\xi} \xi^{\frac{p-N}{p-1}} \int_0^\xi \sigma^{N-1} a(\sigma) d\sigma d\xi \\
&= \frac{p-1}{N-p} (-r^{\frac{p-N}{p-1}} \int_0^r \sigma^{N-1} a(\sigma) d\sigma + \int_0^r \xi^{\frac{(p-2)N+1}{p-1}} a(\xi) d\xi) \\
&\geq \frac{p-1}{N-p} \frac{1}{r^{\frac{N-p}{p-1}}} \int_0^r \Big[r^{\frac{N-p}{p-1}} - (t)^{\frac{N-p}{p-1}} \Big] t^{\frac{(p-2)N+1}{p-1}} p(t) dt \\
&\geq \frac{p-1}{N-p} \frac{1}{r^{\frac{N-p}{p-1}}} (r^{\frac{N-p}{p-1}} - (\frac{r}{2})^{\frac{N-p}{p-1}}) \int_0^{r/2} t^{\frac{(p-2)N+1}{p-1}} a(t) dt \\
&= \frac{p-1}{N-p} (1 - (\frac{1}{2})^{\frac{N-p}{p-1}}) \int_0^{r/2} t^{\frac{(p-2)N+1}{p-1}} a(t) dt \\
&= \infty \quad \text{as } r \to \infty.
\end{aligned}$$

Then

$$\infty > \frac{u(0)+1}{f(u(0))^{1/(p-1)}} \overline{u}(0) \geq \infty,$$

which is a contradiction.

If $\beta < 1$ we have $\frac{u+\beta}{u+1} > \beta \iff u + \beta > \beta u + \beta \iff (1-\beta)u > 0$ is true. In this case we have

$$\begin{aligned}
\overline{u}(0) &\geq \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \frac{f(u(\sigma)) a(\sigma) \sigma^{N-1}}{(u(\sigma)+1)^{p-1}} d\sigma \Big]^{1/(p-1)} d\xi \\
&= \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \frac{f(u(\sigma)) a(\sigma) (u(\sigma)+\beta)^{p-1} \sigma^{N-1}}{(u(\sigma)+\beta)^{p-1} (u(\sigma)+1)^{p-1}} d\sigma \Big]^{1/(p-1)} d\xi \\
&\geq \frac{f(u(0))^{1/(p-1)}}{u(0)+\beta} \beta \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \sigma^{N-1} a(\sigma) d\sigma \Big]^{1/(p-1)} d\xi,
\end{aligned}$$

which implies

$$\infty > \frac{u(0)+\beta}{f(u(0))^{1/(p-1)} \beta} \overline{u}(0) \geq \int_0^r \xi^{\frac{1-N}{p-1}} \Big[\int_0^\xi \sigma^{N-1} a(\sigma) d\sigma \Big]^{1/(p-1)} d\xi \geq \infty,$$

which is a contradiction.

In the first case we observe that we can not have $\int_0^\xi r^{\frac{(p-2)N+1}{p-1}} a(r) dr = \infty$ because

$$\begin{aligned}
\int_0^r \xi^{\frac{1-N}{p-1}} d\xi &> \int_0^r \xi^{\frac{1-N}{p-1}} \int_0^\xi \sigma^{N-1} a(\sigma) d\sigma d\xi \\
&\geq \frac{p-1}{N-p} (1 - (\frac{1}{2})^{\frac{N-p}{p-1}}) \int_0^{r/2} t^{\frac{(p-2)N+1}{p-1}} a(t) dt \to \infty \quad \text{as } r \to \infty
\end{aligned}$$

which is a contradiction.

Remark 6.1. Let $2 \leq p < +\infty$. Then $1 \geq \frac{1}{p-1} > 0$. From the above proofs we observe if that

$$\Big(\int_0^\xi \sigma^{N-1} a(\sigma) d\sigma \Big)^{1/(p-1)} \leq 1$$

then

$$\begin{aligned}
&\frac{p-1}{N-2}(1-(\frac{1}{2})^{\frac{N-2}{p-1}})\int_0^{r/2} t^{1/(p-1)}a^{1/(p-1)}(t)dt\\
&\leq \int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\Big]^{1/(p-1)}d\xi\\
&\leq \int_0^r \xi^{\frac{1-N}{p-1}}d\xi.
\end{aligned}$$

As $r\to\infty$, we have $\int_0^\infty t^{\frac{1}{p-1}}a^{1/(p-1)}(t)dt\neq\infty$.

On the other hand, if

$$\Big(\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\Big)^{1/(p-1)} > 1,$$

then

$$\begin{aligned}
&\frac{p-1}{N-2}(1-(\frac{1}{2})^{\frac{N-2}{p-1}})\int_0^{r/2} t^{1/(p-1)}a^{1/(p-1)}(t)dt\\
&\leq \int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\Big]^{1/(p-1)}d\xi\\
&\leq \frac{p-1}{N-p}\int_0^{r/2} t^{\frac{(p-2)N+1}{p-1}}a(t)dt.
\end{aligned}$$

Then if $\int_0^\infty t^{1/(p-1)}a^{1/(p-1)}(t)dt=\infty$ we have $\int_0^\infty t^{\frac{(p-2)N+1}{p-1}}a(t)dt=\infty$.

Remark 6.2. Let$1<p\leq 2$. Then $1\leq\frac{1}{p-1}<+\infty$. From the above proofs we observe that

$$\int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\Big]^{1/(p-1)}d\xi\leq\frac{p-1}{N-2}\int_0^\infty t^{1/(p-1)}a^{1/(p-1)}(t)dt.$$

If $\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\geq 1$, then

$$\begin{aligned}
&\frac{N-1}{N-p}(1-(\frac{1}{2})^{\frac{N-p}{p-1}})\int_0^{r/2} t^{\frac{(p-2)N+1}{p-1}}a(t)dt\\
&\leq \int_0^r \xi^{\frac{1-N}{p-1}}\Big[\int_0^\xi \sigma^{N-1}a(\sigma)d\sigma\Big]^{1/(p-1)}d\xi\,.
\end{aligned}$$

Then if $\int_0^\infty t^{\frac{(p-2)N+1}{p-1}}a(t)dt=\infty$ we have $\int_0^\infty t^{1/(p-1)}a^{1/(p-1)}(t)dt=\infty$.

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