A FEW RESULTS ABOUT THE P-LAPLACE'S OPERATOR

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Abstract. The aim of this paper is to obtain few results for p-Laplace's operator and these representation an extension of the very know results for laplacian.

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1.THE p-LAPLACE EQUATION

The equation

$$\triangle_p u = 0 \quad \text{on} \quad \Omega, \quad 1 < p < \infty, \tag{1.1.}$$

is called p-Laplace's equation.

Here, $\Omega \subset \mathbb{R}^N$ is an open set, $u : \Omega \longrightarrow \mathbb{R}$ is the unknown, and $\triangle_p$ is the p-Laplace operator defined by

$$\triangle_p u := div(|\nabla u|^{p-2} \nabla u), \tag{1.2.}$$

The previous investigations have led to the equation's critical points

$$D_p(u;\Omega) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx \tag{1.3.}$$

are weak solutions for (1.1.), thus they can be named p-harmonic functions.

2.FUNDAMENTAL SOLUTIONS FOR p-LAPLACE EQUATION

We will first construct a simple radial solution of p-Laplace's equation. To look for radial solutions of p-Laplace's equation on $\Omega = \mathbb{R}^N$ of the form

$$u(x) = v(r);\ r = |x| := \sqrt[2]{x_1^2 + ... + x_N^2}, \tag{2.1.}$$

Here, $v : [0, \infty) \longrightarrow \mathbb{R}$

We note that

$$u_{x_i} = \frac{\partial v(r)}{\partial x_i} = v^{'}(r)\frac{x_i}{r}, \tag{2.2.}$$

and

$$u_{x_i x_i} = \frac{\partial^2 v(r)}{\partial x_i^2} = \frac{x_i^2}{r^2} v^{''}(r) + \frac{1}{r} v^{'}(r) - \frac{x_i^2}{r^3} v^{'}(r), \forall 1 \leq i \leq N, \tag{2.3.}$$

and summation yields

$$\Delta_2 u(x) = v^{''}(r) + \frac{N-1}{r} v^{'}(r), r \neq 0. \tag{2.4.}$$

We have

$$|\nabla u| = \sqrt[2]{\left(\frac{\partial u}{\partial x_1}\right)^2 + ... + \left(\frac{\partial u}{\partial x_N}\right)^2} = \sqrt[2]{\left(v^{'}(r)\frac{x_1}{r}\right)^2 + ... + \left(v^{'}(r)\frac{x_N}{r}\right)^2} = \sqrt[2]{(v^{'}(r))^2} = \left|v^{'}(r)\right|, \tag{2.5.}$$

and

$$\frac{\partial}{\partial x_i}\left|v^{'}(r)\right|^{p-2} = \frac{\partial}{\partial x_i}\left(\sqrt[2]{(v^{'}(r))^2}\right)^{p-2} = (p-2)\left(\sqrt[2]{(v^{'}(r))^2}\right)^{p-3} \frac{v^{'}(r)\frac{x_i}{r}v^{''}(r)}{\left|v^{'}(r)\right|}, \tag{2.6.}$$

But (1.1.) equivalently

$$|\nabla u|^{p-2} \Delta_2 u + \nabla\left(|\nabla u|^{p-2}\right) \cdot \nabla u = 0. \tag{2.7.}$$

We have

$$\nabla\left(|\nabla u|^{p-2}\right)\cdot\nabla u=\nabla\left(\left|v^{'}(r)\right|^{p-2}\right)\cdot\nabla v(r)=$$

$$\left(\frac{\partial}{\partial x_1}\left|v^{'}(r)\right|^{p-2},...,\frac{\partial}{\partial x_N}\left|v^{'}(r)\right|^{p-2}\right)\cdot\left(\frac{\partial v(r)}{\partial x_1},...,\frac{\partial v(r)}{\partial x_N}\right)=$$

$$\frac{(p-2)\left|v^{'}(r)\right|^{p-3}v^{'}(r)\frac{x_1}{r}v^{''}(r)}{\left|v^{'}(r)\right|}\frac{v^{'}(r)x_1}{r}+...+\frac{(p-2)\left|v^{'}(r)\right|^{p-3}v^{'}(r)\frac{x_N}{r}v^{''}(r)}{\left|v^{'}(r)\right|}\frac{v^{'}(r)x_N}{r}=$$

$$\frac{(p-2)\left|v^{'}(r)\right|^{p-3}\left(v^{'}(r)\right)^2v^{''}(r)}{\left|v^{'}(r)\right|r^2}\left(x_1^2+...+x_N^2\right)=$$

$$\frac{(p-2)\left|v^{'}(r)\right|^{p-3}\left(v^{'}(r)\right)^2v^{''}(r)}{\left|v^{'}(r)\right|}. \tag{2.8.}$$

So (2.7.) equivalently

$$\left|v^{'}(r)\right|^{p-2}\left[(p-1)v^{''}(r)+\frac{N-1}{r}v^{'}(r)\right]=0. \tag{2.9.}$$

Assume $\left|v^{'}(r)\right|\neq 0$.
Hence, we have

$$\Delta_p u=0 \text{ for } x\neq 0$$

if and only if

$$(p-1)v^{''}(r)+\frac{N-1}{r}v^{'}(r)=0, \tag{2.10.}$$

In the case (2.10.) note $v^{'}=z$, follows

$$\begin{aligned}
&(p-1)z^{'}+\tfrac{N-1}{r}z=0\Longleftrightarrow\\
&(p-1)\tfrac{dz}{z}=\tfrac{1-N}{r}dr\Longleftrightarrow\\
&(p-1)\ln|z|=(1-N)\ln r+\ln|C|^{p-1}\Longleftrightarrow\\
&z(r)=\sqrt[p-1]{\tfrac{|C|^{p-1}}{r^{N-1}}}=\tfrac{|C|}{r^{\frac{N-1}{p-1}}}.
\end{aligned} \tag{2.11.}$$

We conclude that

$$v^{'}(r)=\frac{C}{r^{\frac{N-1}{p-1}}}, \tag{2.12.}$$

for an arbitrary constant $C \in \mathbb{R}_+$ and thus

$$v(r) = \begin{cases} C \ln r + C_1, & \text{if } N = p \\ C \frac{p-1}{p-N} r^{\frac{p-N}{p-1}} + C_1, & \text{if } \quad N \geq p+1 \end{cases}, r > 0, \qquad (2.13.)$$

with constants $C_1 \in \mathbb{R}$.

3.Gauss-Green, Gauss-Ostrogradski and Green′s formulas for the p-Laplace′ operator

Definition 3.1. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded*

i) We say that Ω has a C^k-boundary, $k \in N \cup \{\infty\}$, if for any $x \in \partial\Omega$ there exists $r > 0$ and a function $\beta \in C^k(R^N)$ such that

$$\Omega \cap B(x;r) = \{y \in B(x;r) : y_N > \beta(y_1, ..., y_{N-1})\},$$

ii)If $\partial\Omega$ is C^k then we can define the unit outer normal field $\upsilon : \partial\Omega \longrightarrow R^N$, where, $\upsilon(x)$, $|\upsilon(x)| = 1$, is the outward pointing unit normal of $\partial\Omega$ at x.

iii)Let $\partial\Omega$ be C^k. We call the directional derivative

$$\frac{\partial u}{\partial \upsilon}(x) := \nu(x) \cdot \nabla u(x), x \in \partial\Omega,$$

the normal derivative of u.

In addition to $C^k(\Omega)$ we define the function space

$$C^k(\overline{\Omega}) := \left\{u \in C^k(\Omega) : D^\alpha u \text{ can be continuously extended to } \partial\Omega \text{ for } |\alpha| \leq k\right\},$$

where

$$D^\alpha u = \frac{\partial^{\alpha_1 + ... + \alpha_N}}{\partial x_1^{\alpha_1} ... \partial x_N^{\alpha_N}} u, \quad |\alpha| = \sum_{i=1}^{N} \alpha_i.$$

We recall the Gauss-Green theorem.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded with C^1-boundary. Then for all $u \in C^1(\overline{\Omega})$*

$$\int_\Omega u_{x_i}(x) dx = \int_{\partial\Omega} u(x) \upsilon_i(x) d\sigma(x).$$

Remark(Gauss-Ostrogradscki): *Let* $\Omega \subset R^N$ *be open and bounded with* C^1*-boundary. Then for all* $\overrightarrow{f} : \overline{\Omega} \longrightarrow R^N$ *such that* $\overrightarrow{f} \in C(\overline{\Omega}) \cap C^1(\Omega)$. *We have*

$$\int_{\Omega} div \overrightarrow{f} dx = \int_{\partial\Omega} \overrightarrow{f} \cdot \upsilon d\sigma(x).$$

Theorem 3.3. I*f* $u \in C^2(\overline{\Omega})$ *such that* $\Delta_p u \in C(\overline{\Omega})$ *then*

$$\int_{\Omega} \Delta_p u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \upsilon} |\nabla u|^{p-2} d\sigma(x). \tag{3.1.}$$

Proof. In theorem Gauss-Ostrogradscki let $\overrightarrow{f} = |\nabla u|^{p-2} \nabla u$.

We have

$$\begin{aligned}
&\int_{\Omega} div \left(|\nabla u|^{p-2} \nabla u\right) dx = \int_{\partial\Omega} \left(|\nabla u|^{p-2} \nabla u\right) \cdot \upsilon d\sigma(x) = \int_{\Omega} \Delta_p u dx = \\
&\int_{\Omega} |\nabla u|^{p-2} \Delta_2 u dx + \int_{\Omega} \nabla \left(|\nabla u|^{p-2}\right) \cdot \nabla u dx = \\
&\int_{\partial\Omega} \tfrac{\partial u}{\partial \upsilon} |\nabla u|^{p-2} d\sigma(x) - \int_{\Omega} \nabla \left(|\nabla u|^{p-2}\right) \cdot \nabla u dx + \\
&\int_{\Omega} \nabla \left(|\nabla u|^{p-2}\right) \cdot \nabla u dx = \int_{\partial\Omega} \tfrac{\partial u}{\partial \nu} |\nabla u|^{p-2} d\sigma(x)
\end{aligned}$$

Moreover, we easily obtain Green's formulas for the p-Laplace operator**:**

Theorem 3.4. *Let* $\Omega \subset R^N$ *be open and bounded with* C^1*-boundary. Then for all* $u, v \in C^2(\overline{\Omega})$ *such that* $\Delta_p u \in C(\overline{\Omega})$, *we have*

$$\begin{aligned}
&G1) \int_{\Omega} (\Delta_p u) v dx = \int_{\partial\Omega} v |\nabla u|^{p-2} \tfrac{\partial u}{\partial \upsilon} d\sigma(x) - \int_{\Omega} \nabla v \cdot \left(|\nabla u|^{p-2} \nabla u\right) dx \\
&G2) \int_{\Omega} [(\Delta_p u) v - (\Delta_p v) u] dx = \int_{\partial\Omega} \left(v |\nabla u|^{p-2} \tfrac{\partial u}{\partial \upsilon} - u |\nabla v|^{p-2} \tfrac{\partial v}{\partial \upsilon}\right) d\sigma(x).
\end{aligned} \tag{3.2.}$$

Proof. $G1$) Let $\overrightarrow{f} = v \left(|\nabla u|^{p-2} \nabla u\right)$.We have

$$div \left[v \left(|\nabla u|^{p-2} \nabla u\right)\right] = v div \left(|\nabla u|^{p-2} \nabla u\right) + \nabla v \cdot \left(|\nabla u|^{p-2} \nabla u\right).$$

So

$$\int_{\Omega} \left[v \Delta_p u + \nabla v \cdot \left(|\nabla u|^{p-2} \nabla u\right)\right] dx = \int_{\partial\Omega} v |\nabla u|^{p-2} \frac{\partial u}{\partial \upsilon} d\sigma(x).$$

Proof. $G2$) By $G1$) we have

$$\int_{\Omega} (\Delta_p u)\, v dx = \int_{\partial\Omega} v\, |\nabla u|^{p-2} \frac{\partial u}{\partial v} d\sigma(x) - \int_{\Omega} \nabla v \cdot \left(|\nabla u|^{p-2} \nabla u \right) dx \qquad (3.3.)$$

we inverse the role u and v,so

$$\int_{\Omega} (\Delta_p v)\, u dx = \int_{\partial\Omega} u\, |\nabla v|^{p-2} \frac{\partial v}{\partial v} d\sigma(x) - \int_{\Omega} \nabla u \cdot \left(|\nabla v|^{p-2} \nabla v \right) dx \qquad (3.4.)$$

Using (3.3.) and (3.4.) we deduce $G2$)

4.Green function, Kelvin transform, or Poisson Kernel?

The following ideas are from [3]: From a physical standpoint equation (1.1.), or rather its generalizations, arises naturally, e.g., in the steady rectilinear motion of incompressible non-Newtonian fluids or in phenomena of phase transition. A glimpse at (1.1.) immediately reveals two unfavorable features:

(i) the operator is badly nonlinear;

(ii) ellipticity is lost at points where $\nabla u = 0$.

The strong nonlinearity makes it impossible to develop a potential theory along the lines of classical one. p-harmonic functions do not enjoy integral representation formulas such as

$$u(x) = \oint_{\partial B_r(x)} u d\sigma = \oint_{B_r(x)} u dy,$$

there is no Green function, or Kelvin transform, or Poisson Kernel. p-subharmonicity is not preserved by the clasical mollification processes, as is the case for subharmonic functions. This makes it impossible to regularize p-subharmonic functions. In retrospect, this obstruction is also deeply connected with (ii) above. The lack of ellipticity results in loss of regularity of p-harmonic functions.

By results of Lewis [4], solutions to the p-Laplacian are $C^{1,\alpha}$ for some $\alpha > 0$, for instance the function

$$u(x) = |x|^{\frac{p}{p-1}}$$

satisfies the equation

$$\Delta_p u = const, \text{but } u \notin C^2, \text{when } p > 2.$$

In particular $|\nabla u|$ is C^{α} in any region where u satisfies the p-Laplace equation

$$\Delta_p u = 0.$$

However the operator $\Delta_p u$, defined above, may fail to have the maximum/comparison principle. The weak maximum principle for the p-Laplace operator is well known and can be find in standard literature in this filed; see [3], [5] and [1], the latter treats the parabolic case.

5.The existence of positive solutions in $C^2(\mathbb{R}^N)$ for the problem with p-Laplacian

Consider the problem

$$\begin{cases} -\Delta_p u = p(x)f(u) \quad \text{in} \quad \mathbf{R}^N \\ u > 0 \quad \text{in} \ \mathbf{R}^N \\ u(x) \to \ 0 \quad \text{as} \quad |\, x\,| \to \infty\,, \end{cases} \tag{5.1.}$$

where $N > 2$, $\Delta_p u$ $(1 < p \leq 2)$ is the p-Laplacian operator and

-*the function* $p(x)$ fulfills the following hypotheses:

$(p1)$ $p(x) \in C(R^N)$ and $p(x) > 0$ in $\mathbb{R}^N$.

$(p2)$ we have

$$\int_0^{\infty} r^{\frac{1}{p-1}} \Phi^{\frac{1}{p-1}}(r)dr < \infty \ \text{ if } \ 1 < p \leq 2$$

where $\Phi(r) := \max\limits_{|x|=r} p(x)$.

-*the function* $f \in C^1((0,\infty),(0,\infty))$ satisfies the following assumptions:

$(f1)$ mapping $u \longrightarrow \frac{f(u)}{u^{p-1}}$ is decreasing on $(0,\infty)$;

$(f2)$ $\lim\limits_{u \searrow 0} \frac{f(u)}{u^{p-1}} = +\infty$;

$(f3)$ $\lim\limits_{u \to 0+} \inf f(u) > 0$.

It easy to prove that

Theorem 5.1. *If* $j : I \subseteq R \longrightarrow R$ *is a integrable nonnegative function, then*

$$\left(\frac{1}{b-a}\int_a^b j(x)dx\right)^h \leq \frac{1}{b-a}\int_a^b j^h(x)dx$$

$\forall\ a,b \in I,\ a < b$ *and* $1 < h < +\infty$

Theorem 5.2. *Under hypotheses* $(f1)-(f3),(p1),(p2)$, *the problem* (5.1.) *has a radially symmetric solution* $u \in C^2(\mathbb{R}^N \backslash \{0\}) \cap C^1(\mathbb{R}^N)$.

Proof. By **Theorem 1.3.** in [2] the problem

$$\begin{cases} -\Delta_p U = p(x)f(U), & \text{if } |x| < k, \\ U = 0, & \text{if } \quad |x| = k. \end{cases}$$

has a radially symmetric solution in $C\left(\overline{B}_k\right) \cap C^1\left(B_k\right) \cap C^2\left(B_k \backslash (0)\right)$

We now prove the existence of a positive function $u \in C^2\left(\mathbb{R}^N\right)$. As in [2] we construct first a positive radially symmetric function w such that $-\Delta_p w = \Phi(r)$, $(r = |x|)$ on $\mathbb{R}^N$ and $\lim\limits_{r \longrightarrow \infty} w(r) = 0$.

We obtain

$$w(r) := K - \int_0^r \left[\xi^{1-N} \int_0^\xi \sigma^{N-1} \Phi(\sigma) d\sigma\right]^{\frac{1}{p-1}} d\xi,$$

where

$$K \leq \int_0^\infty \left[\xi^{1-N} \int_0^\xi \sigma^{N-1} \Phi(\sigma) d\sigma\right]^{\frac{1}{p-1}} d\xi.$$

We first show that $(p2)$ implies that

$$\int_0^{+\infty} \left[\xi^{1-N} \int_0^\xi \sigma^{N-1} \Phi(\sigma) d\sigma\right]^{\frac{1}{p-1}} d\xi,$$

is finite.

Let $1 < p \leq 2$, so $0 < p-1 \leq 1$, follows that $1 \leq \frac{1}{p-1} < +\infty$.

Using **Theorem 5.1.** for any $r > 0$, we have

$$\int_0^r \xi^{\frac{1-N}{p-1}} \left[\frac{\xi}{\xi} \int_0^\xi \sigma^{N-1} \Phi(\sigma) d\sigma\right]^{\frac{1}{p-1}} d\xi = \int_0^r \xi^{\frac{1-N}{p-1}} \xi^{\frac{1}{p-1}} \left[\frac{1}{\xi} \int_0^\xi \sigma^{N-1} \Phi(\sigma) d\sigma\right]^{\frac{1}{p-1}} d\xi \leq$$
$$\int_0^r \xi^{\frac{2-N}{p-1}} \frac{1}{\xi} \int_0^\xi \sigma^{\frac{N-1}{p-1}} \Phi^{\frac{1}{p-1}}(\sigma) d\sigma d\xi = \int_0^r \xi^{\frac{2-N}{p-1}-1} \int_0^\xi \sigma^{\frac{N-1}{p-1}} \Phi^{\frac{1}{p-1}}(\sigma) d\sigma d\xi =$$
$$-\frac{p-1}{N-2} \int_0^r \frac{d}{d\xi} \xi^{\frac{2-N}{p-1}} \int_0^\xi \sigma^{\frac{N-1}{p-1}} \Phi^{\frac{1}{p-1}}(\sigma) d\sigma d\xi =$$
$$\frac{p-1}{N-2} \left[-r^{\frac{2-N}{p-1}} \int_0^r \sigma^{\frac{N-1}{p-1}} \Phi^{\frac{1}{p-1}}(\sigma) d\sigma + \int_0^r \xi^{\frac{1}{p-1}} \Phi^{\frac{1}{p-1}}(\xi) d\xi\right] \leq \frac{p-1}{N-2} \int_0^r \xi^{\frac{1}{p-1}} \Phi^{\frac{1}{p-1}}(\xi) d\xi,$$

so

$$\int_0^r \left[\xi^{1-N} \int_0^\xi \sigma^{N-1} \Phi(\sigma) d\sigma\right]^{\frac{1}{p-1}} d\xi < \infty$$

as $r \longrightarrow \infty$.

Then we obtain

$$K \leq \frac{p-1}{N-2} \cdot \int_0^\infty \xi^{\frac{1}{p-1}} \Phi^{\frac{1}{p-1}}(\xi) d\xi \ \text{ if } \ 1 < p \leq 2,$$

clearly, we have

$$w(r) \leq \frac{p-1}{N-2} \cdot \int_0^\infty \xi^{\frac{1}{p-1}} \Phi^{\frac{1}{p-1}}(\xi) d\xi \ \text{ if } \ 1 < p \leq 2.$$

An upper-solution to (5.1.) will be constructed.

Consider the function

$$\overline{f}(t) = (f(t)+1)^{\frac{1}{p-1}}, t > 0.$$

Note that

$$\begin{array}{ll} \overline{f}(t) \geq f(t)^{\frac{1}{p-1}} & \\ \frac{\overline{f}(t)}{t^{p-1}}, \ \text{ is decreasing}, & (f_1') \\ \lim_{t \longrightarrow 0} \frac{\overline{f}(t)}{t} = \infty, & (f_2') \end{array}$$

Let v be a positive function such that

$$w(r) = \frac{1}{C} \int_0^{v(r)} \frac{t^{p-1}}{\overline{f}(t)} dt \ \text{ where } \ C > 0$$

will be chosen such that

$$KC \leq \int_0^{C^{\frac{1}{p-1}}} \frac{t^{p-1}}{\overline{f}(t)} dt.$$

We prove that we can find $C > 0$ with this property. By our hypothesis (f_2') we obtain that

$$\lim_{x \longrightarrow +\infty} \int_0^x \frac{t^{p-1}}{\overline{f}(t)} dt = +\infty.$$

Now using L'Hopital's rule we have

$$\lim_{x \longrightarrow \infty} \frac{\int_0^x \frac{t^{p-1}}{\overline{f}(t)} dt}{x^{p-1}} = \lim_{x \longrightarrow \infty} \frac{x}{(p-1)\,\overline{f}(x)} = +\infty.$$

From this we deduce that there exists $x_1 > 0$ such that

$$\int_0^x \frac{t^{p-1}}{\overline{f}(t)} dt \geq Kx^{p-1}, \text{for all } x \geq x_1.$$

It follows that for any $C \geq x_1$ we have

$$KC \leq \int_0^{C^{\frac{1}{p-1}}} \frac{t^{p-1}}{\overline{f}(t)} dt.$$

But w is a decreasing function, and this implies that v is a decreasing function too.

Then

$$\int_0^{v(r)} \frac{t^{p-1}}{\overline{f}(t)} dt \leq \int_0^{v(0)} \frac{t^{p-1}}{\overline{f}(t)} dt = Cw(0) = CK \leq \int_0^{C^{\frac{1}{p-1}}} \frac{t^{p-1}}{\overline{f}(t)} dt.$$

It follows that $v(r) \leq C^{\frac{1}{p-1}}$ for all $r > 0$. From $w(r) \longrightarrow 0$ as $r \longrightarrow +\infty$ we deduce $v(r) \longrightarrow 0$ as $r \longrightarrow +\infty$.

By the choice of v we have

$$\nabla w = \frac{1}{C} \cdot \frac{v^{p-1}}{\overline{f}(v)} \nabla v$$

follows that

$$\Delta_p w = \frac{1}{C^{p-1}} \left(\frac{v^{p-1}}{\overline{f}(v)} \right)^{p-1} \Delta_p v + (p-1) \frac{1}{C^{p-1}} \left| \nabla v \right|^p \left(\frac{v^{p-1}}{\overline{f}(v)} \right)^{p-2} \left(\frac{v^{p-1}}{\overline{f}(v)} \right)'. \tag{5.2.}$$

From (5.2.) and $u \longrightarrow \frac{\overline{f}(u)}{u^{p-1}}$ is a decreasing function on $(0, +\infty)$, we deduce that

$$\Delta_p v \leq C^{p-1} \left(\frac{\overline{f}(v)}{v^{p-1}} \right)^{p-1} \Delta_p w = -C^{p-1} \left(\frac{\overline{f}(v)}{v^{p-1}} \right)^{p-1} \Phi(r) \leq -f(v)\Phi(r). \tag{5.3.}$$

It follows that v is a radially symmetric solution of the problem:

$$\begin{cases} -\Delta_p u \geq p(x) f(u) & \text{in } \mathbf{R}^N \\ u > 0 \quad \text{in } \mathbf{R}^N \\ u(x) \to 0 & \text{as } \mid x \mid \to \infty, \end{cases} \tag{5.4.}$$

By the proof of **Theorem 1.1.** in [2] the problem (5.1.) has positive solutions.

Now using

$$u'(r) = \left[r^{1-N} \int_0^r \sigma^{N-1} p(\sigma) f(u(\sigma)) d\sigma\right]^{\frac{1}{p-1}}$$

$$u''(r) = -\frac{p(r)f(u(r)) + (1-N)r^{-N}\int_0^r \sigma^{N-1} p(\sigma) f(u(\sigma)) d\sigma}{p-1} \left[r^{1-N} \int_0^r \sigma^{N-1} p(\sigma) f(u(\sigma)) d\sigma\right]^{\frac{2-p}{p-1}},$$

$$\frac{2-p}{p-1} \geq 0 \Longleftrightarrow 1 < p \leq 2$$

$$\lim_{r \longrightarrow 0} \frac{\int_0^r \sigma^{N-1} p(\sigma) f(u(\sigma)) d\sigma}{r^N} = \frac{p(0) f(u(0))}{N}$$

$$\lim_{r \longrightarrow 0} \frac{\int_0^r \sigma^{N-1} p(\sigma) f(u(\sigma)) d\sigma}{r^{N-1}} = 0$$

we deduce $\lim_{r \longrightarrow 0} u''(r)$ is finite, so $u(r) \in C^2(\mathbb{R}^N)$.

References

[1] E. DiBenedetto, Degenerate and singular parabolic equations, Universitext, Springer-Verlag, New York, 1993.

[2] J.V. Goncalves and C.A. Santos, Positive solutions for a class of quasilinear singular equations, *Journal of Differential Equations*, Vol. 2004(2004), No. 56, pp 1-15.

[3] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear potential theory of degenerate elliptic equations, Oxford Mathematical Monographs, Oxford Science Publications

[4] J.L. Lewis, Capacitary functions in convex rings, Arch. Rat. Mech. Anal. 66 (1977), 201-224

[5] J. Maly, W. P. Ziemer, Fine regularity of solutions elliptic partial differential equations. Math. Surv. Monogr. 51, American Mathematical Society, Providence, RI, 1997.

[6] V. Radulescu, *Partial Differential Equations* , Repografia Univ. Craiova (http://inf.ucv.ro/~radulescu), Craiova, 2005.

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