The equality of the reduced and the full C^*-algebras and the amenability of a topological groupoid

Mădălina Buneci

to appear in Recent advances in operator theory, operator algebras, and their applications, 61-78. Oper. Theory Adv. Appl., 153, Birkhauser, Basel, 2005.

Abstract

C. Anantharaman-Delaroche and J. Renault have proved that the amenability of a topological locally compact groupoid implies the equality of the reduced and the full C^*-algebras. In this paper we shall prove the converse assertion under a technical hypothesis. We shall prove that if G is a locally compact second countable groupoid endowed with a Haar system having "a bounded decomposition over the principal groupoid associated to G", then the equality $C^*_{red}(G) = C^*(G)$ implies the amenability of all quasi-invariant measures. In order to prove this we shall see that the inequality $\|II_\mu(f)\| \leq \|Reg_\mu(f)\|$ for all $f \in C_c(G)$ implies a similar inequality for all $f \in I(G, \nu, \mu)$ (where Reg_μ is the left regular representation of $C_c(G)$ on a quasi invariant measure μ, and II_μ is the trivial representation on μ).

AMS 2000 Subject Classification: 22A22, 43A07, 46L52, 43A65.

Key words and phrases: locally compact groupoid, amenable quasi-invariant measure, full C^*-algebra, reduced C^*-algebra.

1 Introduction

For establishing notation, we include some definitions that can be found in several places (e.g. [5], [7], [9], [12]). A groupoid is a set G endowed with a product map

$$(x, y) \to xy \ \left[: G^{(2)} \to G\right]$$

where $G^{(2)}$ is a subset of $G \times G$ called the set of composable pairs, and an inverse map

$$x \to x^{-1} \ [: G \to G]$$

such that the following conditions hold:

(1) If $(x,y) \in G^{(2)}$ and $(y,z) \in G^{(2)}$, then $(xy,z) \in G^{(2)}$, $(x,yz) \in G^{(2)}$ and $(xy)z = x(yz)$.

(2) $(x^{-1})^{-1} = x$ for all $x \in G$.

(3) For all $x \in G$, $(x,x^{-1}) \in G^{(2)}$, and if $(z,x) \in G^{(2)}$, then $(zx)x^{-1} = z$.

(4) For all $x \in G$, $(x^{-1},x) \in G^{(2)}$, and if $(x,y) \in G^{(2)}$, then $x^{-1}(xy) = y$.

The maps r and d on G, defined by the formulae $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$, are called the range and the source maps. It follows easily from the definition that they have a common image called the unit space of G, which is denoted $G^{(0)}$. Its elements are units in the sense that $xd(x) = r(x)x = x$. Units will usually be denoted by letters as u, v, w while arbitrary elements will be denoted by x, y, z. It is useful to note that a pair (x,y) lies in $G^{(2)}$ precisely when $d(x) = r(y)$, and that the cancellation laws hold (e.g. $xy = xz$ iff $y = z$). The fibres of the range and the source maps are denoted $G^u = r^{-1}(\{u\})$ and $G_v = d^{-1}(\{v\})$, respectively. More generally, given the subsets A, $B \subset G^{(0)}$, we define $G^A = r^{-1}(A)$, $G_B = d^{-1}(B)$ and $G_B^A = r^{-1}(A) \cap d^{-1}(B)$. The reduction of G to $A \subset G^{(0)}$ is $G|A = G_A^A$. The relation $u\tilde{\ }v$ iff $G_v^u \neq \phi$ is an equivalence relation on $G^{(0)}$. Its equivalence classes are called orbits and the orbit of a unit u is denoted $[u]$. A groupoid is called transitive iff it has a single orbit. The quotient space for this equivalence relation is called the orbit space of G and denoted $G^{(0)}/G$. $\pi : G^{(0)} \to G^{(0)}/G, \pi(u) = \dot{u}$ is the canonical projection. A subset of $G^{(0)}$ is said saturated if it contains the orbits of its elements.

A topological groupoid consists of a groupoid G and a topology compatible with the groupoid structure. This means that:

(1) $x \to x^{-1}$ $[: G \to G]$ is continuous.

(2) (x,y) $\left[: G^{(2)} \to G\right]$ is continuous where $G^{(2)}$ has the induced topology from $G \times G$.

We are exclusively concerned with topological groupoids which are second countable, locally compact Hausdorff. It was shown in [10] that measured groupoids (in the sense of Definition 2.3./p. 6 [5]) may be assume to have locally compact topologies, with no loss in generality. A subset U of G is said conditionally compact if for every compact subset K of $G^{(0)}$, $U \cap r^{-1}(K)$ and $U \cap d^{-1}(K)$ is compact in G. If G is locally compact and $G^{(0)}$ is paracompact then $G^{(0)}$ has a fundamental system of conditionally compact neighborhoods (Proposition II.1.9/p.56 [12]). If X is a locally compact space, $C_c(X)$ denotes the space of complex-valuated continuous functions with compact support. The Borel sets of a topological space are taken to be the σ-algebra generated by the open sets.

Let G be a locally compact second countable groupoid equipped with a Haar system,.i.e. a family of positive Radon measures on G, $\left\{\nu^u,\, u \in G^{(0)}\right\}$, such that

1) For all $u \in G^{(0)}$, $supp(\nu^u) = G^u$.

2) For all $f : G \to \mathbf{C}$ continuous with compact support,

$$u \to \int f(x)\, d\nu^{u}(x) \quad \left[: G^{(0)} \to \mathbf{C}\right]$$

is continuous.

3) For all $f : G \to \mathbf{C}$ continuous with compact support, and all $x \in G$,

$$\int f(y)\, d\nu^{r(x)}(y) = \int f(xy)\, d\nu^{d(x)}(y)$$

As a consequence of the existence of continuous Haar systems, $r, d : G \to G^{(0)}$ are open maps ([14]).

The construction of the C^*-algebra of a groupoid extends the well-known case of a group. The space of continuous functions with compact support on groupoid is made into a $*$-algebra and endowed with the smallest C^*-norm making its representations continuous. Let $C_c(G)$ be the space of continuous functions with compact support on the groupoid G. For $f,\ g \in C_c(G)$ the convolution is defined by:

$$f * g(x) = \int f(xy)\, g\left(y^{-1}\right) d\nu^{d(x)}(y)$$

and the involution by

$$f^{*}(x) = \overline{f\left(x^{-1}\right)}.$$

Under these operations, $C_c(G)$ becomes a topological $*$-algebra. A representation of $C_c(G)$ is a $*$-homomorphism from $C_c(G)$ into $\mathcal{B}(H)$, for some Hilbert space H, that is continuous with respect to the inductive limit topology on $C_c(G)$ and the weak operator topology on $\mathcal{B}(H)$. The full C^*-algebra $C^*(G)$ is defined as the completion of the involutive algebra $C_c(G)$ with respect to the full C^*-norm

$$\|f\| = \sup \|L(f)\|$$

where runs over all non-degenerate representation of $C_c(G)$ which are continuous for the inductive limit topology. Let us single out a special class of representations of $C_c(G)$ that serve as analogues of the regular representation of a group. If μ is a measure on $G^{(0)}$, then the measure $\nu = \int \nu^{u} d\mu(u)$, defined by

$$\int f(y)\, d\nu(y) = \int \left(\int f(y)\, d\nu^{u}(y)\right) d\mu(u), \quad f \geq 0 \text{ Borel}$$

is called the measure on G induced by μ. The image of ν by the inverse map $x \to x^{-1}$ is denoted ν^{-1}. μ is said quasi-invariant if its induced measure ν is equivalent to its inverse ν^{-1} A measure belongings to the class of a quasi-invariant measure is also quasi-invariant. We say that the class is invariant. If

μ quasi-invariant measure, then $Ind\mu(f)$ is the operator on $L^2\left(G,\nu^{-1}\right)$ defined by formula

$$Ind\mu(f)\,\xi(x) = f * \xi(x)$$

and $Ind_u(f)$ is the operator on $L^2\left(G,(\nu^u)^{-1}\right)$ defined by formula

$$Ind_u(f)\,\xi(x) = f * \xi(x) = \int f(xy)\,\xi\left(y^{-1}\right) d\nu^u(y)$$

The reduced C^*-algebra $C^*_{red}(G)$ is defined as the completion of the involutive algebra $C_c(G)$ with respect to the reduced C^*-norm

$$\|f\| = \|Ind_u(f)\|$$

If μ is a quasi invariant measure with $supp(\mu) = G^{(0)}$, then

$$\|f\|_{red} = \|Ind\mu(f)\|$$

If μ is a quasi-invariant measure on $G^{(0)}$ and ν is the measure induced on G, then the Radon-Nikodym derivative $\Delta = \frac{d\nu}{d\nu^{-1}}$ is called the modular function of μ. Let $\nu_0 = \Delta^{-\frac{1}{2}}\nu$.

For $f \in L^1(G,\nu_0)$ define

$$\|f\|_{I,\mu} = \max\left\{\left\|u \to \int |f| d\nu^u\right\|_\infty, \left\|u \to \int |f^*| d\nu^u\right\|_\infty\right\}$$

Let $I(G,\nu,\mu) = \left\{f \in L^1(G,\nu_0), \|f\|_{I,\mu} < \infty\right\}$.

Under the convolution and the involution, $I(G,\nu,\mu)$ becomes a Banach $*$-algebra.

Every representation $\left(\mu, G^{(0)} * \mathcal{H}, L\right)$(see Definition 3.20/p.68 [7]) of G can be integrated into a representation, still denoted by L, of $I(G,\nu,\mu)$. The relation between the two representation is:

$$\langle L(f)\,\xi_1,\xi_2\rangle = \int f(x)\,\langle L(x)\,\xi_1(d(x)),\xi_2(r(x))\rangle\,\Delta^{-\frac{1}{2}}(x)\,d\nu^u(x)\,d\mu_1(u)$$

where $f \in I(G,\nu,\mu)$, $\xi_1,\xi_2 \in \int^{\oplus}_{G^{(0)}} \mathcal{H}(u)\,d\mu(u)$.

Conversely, every non-degenerate $*$-representation of any suitably large $*$-algebra of $I(G,\nu,\mu)$ is obtained in this fashion (see [6], [12]).

We denote by $C^*(G,\nu,\mu)$ the completion of $I(G,\nu,\mu)$ with respect to the norm

$$\|f\| = sup\,\|L(f)\|$$

where L ranges over all representation of (G,ν,μ).

The (left) regular representation of G on μ is the representation

$$\left(\mu, G^{(0)} * L^{(2)}(\nu), Reg\mu\right)$$

where

$$Reg(x) : L^2\left(\nu^{d(x)}\right) \to L^2\left(\nu^{r(x)}\right)$$

is defined by the formula

$$Reg(x)\,\xi(d(x))(y) = \xi\left(x^{-1}y\right).$$

The integrated form of this representation is called the (left) regular representation of $C_c(G)$ (also, of $C^*(G)$ on μ). The map

$$W : L^2(\nu) \to L^2\left(\nu^{-1}\right), W\xi = \xi\Delta^{\frac{1}{2}}$$

is a Hilbert space isomorphism that implements a unitary equivalence between Reg_μ and $Ind\mu$ (II.1.10 [12]). Therefore, if μ is a quasi invariant measure with $supp(\mu) = G^{(0)}$ and Reg_μ is the left regular representation of $C_c(G)$ on μ, then

$$\|f\|_{red} = \|Reg_\mu(f)\|.$$

In [1], C. Anantharaman-Delaroche and J. Renault have proved the following theorem:

Theorem 1 *(Proposition 6.1.8/p. 146 [1]). Let G be a locally compact groupoid equipped with a continuous Haar system $\{\nu^u, u \in G^{(0)}\}$. If G is measurewise amenable (amenable with respect to all quasi invariant measures), then $C^*(G) = C^*_{red}(G)$.*

We shall prove that under a technical hypothesis the converse assertion is true. We shall use the following result:

Theorem 2 *(Theorem 6.1.4/p.142 [1]). The following conditions are equivalent:*

(1) (G, ν, μ) is amenable

(2) The trivial representation of $C^*(G, \nu, \mu)$ is weakly contained in the regular representation.

(3) The regular representation is faithful on $C^*(G, \nu, \mu)$.

We shall also use the decomposition

$$\nu^u = \int \nu_{u,v}\, d\eta_u(v) \quad \text{for all } u \in G^{(0)}$$

of the Haar system for G over the principal groupoid associated to G (see Section1 [13], $\nu_{u,v}$ is supported on G^u_v for all $u\tilde{\ }v$, η_u is supported on $[u]$ for all u and $\eta_u = \eta_v$ for all $u\tilde{\ }v$). We shall assume that the decomposition of the Haar system over the principal groupoid is bounded. This means that there is a positive Radon measure η_0 on $G^{(0)}$ such that the family $\{\eta_u\}_{u\in G^{(0)}}$ is dominated by η_0, i.e.

$$\eta_u(f) \leq \eta_0(f), \text{ for all positive function } f \in C_c\left(G^{(0)}\right),$$

Obviously, locally compact second countable transitive groupoids and locally compact groupoids for which the applications $d_u : G^u \to G^{(0)}, d_u(x) = d(x)$ are open (in particular, locally trivial groupoids) satisfy the hypothesis.

We shall prove that if the decomposition of the Haar system over the principal groupoid is bounded, then the equality $C^*_{red}(G) = C^*(G)$ implies the amenability of all quasi-invariant measures. In [3] we have proved the same result for transitive locally compact second countable groupoids. In order to prove the result for groupoids having the decomposition of the Haar system over the principal groupoid is bounded, we shall show that the inequality $\|II\mu(f)\| \leq \|Reg_\mu(f)\|$ for all $f \in C_c(G)$ implies a similar inequality for all $f \in I(G,\nu,\mu)$ (where Reg_μ is the left regular representation of $C_c(G)$ on a quasi invariant measure μ, and II_μ is the trivial representation on μ).

2 Functions approximated by continuous functions in the norm of $C^*(G,\nu,\mu)$

First we present some results on the structure of the Haar systems, as developed by J. Renault in Section 1 of [13] and also by A. Ramsay and M.E. Walter in Section 2 of [11]. Let $\left\{\nu^u, u \in G^{(0)}\right\}$ be a continuous Haar system on G.

In Section 1 of [13] Jean Renault constructs a Borel Haar system for G'. One way to do this is to choose a function F_0 continuous with conditionally support which is nonnegative and equal to 1 at each $u \in G^{(0)}$. Then for each $u \in G^{(0)}$ choose a left Haar measure $\nu_{u,u}$ on G^u_u so the integral of F_0 with respect to $\nu_{u,u}$ is 1.

Renault defines $\nu_{u,v} = x\nu_{v,v}$ if $x \in G^u_v$ (where $x\nu_{u,u}(f) = \int f(xy)\, d\nu_{v,v}(y)$ as usual). If z is another element in G^u_v, then $x^{-1}z \in G^v_v$, and since $\nu_{v,v}$ is a left Haar measure on G^v_v, it follows that $\nu_{u,v}$ is independent of the choice of x. If K is a compact subset of G, then $\sup\limits_{u,v} \nu_{u,v}(K) < \infty$. Renault also defines a 1-cocycle δ on G such that for every $u \in G^{(0)}$, $\delta|_{G^u_u}$ is the modular function for $\nu_{u,u}$. δ and $\delta^{-1} = 1/\delta$ are bounded on compact sets in G.

Let

$$R = (r,d)(G) = \{(r(x), d(x)),\ x \in G\}$$

be the graph of the equivalence relation induced on $G^{(0)}$. This R is the image of G under the homomorphism (r,d), so it is a σ-compact groupoid. With this apparatus in place, Renault describes a decomposition of the Haar system

$\{\nu^u, u \in G^{(0)}\}$ for G over the equivalence relation R (the principal groupoid associated to G). He proves that there is a unique Borel Haar system α for R with the property that

$$\nu^u = \int \nu_{s,t} d\alpha^u(s,t) \quad \text{for all } u \in G^{(0)}.$$

In Section 2 [11] A. Ramsay and M.E. Walter prove that

$$\sup_u \alpha^u((r,d)(K)) < \infty, \text{for all compact } K \subset G$$

If μ is a quasi-invariant measure for $\{\nu^u, u \in G^{(0)}\}$, then μ is a quasi-invariant measure for $\{\alpha^u, u \in G^{(0)}\}$. Also if Δ_R is the modular function associated to $\{\alpha^u, u \in G^{(0)}\}$ and μ, then $\Delta = \delta\Delta_R \circ (r,d)$ can serve as the modular function associated to $\{\nu^u, u \in G^{(0)}\}$ and μ.

For each $u \in G^{(0)}$ the measure α^u is concentrated on $\{u\} \times [u]$. Therefore there is a measure η_u concentrated on $[u]$ such that $\alpha^u = \varepsilon_u \times \eta_u$, where ε_u is the unit point mass at u. Since $\{\alpha^u, u \in G^{(0)}\}$ is a Haar system, we have $\eta_u = \eta_v$ for all $(u,v) \in R$, and the function

$$u \rightarrow \int f(s)\, \eta_u(s)$$

is Borel for all $f \geq 0$ Borel on $G^{(0)}$. For each u the measure η_u is quasi-invariant (Section 2 [11]) Therefore η_u is equivalent to $d_*(v^u)$ (Lemma 4.5/p. 277 [9]).

Let μ be a quasi-invariant measure and let $\mu_1 = \int \eta_u d\mu(u)$. Then μ_1 is equivalent to μ. Indeed, let $f \geq 0$ Borel on $G^{(0)}$ such that $\mu(f) = 0$.Since μ is quasi-invariant, it follows that for $\mu\, a.a.\ u$ $\nu^u(f \circ d) = 0$, and since η_u is equivalent to $d_*(v^u)$, it results $\eta_u(f) = 0$ for μ $a.a.\ u$. Conversely if $\mu_1(f) = 0$, then $\eta_u(f) = 0$ for μ $a.a.\ u$, and therefore $\nu^u(f \circ d) = 0$. Thus the quasi-invariance of μ implies $\mu(f) = 0$.

Let α the measure induced by μ_1 on R, and let Δ_R be the modular function of μ_1. Then $\Delta_R = \frac{d\alpha}{d\alpha^{-1}}$. It is easy to note that α is symmetric. Hence $\Delta_R = 1$. If Δ is the modular function associated to $\{\nu^u, u \in G^{(0)}\}$ and μ_1, then $\Delta = \delta\Delta_R \circ (r,d) = \delta$.

The next lemma contains the properties of the decomposition of a Haar system. For its proof see Section 1 [13], Section 2 [11], Theorem 4.4./p. 23 [5], or [2].

Lemma 3 *Let $\{\nu^u, u \in G^{(0)}\}$ be a continuous Haar system on G. Let*

$$\nu^u = \int \nu_{u,v} d\eta_u(v) \quad \textit{for all } u \in G^{(0)}$$

be the decomposition of the Haar system for G over the equivalence relation R. Let μ be a quasi-invariant measure and $\mu_1 = \int \eta_u d\mu(u)$. Let Δ be the modular function associated to $\{\nu^u, u \in G^{(0)}\}$ and μ_1. Then

1) $\nu_{u,v}$ *is concentrated on* G_v^u, *and* $\nu_{u,v} \neq 0$, *for all* $(u,v) \in R$.

2) *For all* $f \geq 0$ *Borel on* G,

$$(u,v) \mapsto \int f(y)\, d\nu_{u,v}(y) \; \left[: R \rightarrow \overline{\mathbf{R}}\right]$$

is an extended real-valued Borel function.

3) $\sup\limits_{u,v} \nu_{u,v}(K) < \infty$, *for all compact* $K \subset G$.

4) *For all* $f \geq 0$ *Borel on* G,

$$\int f(xy)\, d\nu_{d(x),v}(y) = \int f(y)\, d\nu_{r(x),v}(y) \quad \textit{for all } x \in G,\ v \in [d(x)]$$

5) *For all* $f \geq 0$ *Borel on* G,

$$\Delta(x) \int f(yx)\, d\nu_{u,r(x)}(y) = \int f(y)\, d\nu_{u,d(x)}(y) \quad \textit{for all } x \in G_0,\ u \in [d(x)]$$

6) $\Delta : G \rightarrow \mathbf{R}_+^*$ *is a homomorphism.*

7) Δ *and* $\Delta^{-1} = 1/\Delta$ *are bounded on compact sets in* G.

8) *For all* $f \geq 0$ *Borel on* G,

$$\int f(y)\, d\nu_{u,v}(y) = \int f\left(y^{-1}\right) \Delta\left(y^{-1}\right) d\nu_{v,u}(y) \;\; \textit{for all } \; (u,v) \in R$$

9) $\sup\limits_{u} \varepsilon_u \times \eta_u((r,d)(K)) < \infty$, *for all compact* $K \subset G$

Throughout this section the Haar system $\left\{\nu^u, u \in G^{(0)}\right\}$ and the systems of measures in its decomposition (as in preceding lemma) will be considered fixed.

We shall need that $\sup\limits_{u} \eta_u(K) < \infty$, for all compact $K \subset G^{(0)}$. If there is a positive Radon measure η_0 on $G^{(0)}$ such that the family $\{\eta_u\}_{u \in G^{(0)}}$ is dominated by η_0, i.e.

$$\eta_u(f) \leq \eta_0(f), \text{ for all positive function } f \in C_c\left(G^{(0)}\right),$$

we shall say that *the decomposition of the Haar system over the principal groupoid is bounded.*

Notation 4 *Let* B *be a set with the property that it intersects each orbit in exactly one element. Let* $e : G^{(0)} \rightarrow G^{(0)}$ *be the function defined by* $e(u)$ *is the only element in* $B \cap [u]$. *Let* d_B *be the function*

$$d_B : G^B \rightarrow G^{(0)},\ d_B(x) = d(x).$$

Lemma 5 *If the function d_B is open, then*

$$\sup_u \eta_u(K) < \infty, \text{ for all compact } K \subset G^{(0)}.$$

Proof. Let K be a compact subset of $G^{(0)}$. Since G is locally compact and d_B is continuous and open from G^B onto $G^{(0)}$, there is a compact $L \subset G$, such that $K \subset d_B\left(L \cap G^B\right)$. We have

$$\begin{aligned} \int 1_K(v)\, d\eta_u(v) &= \int 1_K(v)\, d\eta_{e(u)}(v) \\ &\leq \int 1_{d_B(L\cap G^B)}(v)\, d\eta_{e(u)}(v) \\ &= \int 1_{(r,d)(L\cap G^B)}(e(u), v)\, d\eta_{e(u)}(v) \end{aligned}$$

Hence $\sup_u \eta_u(K) < \infty$. ■

Example 6 *Groupoids which satisfy the hypothesis (d_B is open):*

1) locally compact transitive groupoids

2) locally compact groupoids for which the applications $d_u : G^u \to G^{(0)}, d_u(x) = d(x)$ are open. In particular, locally trivial groupoids satisfy the hypothesis.

3) locally compact group bundles having the bundle maps open.

Remark 7 *If G is a locally transitive groupoid, then any Haar system on G have the bounded decomposition over the principal groupoid.*

Notation 8 *Let μ be a quasi-invariant probability measure and let $\mu_1 = \int \eta_u d\mu(u)$. Then μ_1 is equivalent to μ. We replace μ with μ_1 (see [4]). Let Δ be the modular function associated to $\left\{\nu^u, u \in G^{(0)}\right\}$ and μ_1 Let ν be the measure induced by μ_1 on G, and $\nu_0 = \Delta^{-\frac{1}{2}}\nu$.*

For $f \in L^1(G, \nu_0)$ define f^* by $f^*(y) = \overline{f(y^{-1})}$ and

$$\|f\|_{I,\mu} = \max\left\{\left\|u \to \int |f| d\nu^u\right\|_\infty, \left\|u \to \int |f^*| d\nu^u\right\|_\infty\right\}$$

$$\|f\|_{II,\mu} = \sup\{\int |f(y)\, j(d(y)\, k(r(y)))| \Delta(y)^{-\frac{1}{2}}\, d\nu(y) : \\ j, k \in L^2\left(G^{(0)}, \mu_1\right), \int |j|^2 d\mu_1 = \int |k|^2 d\mu_1 = 1\}.$$

It is easy to see that $\|f\|_{L^1(G,\nu_0)} = \|f^*\|_{L^1(G,\nu_0)} \leq \|f\|_{II,\mu} = \|f^*\|_{II,\mu} \leq \|f\|_{I,\mu} = \|f^*\|_{I,\mu}$.

Let $I(G, \nu, \mu) = \left\{f \in L^1(G, \nu_0), \|f\|_{I,\mu} < \infty\right\}$

Let $II(G, \nu, \mu) = \left\{f \in L^1(G, \nu), \|f\|_{II,\mu} < \infty\right\}$

Remark 9 *If L is the integrated form of a representation, $\left(\mu, G^{(0)} * \mathcal{H}, L\right)$, of the groupoid G, then*

$$|\langle L(f)\xi,\eta\rangle| \leq \left\langle II_\mu(|f|)\widetilde{\xi},\widetilde{\eta}\right\rangle$$

where $\widetilde{\xi}(u) = \|\xi(u)\|$. Therefore $\|L(f)\| \leq \|II_\mu(|f|)\| = \|f\|_{II,\mu} \leq \|f\|_{I,\mu}$.

Lemma 10 *Let us assume that the decomposition of the Haar system over the principal groupoid is bounded. Let $f \in L^1(G,\nu_0)$ such that $f_\Delta \in L^\infty\left(G^{(0)},\mu\right)$, where for μ-a.a $w \in G^{(0)}$*

$$f_\Delta(w) = \int\int\left(\int |f(x)|\,\Delta(x)^{-\frac{1}{2}}\,d\nu_{u,v}(x)\right)^2 d\eta_{e(w)}(u)\,d\eta_{e(w)}(v).$$

Then there is a sequence $(f_n)_n$, in $C_c(G)$ such that

$$\lim_n \|f - f_n\|_{II,\mu} = 0.$$

Proof. Let $g \in L^1(G,\nu_0)$ such that $g_\Delta \in L^\infty\left(G^{(0)},\mu\right)$, where

$$g_\Delta(w) = \int\int\left(\int |g(x)|\,\Delta(x)^{-\frac{1}{2}}\,d\nu_{u,v}(x)\right)^2 d\eta_{e(w)}(v)\,d\eta_{e(w)}(v),\ w \in G^{(0)}.$$

We claim that

$$\|g\|^2_{II,\mu} \leq \|g_\Delta\|_\infty \quad \left(\text{in } L^\infty\left(G^{(0)},\mu\right)\right)$$

Indeed, let $j,k \in L^2\left(G^{(0)},\mu_1\right)$ with $\int |j|^2 d\mu_1 = \int |k|^2 d\mu_1 = 1$. We have

$$\begin{aligned}
&\int |g(x)\,j(d(x))\,k(r(x))|\,\Delta(x)^{-\frac{1}{2}}\,d\nu(x) \\
&\quad = \int\int\int\int |g(x)|\,\Delta(x)^{-\frac{1}{2}}\,d\nu_{u,v}(x)\,|j(v)|\,|k(u)|\,d\eta_{e(w)}(u)\,d\eta_{e(w)}(v)\,d\mu(w) \\
&\quad \leq \int(\left(\int\int\left(\int |g(x)|\,\Delta(x)^{-\frac{1}{2}}\,d\nu_{u,v}(x)\right)^2 d\eta_{e(w)}(u)\,d\eta_{e(w)}(v)\right)^{\frac{1}{2}}\cdot \\
&\qquad\qquad \left(\int\int |j(v)|^2\,|k(u)|^2\,d\eta_{e(w)}(u)\,d\eta_{e(w)}(v)\right)^{\frac{1}{2}})d\mu(w) \\
&\quad \leq \sqrt{\|g_\Delta\|_\infty}\cdot\left(\int\int |j(v)|^2\,d\eta_{e(w)}(v)\,d\mu(w)\right)^{\frac{1}{2}}\left(\int\int |k(u)|^2\,\eta_{e(w)}(u)\,d\mu(w)\right)^{\frac{1}{2}} \\
&\quad = \sqrt{\|g_\Delta\|_\infty}.
\end{aligned}$$

Consequently,

$$\|g\|_{II,\mu} \leq \|g_\Delta\|_\infty$$

Let $B_c(G)$ be the space of bounded Borel functions on G, each with compact support. Let η_0 be a dominant for $\{\eta_u\}$. If $f \in B_c(G)$, then f is the limit almost everywhere with respect to $\int \nu_{u,v} d(\eta_0 \times \eta_0)(u,v)$, of a sequence, $(f_n)_n$, in $C_c(G)$ that is uniformly bounded and supported on some compact set supporting f. Let K be the support of f. Since

$$\int\int |f(x) - f_n(x)| \Delta(x)^{-\frac{1}{2}} d\nu_{u,v}(x) d(\eta_0 \times \eta_0)(u,v) \to 0 \ (n \to \infty)$$

it follows that there is a subsequence of $(f_n)_n$ such that

$$\int |f(x) - f_{n_k}(x)| \Delta(x)^{-\frac{1}{2}} d\nu_{u,v}(x) \to 0 \ (k \to \infty) \text{ a.e}$$

Because of the boundedness properties of f, $(f_n)_n$ and the boundedness of the system $\{\nu_{u,v}, (u,v) \in R\}$ and Δ on compact sets, it results that there exists $M > 0$ such that

$$\left(\int |f(x) - f_{n_k}(x)| \Delta(x)^{-\frac{1}{2}} d\nu_{u,v}(x)\right)^2 \leq M 1_{r(K)}(u) 1_{d(K)}(v)$$

Since

$$\sup_w \int\int \left(\int |f(x) - f_{n_k}(x)| \Delta(x)^{-\frac{1}{2}} d\nu_{u,v}(x)\right)^2 d\eta_{e(w)}(v) d\eta_{e(w)}(u)$$
$$\leq \int\int \left(\int |f(x) - f_{n_k}(x)| \Delta(x)^{-\frac{1}{2}} d\nu_{u,v}(x)\right)^2 d\eta_0(v) d\eta_0(u)$$

which converges to zero, by the Dominated Convergence Theorem, it follows that

$$\lim_k \|f - f_{n_k}\|_{II,\mu} = 0.$$

Let $f \in L^1(G, \nu_0)$ such that

$$\|f_\Delta\|_\infty < \infty$$

Let $g_n(x) = f(x)$ if $|f(x)| \leq n$, $g_n(x) = 0$ otherwise. Let $(K_n)_n$ be an increasing sequence of compact sets with $\bigcup_n K_n = G$. Let $f_n = g_n 1_{K_n}$. $|f - f_n|$ converges pointwise to zero, dominated by $|f|$. Hence

$$\left\| w \to \int\int \left(\int |f(x) - f_n(x)| \Delta(x)^{-\frac{1}{2}} d\nu_{u,v}(x)\right)^2 d\eta_{e(w)}(v) d\eta_{e(w)}(u) \right\|_\infty$$
$$\leq \int\int \left(\int |f(x) - f_n(x)| \Delta(x)^{-\frac{1}{2}} d\nu_{u,v}(x)\right)^2 d\eta_0(v) d\eta_0(u)$$

which converges to zero, by the Dominated Convergence Theorem. It follows that there is a sequence in $B_c(G)$ such that

$$\lim_n \|f - f_n\|_{II,\mu} = 0.$$

■

Proposition 11 *Let us assume that the decomposition of the Haar system over the principal groupoid is bounded. If $f \in I(G,\nu,\mu)$ and $g \in C_c(G)$, then the function $f * g \in L^1(G,\nu_0)$ and the function*

$$w \to \int\int\left(\int |f*g(x)|\,\Delta(x)^{-\frac{1}{2}}\,d\nu_{u,v}(x)\right)^2 d\eta_{e(w)}(v)\,d\eta_{e(w)}(v)\,,\ w\in G^{(0)}$$

is in $L^\infty\left(G^{(0)},\mu\right)$.

Proof. Let K be the support of g. Let $M=\sup\limits_{v,w}\left(\int |g(y)|\,\Delta(y)^{-\frac{1}{2}}\,d\nu_{w,v}(y)\right)$.

$$\begin{aligned}
&\nu_{u,v}\left(|f*g|\,\Delta^{-\frac{1}{2}}\right)\\
&=\int\left|\int\int f(xy)\,g\left(y^{-1}\right)d\nu_{d(x),w}(y)\,d\eta_{e(d(x))}(w)\right|\Delta(x)^{-\frac{1}{2}}\,d\nu_{u,v}(x)\\
&\le\int\int\int\left|f(xy)\,g\left(y^{-1}\right)\right|d\nu_{d(x),w}(y)\,d\eta_{e(u)}(w)\,\Delta(x)^{-\frac{1}{2}}\,d\nu_{u,v}(x)\\
&=\int\int\int\left|f(xy)\,g\left(y^{-1}\right)\right|\Delta(x)^{-\frac{1}{2}}\,d\nu_{v,w}(y)\,d\nu_{u,v}(x)\,d\eta_{e(u)}(w)\\
&=\int\int\int\left|f(xy)\,g\left(y^{-1}\right)\right|\Delta(x)^{-\frac{1}{2}}\,d\nu_{u,v}(x)\,d\nu_{v,w}(y)\,d\eta_{e(u)}(w)\\
&=\int\int\int\left|f(x)\,g\left(y^{-1}\right)\right|\Delta\left(y^{-1}\right)\Delta\left(xy^{-1}\right)^{-\frac{1}{2}}\,d\nu_{u,d(y)}(x)\,d\nu_{v,w}(y)\,d\eta_{e(u)}(w)\\
&=\int\int\int\left|f(x)\,g\left(y^{-1}\right)\right|\Delta(x)^{-\frac{1}{2}}\,\Delta(y)^{-\frac{1}{2}}\,d\nu_{u,w}(x)\,d\nu_{v,w}(y)\,d\eta_{e(u)}(w)\\
&=\int\left(\int|f(x)|\,\Delta(x)^{-\frac{1}{2}}\,d\nu_{u,w}(x)\int g\left(y^{-1}\right)\Delta(y)^{-\frac{1}{2}}\,d\nu_{v,w}(y)\right)d\eta_{e(u)}(w)\\
&=\int\left(\int|f(x)|\,\Delta(x)^{-\frac{1}{2}}\,d\nu_{u,w}(x)\int|g(y)|\,\Delta(y)^{-\frac{1}{2}}\,d\nu_{w,v}(y)\right)d\eta_{e(u)}(w)\\
&=\int\left(\int|f(x)|\,\Delta(x)^{-\frac{1}{2}}\,d\nu_{u,w}(x)\int|g(y)|\,\Delta(y)^{-\frac{1}{2}}\,d\nu_{w,v}(y)\right)\cdot\\
&\qquad\qquad\cdot\,1_{r(K)}(w)\,d\eta_{e(u)}(w)\,1_{d(K)}(v)\\
&\le\sup_{v,w}\left(\int|g(y)|\,\Delta(y)^{-\frac{1}{2}}\,d\nu_{w,v}(y)\right)\int\int|f(x)|\,\Delta(x)^{-\frac{1}{2}}\,d\nu_{u,w}(x)\cdot\\
&\qquad\qquad\cdot 1_{r(K)}(w)\,d\eta_{e(u)}(w)\,1_{d(K)}(v)\\
&=M1_{d(K)}(v)\int\int|f(x)|\,\Delta(x)^{-\frac{1}{2}}\,d\nu_{u,w}(x)\,1_{r(K)}(w)\,d\eta_{e(u)}(w)\\
&\le M1_{d(K)}(v)\int\left(\int|f(x)|\,d\nu_{u,w}(x)\right)^{\frac{1}{2}}\left(\int|f(x)|\,\Delta(x)^{-1}\,d\nu_{u,w}(x)\right)^{\frac{1}{2}}\cdot\\
&\qquad\qquad\cdot 1_{r(K)}(w)\,d\eta_{e(u)}(w)\\
&\le M1_{d(K)}(v)\int\left(\int|f(x)|\,d\nu_{u,w}(x)\right)^{\frac{1}{2}}\left(\int\left|f\left(x^{-1}\right)\right|d\nu_{w,u}(x)\right)^{\frac{1}{2}}\cdot\\
&\qquad\qquad\cdot 1_{r(K)}(w)\,d\eta_{e(u)}(w)\\
&\le M1_{d(K)}(v)\left(\int\int|f(x)|\,d\nu_{u,w}(x)\,d\eta_{e(u)}(w)\right)^{\frac{1}{2}}\cdot\\
&\qquad\qquad\cdot\left(\int\int\left|f\left(x^{-1}\right)\right|d\nu_{w,u}(x)\,1_{d(K)}(w)\,d\eta_{e(u)}(w)\right)^{\frac{1}{2}}\\
&\le M1_{d(K)}(v)\left(\int\int|f(x)|\,d\nu^{u}(x)\right)^{\frac{1}{2}}\cdot\\
&\qquad\qquad\cdot\left(\int\int\left|f\left(x^{-1}\right)\right|d\nu_{w,u}(x)\,1_{r(K)}(w)\,d\eta_{e(u)}(w)\right)^{\frac{1}{2}}\\
&\le M1_{d(K)}(v)\left(\|f\|_I\right)^{\frac{1}{2}}\left(\int\int\left|f\left(x^{-1}\right)\right|d\nu_{w,u}(x)\,1_{r(K)}(w)\,d\eta_{e(u)}(w)\right)^{\frac{1}{2}}\ \mu\text{-a.e.}
\end{aligned}$$

Then

$$\int\int\left(\int\left|f*g\left(x\right)\right|\Delta\left(x\right)^{-\frac{1}{2}}d\nu_{u,v}\left(x\right)\right)^{2}d\eta_{e(s)}\left(v\right)d\eta_{e(s)}\left(u\right)$$

$$= \int\int M^{2}\left\|f\right\|_{I}1_{d(K)}\left(v\right)\cdot$$
$$\cdot\left(\int\left|f\left(x^{-1}\right)\right|d\nu^{w}\left(x\right)\right)1_{d(K)}\left(w\right)d\eta_{e(s)}\left(v\right)d\eta_{e(s)}\left(w\right)$$

$$= M^{2}\left\|f\right\|_{I}\int\int 1_{d(K)}\left(v\right)\cdot$$
$$\cdot\left(\int\left|f\left(x^{-1}\right)\right|d\nu^{w}\left(x\right)\right)1_{d(K)}\left(w\right)d\eta_{e(s)}\left(v\right)d\eta_{e(s)}\left(w\right)$$

$$= M^{2}\left\|f\right\|_{I}\int\int 1_{d(K)}\left(v\right)\cdot$$
$$\cdot\int\left|f\left(x^{-1}\right)\right|d\nu_{w,u}\left(x\right)d\eta_{e(s)}\left(u\right)d\eta_{e(s)}\left(v\right)1_{r(K)}\left(w\right)d\eta_{e(s)}\left(w\right)$$

$\leq M^{2}\left\|f\right\|_{I}^{2}\int\int 1_{d(K)}\left(v\right)1_{d(K)}\left(w\right)d\eta_{e(s)}\left(v\right)d\eta_{e(s)}\left(w\right)$
$<\infty$, μ-a.e. ■

3 The inequality $\|II_{\mu}(\cdot)\|\leq\|Reg_{\mu}(\cdot)\|$

Proposition 12 *Let G be a locally compact groupoid equipped with a continuous Haar system. Assume that the decomposition of the Haar system over the principal groupoid is bounded and that $C^{*}_{red}(G)=C^{*}(G)$. Let μ be a quasi-invariant probability measure with $supp(\mu)=G^{(0)}$. Then*

$$||II_{\mu}(f)||\leq||Reg_{\mu}(f)||$$

for all $f\in I(G,\nu,\mu)$.(Reg_{μ} is the left regular representation of $C_{c}(G)$ on μ)

Proof. Let

$$\nu^{u}=\int\nu_{u,v}d\eta_{u}\left(v\right)\quad\text{for all }u\in G^{(0)}$$

be the decomposition of the Haar system for G over the principal groupoid associated to G Let $\mu_{1}=\int\eta_{u}d\mu\left(u\right)$. Then μ_{1} is equivalent to μ.Let Δ be the modular function associated to $\left\{\nu^{u},u\in G^{(0)}\right\}$ and μ_{1} Let ν be the measure induced by μ_{1} on G, and $\nu_{0}=\Delta^{-\frac{1}{2}}\nu$.

If $f \in C_c(G)$, then

$$||f||_{red} = ||Ind\mu(f)||$$

where $Ind\mu(f)$ is the operator on $L^2(G, \nu^{-1})$ defined by formula

$$Ind\mu(f)\xi(x) = f * \xi(x)$$

([7], pg.50). The map $W : L^2(G, \nu) \to L^2(G, \nu^{-1})$ defined by the formula, $W\xi = \xi\Delta^{\frac{1}{2}}$ is a Hilbert space isomorphism that implements a unitary equivalence between Reg_μ (the left regular representation of $C_c(G)$ on μ) and $Ind\mu$.([12], II.1.10). Therefore, if μ is a quasi-invariant measure with supp$(\mu) = G^{(0)}$, then

$$||f||_{red} = ||Reg_\mu(f)|| \qquad (1)$$

Since $\mu_1 = \int\int \eta_{e(u)}(w)\, d\mu(u)$ is equivalent to μ, it follows that $||Reg_\mu(f)|| = ||Reg_{\mu_1}(f)||$ and $||Ind\mu(f)|| = ||Ind\mu_1(f)||$.

Since $C^*_{red}(G) = C^*(G)$, for all $f \in C_c(G)$

$$||II_\mu(f)|| \le ||f||_{red} \quad (2)$$

From (1) and (2) it follows that

$$||II_\mu(f)|| \le ||Reg_\mu(f)|| = ||Ind\mu(f)|| \quad (3)$$

for all $f \in C_c(G)$. If $f \in C_c(G)$, $f \ge 0$, then $||II_\mu(f)|| = ||Reg_\mu(f)|| = ||Ind\mu(f)||$.

If $f \in L^1(G, \nu_0)$ and

$$\int\int\int \left(\int |f(x)|\, \Delta(x)^{-\frac{1}{2}}\, d\nu_{u,v}(x)\right)^2 d\eta_{e(w)}(v)\, d\eta_{e(w)}(v)\, d\mu(w) < \infty.$$

then there is a sequence a sequence $(f_n)_n$, in $C_c(G)$ such that

$$\lim_n \|f - f_n\|_{II,\mu} = 0.$$

Hence

$$\lim_n \|L(f - f_n)\| = 0.$$

for any representation L of G. From (3) it follows that $||II\mu(f)|| \le ||Reg_\mu(f)||$ for all f with the above property.

Let $f \in I(G, \nu, \mu)$ and $g \in C_c(G)$. By the preceding lemma, the function $f * g$ has the property that

$$\int\int\int \left(\int |f * g(x)|\, \Delta(x)^{-\frac{1}{2}}\, d\nu_{u,v}(x)\right)^2 d\eta_{e(w)}(v)\, d\eta_{e(w)}(v)\, d\mu(w) < \infty.$$

It follows that

$$||II_{\mu}\left(f * g\right)|| \leq ||Reg_{\mu}\left(f * g\right)||$$

Let $(K_n)_n$ be a sequence of compact subsets of $G^{(0)}$ with $\cup K_n = G^{(0)}$. By Corollary 2.11/p, 12 [8] (or Lemma 5.48/p. 167 [7]), the *-algebra $C_c(G)$ has a two-sided approximate identity with respect to the inductive limit topology, $(e_n)_n$, with the following properties:

1) $e_n(x) \geq 0$, for all $x \in G$

2) $\left|\int e_n(x)\, d\nu^u(x) - 1\right| < \frac{1}{n}$, for all $u \in K_n$

3) $e_n(x) = e_n\left(x^{-1}\right)$, for all $x \in G$.

Let $\xi \in L^2\left(G, \nu^{-1}\right)$. Since $\|Ind\mu(e_n)\xi - \xi\|_2 = \|e_n * \xi - \xi\|_2 \to 0\ (n \to \infty)$, Banach Steinhaus Theorem implies that $(\|Ind\mu(e_n)\|)_n$ is bounded. Thus there is $M > 0$, such that $\|e_n\|_{II,\mu} = \|II_\mu(e_n)\| = \|Ind\mu(e_n)\| \leq M$ for all n.

Let $f \in I(G, \nu, \mu)$. First, we shall assume that $d(supp(f))$ is a compact set in $G^{(0)}$.

For each n, $e_n \in L^2\left(G, \nu^{-1}\right)$ because $e_n \in C_c(G)$. Hence $f * e_n \in L^2\left(G, \nu^{-1}\right)$ $(\|f * e_n\|_2 \leq \|f\|_I \|e_n\|_2)$.

Since $f \in I(G, \nu, \mu) \subset L^1(G, \nu)$, it follows that for each ε, there is $g \in C_c(G)$, such that $\int |f - g|\, d\nu < \varepsilon$. We have

$$\int |f - f * e_n|\, d\nu \leq \int |f - g|\, d\nu + \int |g - g * e_n|\, d\nu + \int |(f - g) * e_n|\, d\nu$$

and

$\int |(f - g) * e_n|\, d\nu = \int \left|\int (f - g)(y)\, e_n\left(y^{-1}x\right) d\nu^{r(x)}(y)\right| d\nu(x)$
$\leq \int \int |f - g|(y)\, e_n\left(y^{-1}x\right) d\nu^{r(x)}(y)\, d\nu(x)$
$\leq \int \int \int |f - g|(y)\, e_n\left(y^{-1}x\right) d\nu^{r(x)}(y)\, d\nu^u(x)\, d\mu(u)$
$= \int \int \int |f - g|(y)\, e_n\left(y^{-1}x\right) d\nu^u(x)\, d\nu^u(y)\, d\mu(u)$
$= \int \int \int |f - g|(y)\, e_n(x)\, d\nu^{d(y)}(x)\, d\nu^u(y)\, d\mu(u)$
$\leq \left(1 + \frac{1}{n}\right) \int |f - g|\, d\nu$ for n with the property that $d(supp(f) \cup supp(g)) \subset K_n$

Hence $\lim_n \int |f - f * e_n|\, d\nu = 0$. Since $\lim_n \int \int |f - f * e_n|\, d\nu^u d\mu_1(u) = 0$, passing to a subsequence we may assume that $\lim_n \int |f - f * e_n|\, d\nu^u = 0$ *a.e.*, dominated by $3\|f\|_I$ for all n such that $d(supp(f)) \subset K_n$.

If $a \in C_c\left(G^{(0)}\right)$, then

$$
\begin{aligned}
& \int \left|f * e_n\right|(x)\left|a(d(x))\right|^2 \Delta^{-1}(x)\, d\nu(x) = \\
= & \int \left|f * e_n\right|\left(x^{-1}\right)\left|a(r(x))\right|^2 d\nu(x) \\
= & \int\int \left|e_n * \overline{f^*}\right|(x)\left|a(r(x))\right|^2 d\nu^u(x)\, d\mu_1(u) \\
\leq & \int\int\int e_n(y)\left|f^*\left(y^{-1}x\right)\right| d\nu^{r(x)}(y)\left|a(r(x))\right|^2 d\nu^u(x)\, d\mu_1(u) \\
= & \int\int\int e_n(y)\left|f^*\left(y^{-1}x\right)\right| \left|a(u)\right|^2 d\nu^u(y)\, d\nu^u(x)\, d\mu_1(u) \\
= & \int\int e_n(y)\left(\int \left|f^*\left(y^{-1}x\right)\right| d\nu^u(x)\right)\left|a(u)\right|^2 d\nu^u(y)\, d\mu_1(u) \\
= & \int\int e_n(y)\left(\int \left|f^*(x)\right| \nu^{d(y)}(x)\right)\left|a(u)\right|^2 d\nu^u(y)\, d\mu_1(u) \\
\leq & \left\|f\right\|_I \int\int e_n(y)\, d\nu^u(y)\left|a(u)\right|^2 d\mu_1(u) \\
\leq & \left(1+\frac{1}{n}\right)\left\|f\right\|_I \left\|a\right\|_2^2
\end{aligned}
$$

for all n such that $supp(a) \subset K_n$.

If $a \in C_c\left(G^{(0)}\right)$ and $b \in L^2\left(G^{(0)}, \mu\right)$, then $\lim_n \left|\left\langle II\left(f - f * e_n\right) a, b\right\rangle\right| = 0$. Indeed,

$$
\begin{aligned}
\left|\left\langle II_\mu\left(f - f * e_n\right) a, b\right\rangle\right| \leq & \\
\leq & \int \left|f(x) - f * e_n(x)\right|\left|a(d(x))\right|\left|b(r(x))\right| d\nu_0(x) \\
\leq & \left(\int \left|f(x) - f * e_n(x)\right| \left|a(d(x))\right|^2 \Delta^{-1}(x)\, d\nu(x)\right)^{\frac{1}{2}} \cdot \\
& \cdot \left(\int \left|f(x) - f * e_n(x)\right| \left|b(r(x))\right|^2 d\nu(x)\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\leq & \left(\int\int \left|f(x) - f * e_n(x)\right| d\nu_u(x)\, \left|a(u)\right|^2 d\mu(u)\right)^{\frac{1}{2}} \cdot \\
& \cdot \left(\int \left|f(x) - f * e_n(x)\right| d\nu^u(x)\left|b(u)\right|^2 d\mu(u)\right)^{\frac{1}{2}} \\
\leq & \left\|a\right\|_2 \left(3\left\|f\right\|_I\right)^{\frac{1}{2}}\left(\int \left|f(x) - f * e_n(x)\right| d\nu^u(x)\left|b(u)\right|^2 d\mu_1(u)\right)^{\frac{1}{2}}.
\end{aligned}
$$

Since $C_c\left(G^{(0)}\right)$ is dense in $L^2\left(G^{(0)}, \mu_1\right)$ and $\left(\left\|II_\mu\left(f - f * e_n\right)\right\|\right)_n$ is bounded, it follows that $\lim_n \left|\left\langle II_\mu\left(f - f * e_n\right) a, b\right\rangle\right| = 0$ for all $a, b \in L^2\left(G^{(0)}, \mu_1\right)$.

Finally, we have

$$\begin{aligned}
|\langle II_\mu((f)\,a,b\rangle| &\leq \\
&\leq \lim_n |\langle II_\mu((f*e_n-f)\,a,b\rangle| + \lim_n |\langle II_\mu\left((f*e_n)\,a,b\right\rangle| \\
&\leq \lim_n \|II_\mu\left((f*e_n)\right\| \|a\|_2 \|b\|_2 \\
&\leq \lim_n \|Ind\mu\left((f*e_n)\right\| \|a\|_2 \|b\|_2 \\
&\leq \lim_n \|Ind\mu\,(f)\,Ind\mu\,(e_n)\| \|a\|_2 \|b\|_2 \\
&\leq \lim_n \|Ind\mu\,(f)\| \|Ind\mu\,(e_n)\| \|a\|_2 \|b\|_2 .
\end{aligned}$$

Thus $\|II_\mu\,(f)\| \leq \|Ind\mu((f)\|$.

Let $f \in I\,(G,\nu,\mu)$. Let $(K_n)_n$ be a sequence of compact subsets of $G^{(0)}$ with $\cup K_n = G^{(0)}$. If $f_n = f 1_{K_n} \circ d$, then $|f_n - f|$ converges to zero dominated by $|f|$. Hence, if $a,b \in L^2\left(G^{(0)},\mu_1\right)$, then $\lim_n |\langle II_\mu\,(f-f_n)\,a,b\rangle| = 0$. On the other hand, if $\xi \in L^2\left(G,\nu^{-1}\right)$,

$$\begin{aligned}
\|Ind\mu\,(f_n)\,\xi\|_2 &= \|f_n * \xi\|_2 = \|(f 1_{K_n} \circ d) * \xi\|_2 = \\
&= \|f * (\xi 1_{K_n} \circ r)\|_2 \leq \|Ind\mu\,(f)\,(\xi 1_{K_n} \circ r)\|_2 \\
&\leq \|Ind\mu\,(f)\|_2 \|(\xi 1_{K_n} \circ r)\|_2 \leq \|Ind\mu\,(f)\|_2 \|\xi\|_2
\end{aligned}$$

Thus $\|Ind\mu\,(f_n)\| \leq \|Ind\mu\,(f)\|$ .For $a,b \in L^2\left(G^{(0)},\mu\right)$, we have

$$\begin{aligned}
|\langle II_\mu((f)\,a,b\rangle| &= \\
&\leq \lim_n |\langle II_\mu(f_n - fa,b\rangle| + \lim_n |\langle II_\mu\left((f_n)\,a,b\right\rangle| \\
&\leq \lim_n \|II_\mu\left((f_n)\right\| \|a\|_2 \|b\|_2 \\
&\leq \lim_n \|Ind\mu\left((f_n)\right\| \|a\|_2 \|b\|_2 \\
&\leq \lim_n \|Ind\mu\,(f)\| \|a\|_2 \|b\|_2 .
\end{aligned}$$

It follows that $\|II_\mu((f)\| \leq \|Ind\mu\,(f)\| = ||Reg_\mu\,(f)\,||$ for all $f \in I\,(G,\nu,\mu)$. ■

Proposition 13 *Let G be a locally compact groupoid equipped with a continuous Haar system $\{\nu^u, u \in G^{(0)}\}$. Assume that the decomposition of the Haar system over the principal groupoid is bounded and that $C^*_{red}(G) = C^*(G)$. Then G is measurewise amenable.*

Proof. Let μ be a quasi-invariant probability measure. First, we shall assume that μ is a quasi-invariant probability measure with $\text{supp}(\mu) = G^{(0)}$. From the preceding propositions it follows that the trivial representation of $C^*(G,\nu,\mu)$ is weakly contained in the regular representation. Therefore (G,ν,μ) is amenable (Theorem 6.1.4/p. 142 [1]).

Let μ be an arbitrary quasi-invariant measure, μ_0 be a quasi-invariant measure with $\text{supp}(\mu_0) = G^{(0)}$, and $\mu_1 = \mu_0 + \mu$. Then μ_1 is quasi-invariant measure with $\text{supp}(\mu_1) = G^{(0)}$, and therefore (G,ν,μ_1) is amenable. Proposition 3.2.14, (iii)/p. 74 [1] shows that there exists a sequence (g_n) in $B_b(G,\nu)^+$ normalized and such that

$$\lim_n \int h(u) \left[\int |f * g_n - (\nu(f) \circ r)\, g_n|\, d\nu^u\right] d\mu_1(u) = 0$$

for all $f \in B_b(G, \nu)$ and $h \in L^1(\mu_1)$.

Since $\mu << \mu_1$, there exists a Borel function g such that $\mu = g\mu_1$. If $h \in L^1(\mu)$, then $hg \in L^1(\mu_1)$ and

$$\lim_n \int h(u)\, g(u) \left[\int |f * g_n - (\nu(f) \circ r)\, g_n|\, d\nu^u \right] d\mu_1(u) = 0$$

$$\lim_n \int h(u) \left[\int |f * g_n - (\nu(f) \circ r)\, g_n|\, d\nu^u \right] d\mu(u) = 0$$

Therefore (G, ν, μ) is amenable. ■

Acknowledgement. This work is partially supported by the CNCSIS grant At 127/2004.

References

[1] C. Anantharaman-Delaroche, J. Renault, *Amenable groupoids*, Monographie de L'Enseignement Mathematique No **36**, Geneve, 2000.

[2] M. Buneci, *Consequences of Hahn structure theorem for the Haar measure*, Math. Reports, **4**(54), 4(2002), 321-334.

[3] M. Buneci, *C*-algebras associated to the transitive groupoids*, An. Univ. Craiova Ser. Mat. Inform, Vol. XXVIII, 2001, 79-92.

[4] M. Buneci, *The structure of the Haar systems on locally compact groupoids*, Math. Phys. Electron. J, Vol. **8** (4) (2002), 1-11.

[5] P. Hahn, *Haar measure for measure groupoids*, Trans. Amer. Math. Soc. **242**(1978), 1-33.

[6] P. Hahn, *The regular representations of measure groupoids*, Trans. Amer. Math. Soc. **242** (1978), 34-72.

[7] P. Muhly, *Coordinates in operator algebra*, Book in preparation.

[8] P. Muhly, J. Renault and D. Williams, *Equivalence and isomorphism for groupoid C*-algebras*, J. Operator Theory **17**(1987), 3-22

[9] A. Ramsay, *Virtual groups and groups actions*, Adv. in Math. **6**(1971), 253-322.

[10] A. Ramsay, *Topologies on measured groupoids*, J. Funct. Anal. **47**(1982), 314-343.

[11] A. Ramsay and M. E. Walter, *Fourier-Stieltjes Algebras of locally compact groupoids*, J. Funct. Anal. **148**(1997), 314-367.

[12] J. Renault, *A groupoid approach to C^*- algebras*, Lecture Notes in Math., Springer-Verlag, **793**, 1980.

[13] J. Renault,*The ideal structure of groupoid crossed product algebras*, J. Operator Theory, **25**(1991), 3-36.

[14] J. Westman, *Nontransitive groupoid algebras*, Univ. of California at Irvine, 1967.

University Constantin Brâncuşi of Târgu-Jiu
Bulevardul Republicii, Nr. 1,
210152 Târgu-Jiu , Gorj
Romania
e-mail: ada@utgjiu.ro