AMENABLE EQUIVARIANT MAPS DEFINED ON A GROUPOID

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ABSTRACT. We prove the equivalence between the amenability of an equivariant map defined on a groupoid and the amenability of a semidirect product. Also, we give a generalization of the following well known result (J.P. Pier, 21.2 of [12]): if G is an amenable locally compact non-compact group, and m is an invariant mean on $L^\infty(G)$, then $m(f) = 0$ for every $f \in C_0(G)$. We establish a similar property of an approximate strongly invariant mean [1, Definition 3.1.24] for the map $x \mapsto (r(x), d(x))$, where r and d are the range, respectively the domain map, of a locally compact groupoid G with non-compact isotropy groups.

1. INTRODUCTION

The notion of amenability for groupoids was introduced in [13] and was extensively studied in [1]. In [1] a more general notion of amenability for equivariant maps was introduced. We shall establish that an equivariant map π from a Borel groupoid G to a Borel G-space S is amenable if and only if the semidirect product $S * G$ is amenable. We shall also extend from groups to groupoids the following well known result: if G is an amenable non-compact locally compact Hausdorff topological group endowed with a Haar measure ν, and m is an invariant mean on $L^\infty(G)$, then $m(f) = 0$ for all $f \in C_0(G)$, the space of complex-valued continuous functions on G vanishing at infinity [12, Section 21.1]. In this paper we shall consider the principal groupoid R associated with G, and a decomposition

$$\lambda^u = \int \beta_t^s \, d\alpha^u(s,t)$$

of the Haar system λ on G over the principal groupoid R. We shall assume that there is an approximate strongly invariant mean $(g_n)_n$ for the map $\theta : G \to R$, $\theta(x) = (r(x), d(x))$ with respect to $(\lambda, \beta, \mu \circ \alpha)$, where μ is a quasi-invariant measure for the Haar system λ. This condition is equivalent to the amenability of θ and is weaker than the amenability of $r : G \to G^{(0)}$. We shall prove that there is a subsequence of $(g_n)_n$, still denoted $(g_n)_n$, such that

$$\lim_{n\to\infty} \int \varphi(x) g_n(x) \, d\beta_v^u(x) = 0$$

for any compactly supported Borel bounded function φ on G and $\mu \circ \alpha$-a.e. $(u, v) \in R$.

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In order to establish notation and to give the reader some essential information about groupoids and amenable maps, we include here a list of definitions that can be found in [9, 10, 13, 8, 1].

1.1. **Groupoids and semi-direct products.** We shall use the definition of a topological groupoid given by Renault in [13, Definition I.2.1]. For a groupoid G, $G^{(0)}$ will denote its *unit space* and $G^{(2)}$ the *set of the composable pairs*. Usually, elements of G will be denoted by letters as x, y, or z, and the elements of $G^{(0)}$ by letters such as u, v, or w. The *inverse map* is written $x \mapsto x^{-1}$ $[: G \to G]$ and the *product map* is written $(x, y) \mapsto xy$ $[: G^{(2)} \to G]$. The *range* and the *source* maps from G to $G^{(0)}$ will be denoted respectively by r and d. The *fibers of the range* and the *source maps* are denoted $G^u = r^{-1}(\{u\})$ and $G_v = d^{-1}(\{v\})$, respectively. More generally, given the subsets $A, B \subset G^{(0)}$, we define $G^A = r^{-1}(A)$, $G_B = d^{-1}(B)$, and $G^A_B = r^{-1}(A) \cap d^{-1}(B)$. The *reduction* of G is $G|A = G^A_A$.

A *Borel groupoid* is a groupoid G endowed with a Borel structure, such that $G^{(2)}$ is a Borel set in the product structure $G \times G$, and the product map and the inverse map are Borel functions. We are exclusively concerned with analytic Borel structure.

A *topological groupoid* consists of a groupoid G and a topology compatible with the groupoid structure. This means that the product map and the inverse map are continuous functions, where $G^{(2)}$ has the induced topology from $G \times G$. We are concerned with topological groupoids which are second countable, locally compact, and Hausdorff. It was shown in [10] that measured groupoids (in the sense of [6, Definition 2.3]) may be assumed, with no loss of generality, to have locally compact topologies.

If X is a locally compact space, $C_c(X)$ denotes the space of complex-valued continuous functions with compact support. The *Borel sets* of a topological space are taken to be the elements of the σ-algebra generated by the open sets.

A *Haar system* on a locally compact groupoid G is a family of positive Radon measures on G, $\{\lambda^u : u \in G^{(0)}\}$, having the following properties [13, Definition I.2.2]:

(1) For all $u \in G^{(0)}$, $\operatorname{supp}(\lambda^u) = G^u$.
(2) For all $f \in C_c(G)$

$$u \mapsto \int f(x)\, d\lambda^u(x) \quad [: G^{(0)} \to \mathbb{C}]$$

is continuous.
(3) For all $f \in C_c(G)$ and all $x \in G$,

$$\int f(y)\, d\lambda^{r(x)}(y) = \int f(xy)\, d\lambda^{d(x)}(y).$$

In the Borel setting, a system of measures $\{\lambda^u : u \in G^{(0)}\}$ will be called *Borel Haar system* (in short *BHS*) on the Borel groupoid G if each λ^u has its support in G^u, is invariant in the sense of (3), is proper in the sense that there is a positive Borel function f_0 on G such that $\lambda^u(f_0) = 1$ for all u, and for all $f \geq 0$ Borel on G,

$$u \mapsto \int f(x)\, d\lambda^u(x) \quad [: G^{(0)} \to \overline{\mathbb{R}}]$$

is a real-extended Borel map.

The image of λ^u by the inverse map $x \mapsto x^{-1}$ is denoted λ_u.

If μ is a measure on $G^{(0)}$, then the measure $\nu = \int \lambda^u \, d\mu(u)$, defined by

$$\int f(y)\, d\nu(y) = \int \left(\int f(y)\, d\lambda^u(y) \right) d\mu(u), \qquad f \geq 0 \text{ Borel,}$$

is called the measure on G induced by μ. The image of ν by the inverse map $x \mapsto x^{-1}$ is denoted ν^{-1}. The measure μ is said *quasi-invariant* if its induced measure ν is equivalent to its inverse ν^{-1}. If μ is a quasi-invariant measure on $G^{(0)}$ and λ is the measure induced on G, then the Radon-Nikodym derivative $\Delta = d\nu / d\nu^{-1}$ is called the *modular function of* μ.

A (*left*) *Borel* G*-space* is a Borel space S endowed with a Borel surjection $r : S \to G^{(0)}$ and a Borel map $(x, s) \mapsto xs$ from the space

$$G * S = \{(x, s) : d(x) = r(s)\}$$

to S, satisfying the following conditions:

(1) $r(xs) = r(x)$ for all $(x, s) \in G * S$, and $r(s)s = s$ for all $s \in S$.
(2) If $(x_1, x_2) \in G^{(2)}$ and $(x_2, s) \in G * S$, then $(x_1 x_2)s = x_1(x_2 s)$.

The space

$$S * G = \{(s, x) : r(s) = r(x)\}$$

has a groupoid structure, called *semi-direct product*, with the following operations

$$(s, x)(x^{-1}s, y) = (s, xy), \qquad (s, x)^{-1} = (x^{-1}s, x^{-1}).$$

Let us fix a BHS $\lambda = \{\lambda^u : u \in G^{(0)}\}$ on G. Let S be a Borel G-space. Then $\{\varepsilon_s \times \lambda^{r(s)} : s \in S\}$ is a BHS for $S * G$, where ε_s is the unit point mass at s. A positive measure μ on S is called *quasi-invariant with respect to* ν if the measure $\mu \circ \lambda$ on $S * G$ defined by

$$\int f \, d\mu \circ \lambda = \int f(s, x)\, d\lambda^{r(s)}(x)\, d\mu(s)$$

for all Borel nonnegative function on $S * G$, is equivalent to its image under the inverse map $(s, x) \to (x^{-1}s, x^{-1})$.

Let us denote by $B_b(G, \nu)$ the space of Borel functions f on G with

$$u \mapsto \int f \, d\lambda^u$$

bounded. $B_b(G, \nu)$ acts by convolution on $L^\infty(S)$ according to the formula

$$f * \varphi(s) = \int f(x) \varphi(x^{-1}s)\, d\lambda^{r(s)}(x), \qquad f \in B_b(G, \nu),\ \varphi \in L^\infty(S).$$

1.2. **Amenable Borel equivariant maps.** Concerning amenability, we shall use definitions and notation from [1], which we recall below.

Let T, S be two Borel spaces and $\pi : T \to S$ a Borel map. A *Borel* π*-system* (or a *Borel system of measures for* π) is a family $\beta = \{\beta^s : s \in S\}$ of measures on $\pi^{-1}(\{s\})$ such that

(1) For every nonnegative Borel function f on T,

$$s \overset{\beta(f)}{\mapsto} \int f(t)\, d\beta^s(t)$$

defines a Borel map.
(2) There is a positive Borel function g on T such that $\beta(g) = 1$.

Condition (2) is equivalent to:

($*$): Y is the union of an increasing sequence $(A_n)_n$ of Borel subsets such that $s \mapsto \beta^s(A_n)$ is bounded for all n, and $\beta^s \neq 0$ for all $s \in S$ [5, Lemme 3, p. 37].

Let β be such a π-system and c be a class of measures on S. We shall fix a positive measure μ belonging to c, and the space $L^\infty(S, c)$ will be identified to $L^\infty(S, \mu)$, which will be denoted by $L^\infty(S)$. We introduce the measure $\mu \circ \beta$ on T defined by

$$\int f \, d\mu \circ \beta = \int \beta(f) \, d\mu$$

for every nonnegative Borel function $f : T \to \mathbb{R}$.

We define the Banach space $L^\infty(S, L^1(T, \beta))$ of $\mu \circ \beta$ -measurable functions $f : T \to \mathbb{C}$ such that $s \mapsto \int |f| d\beta^s$ is μ-essentially bounded, normed by

$$\|f\|_{\pi,1} = \|\beta(|f|)\|_\infty,$$

and the set $L^\infty(S, L^1(T, \beta))_1^+$ of nonnegative elements with norm ≤ 1 in $L^\infty(S, L^1(T, \beta))$. We denote $L^\infty(T) = L^\infty(T, \mu \circ \beta)$. In this context a *mean* is a positive unital $L^\infty(S)$-linear map [1, Definition 1.3.4]

$$m : L^\infty(T) \to L^\infty(S).$$

Now, let us fix two (left) Borel G-spaces T and S. A Borel map π from T to S is said to be *equivariant* if $r(\pi(t)) = r(t)$ for all $t \in T$, and $\pi(xt) = x\pi(t)$ for all $(x, t) \in G * T$. If $\pi : T \to S$ is an equivariant Borel map, then a Borel π-system $\beta = \{\beta^s : s \in S\}$ of measures is called *invariant* (or G-*invariant*) if $x^{-1}\beta^s = \beta^{x^{-1}s}$ for every $(s, x) \in S * G$, where $x^{-1}\beta^s$ is defined by

$$\int f(t) \, d(x^{-1}\beta^s)(t) = \int f(x^{-1}t) \, d\beta^s(t).$$

The Borel π-system $\beta = \{\beta^s : s \in S\}$ of measures is G-*quasi-invariant* if there is a Borel cocycle (i.e. a groupoid homomorphism) $q : T * G \to R_*^+$ such that

$$\int f(x^{-1}t) q(t, x) \; d\beta^s(t) = \int f(t) \; d\beta^{x^{-1}s}(t)$$

for all $(s, x) \in S * G$, and $f \geq 0$ Borel on T. It is not hard to see that if a π-system $\beta = \{\beta^s : s \in S\}$ is G-quasi-invariant and if μ is a quasi-invariant measure on S, then $\mu \circ \beta$ is a quasi-invariant measure on T. Conversely, any quasi-invariant measure on T can be obtained in that way.

An *invariant mean for* π is a mean $m : L^\infty(T) \to L^\infty(S)$ which satisfies the invariance property

$$m(f * \varphi) = f * m(\varphi)$$

for all $f \in B_b(G, \nu)$ and $\varphi \in L^\infty(T)$ [1, Definition 3.1.4]. The G-equivariant map $\pi : T \to S$ is called *amenable with respect to* (λ, μ, β) if there is an invariant mean $m : L^\infty(T) \to L^\infty(S)$ [1, Definition 3.2.1].

An *approximate strongly invariant mean for* π is a net $(g_i)_i$ satisfying the following [1, Definition 3.1.24]:

(1) $\beta(g_i) = 1$ for all i.

(2) For all $f \in L^1(S * G)$,

$$\lim_i \int f(s,x)|g_i(x^{-1}t) - g_i(t)| \, d\lambda^{r(s)}(x) \, d\beta^s(t) d\mu(s) = 0.$$

The existence of invariant mean $m : L^\infty(T) \to L^\infty(S)$ for the G-equivariant map $\pi : T \to S$ is equivalent with the existence of an approximate strongly invariant mean $(g_i)_i$ for π (Propositions 3.1.7, 3.1.21, 3.1.22 of [1]).

Also the amenability of the G-equivariant map $\pi : T \to S$ (with respect to(λ, μ, β)) is equivalent to the existence of an invariant measurable π-system of means (Propositions 3.1.27, 3.2.5 of [1]). An invariant measurable π-system of means is a family $m = \{m^s : s \in S\}$ of states (or means) m^s on $L^\infty(T, \beta^s)$ such that, for every $\phi \in L^\infty(T)$,

(1) $s \mapsto m^s(\phi)$ is μ-measurable.
(2) $xm^{x^{-1}s}(\phi) = m^s(\phi)$ for $\mu \circ \lambda$-a.e. $(s,x) \in S * G$, where

$$xm^{x^{-1}s}(\phi) := m^{x^{-1}s}(t \mapsto \phi(xt)).$$

In [1] the amenability of a measured groupoid (G, λ, μ) (a groupoid G endowed with a Haar system λ and a quasi-invariant measure μ) was defined as the amenability of the range map with respect to (λ, μ) [1, Definition 3.2.8].

1.3. **Groupoid representations.** Let $\mathcal{H} = \{\mathcal{H}(s)\}_{s \in S}$ be a family of Hilbert spaces indexed by a set S. Let us form the disjoint union

$$S * \mathcal{H} = \{(s, \xi) : \xi \in \mathcal{H}(s)\},$$

and let $p : S * \mathcal{H} \to S$ be the natural projection, $p(s,\xi) = s$. A pair $(S * \mathcal{H}, p)$ is called a *Hilbert bundle over S*. For each $s \in S$, the space $\mathcal{H}(s)$, which can be identified with $p^{-1}(\{s\}) = \{s\} \times \mathcal{H}(s)$, is called the *fibre over s*. A *section* of the bundle is a function $f : S \to S * \mathcal{H}$ such that $p(f(s)) = s$ for all $s \in S$. Given a section f, we may write $f(s) = (s, \hat{f}(s))$ for a uniquely determined element

$$\hat{f} \in \prod_{s \in S} \mathcal{H}(s) = \left\{ \phi : S \to \bigcup_{s \in S} \mathcal{H}(s) \ : \ \phi(s) \in \mathcal{H}(s) \text{ for all } s \right\};$$

and given an element $\hat{f} \in \prod_{s \in S} \mathcal{H}(s)$ we may define a section $f(s) = (s, \hat{f}(s))$. Because of this link between sections of $S * \mathcal{H}$ and elements of $\prod_{s \in S} \mathcal{H}(s)$, we shall often abuse notation and write $f(s)$ instead of $\hat{f}(s)$. An *analytic Borel Hilbert bundle* is a Hilbert bundle $(S * \mathcal{H}, p)$, where $S * \mathcal{H}$ is endowed with an analytic Borel structure which satisfies the axioms:

(1) A subset E is Borel if and only if $p^{-1}(E)$ is Borel.
(2) There is a sequence $\{f_n\}_n$ of sections, called a *fundamental sequence*, such that
   (a) each function $\tilde{f}_n : S * \mathcal{H} \to \mathbb{C}$, defined by $\tilde{f}_n(s, \xi) = (f_n(s), \xi)_{\mathcal{H}(s)}$, is Borel.
   (b) for each pair (f_n, f_m) of fundamental sections, $s \mapsto (f_n(s), f_m(s))_{\mathcal{H}(s)}$ defines a Borel function.
   (c) the functions $\{\tilde{f}_n\}_n$ and p separate the points of $S * \mathcal{H}$.

Let G be a groupoid and $\{\lambda^u : u \in G^{(0)}\}$ be a Haar system on G. Let $G^{(0)} * \mathcal{H}$ be a Hilbert bundle. We write $\mathrm{Iso}(G^{(0)} * \mathcal{H})$ for

$$\{(u, L, v) \ : \ L : \mathcal{H}(v) \to \mathcal{H}(u) \text{ Hilbert space isomorphism}\}$$

endowed with the weakest Borel structure so that the maps

$$(u, L, v) \mapsto (Lf_n(v), f_m(u))$$

are Borel for every n and m, where $(f_n)_n$ is a fundamental sequence for $G^{(0)} * \mathscr{H}$.

$\mathrm{Iso}(G^{(0)} * \mathscr{H})$ is a groupoid in the operations:

$$(u, L_1, v)(v, L_2, w) = (u, L_1 L_2, w), \qquad (u, L, v)^{-1} = (v, L^{-1}, u).$$

A *unitary representation of* G consists of a quasi-invariant measure μ, a Hilbert bundle $G^{(0)} * \mathscr{H}$, a conull subset U of $G^{(0)}$, and a Borel map

$$L\colon G|_U \to \mathrm{Iso}\big(G^{(0)} * \mathscr{H}|_U\big),$$

where $G^{(0)} * \mathscr{H}|_U$ is the restriction of $G^{(0)} * \mathscr{H}$ to U, such that

(1) $L(x) = (d(x), \hat{L}(x), r(x))$ and $\hat{L}(x) : \mathscr{H}(d(x)) \to \mathscr{H}(r(x))$ is a Hilbert space isomorphism for $\mu \circ \lambda$-a.e. $x \in G|_U$.
(2) $\hat{L}(u) = I_u$, the identity operator on $\mathscr{H}(u)$, for μ-a.e. $u \in U$.
(3) $\hat{L}(x)\hat{L}(y) = \hat{L}(xy)$ for $\int(\lambda^u \times \lambda_u)\, d\mu(u)$-a.e. $(x, y) \in G^{(2)}$.
(4) $\hat{L}(x^{-1}) = \hat{L}(x)^{-1}$ for $\mu \circ \lambda$-a.e. x.

If $(G^{(0)} * \mathscr{H}, L, \mu)$ is a representation of the groupoid G, we abuse notation and write $L(x)$ instead of $\hat{L}(x)$. For any representation L, there is a Borel homomorphism $L_0 : G \to \mathrm{Iso}(G^{(0)} * \mathscr{H})$ that preserves the unit space $G^{(0)}$, in the sense that

$$L_0(x) = (d(x), \hat{L}_0(x), r(x)),$$

where $\hat{L}_0(x) : \mathscr{H}(d(x)) \to \mathscr{H}(r(x))$ is a Hilbert space isomorphism, such that L_0 agrees $\mu \circ \lambda$-a.e. with L.

2. The equivalence between the amenability of an equivariant map defined on a groupoid and the amenability of a semidirect product

We first set up some notation. Denote by $\mathscr{B}(H)$ the space of the bounded operators on the Hilbert space H. Let $S * \mathscr{H}$ be an analytic Borel Hilbert bundle over S, and let $(f_n)_n$ be a fundamental sequence for this bundle. We write $S * \mathscr{B}(\mathscr{H})$ for the set of $A\colon S \to \bigcup_{s\in S}\mathscr{B}(\mathscr{H}(s))$ with the properties:

(1) $A(s) \in \mathscr{B}(\mathscr{H}(s))$ for all $s \in S$.
(2) $s \mapsto \langle A(s) f_n(s), f_m(s)\rangle$ is Borel for all $m, n \in \mathbb{N}$.
(3) $\sup_s \|A(s)\| < \infty$.

Let μ be a measure on S. We write $L^\infty(S, \mathscr{H}, \mu)$ for the set of $A : S \to \bigcup_{s\in S}\mathscr{B}(\mathscr{H}(s))$ for which

(1) $A(s) \in \mathscr{B}(\mathscr{H}(s))$ for μ-a.e. $s \in S$.
(2) $s \mapsto \langle A(s) f_n(s), f_m(s)\rangle$ is μ-measurable for all $m, n \in \mathbb{N}$.
(3) $\|s \mapsto \|A(s)\|\|_\infty < \infty$.

In order to prove the equivalence between the amenability of an equivariant map defined on a groupoid and the amenability of a semidirect product, we shall use the notion of amenable representation of a groupoid introduced in [4, Definition 2].

Definition 1. Let G be a Borel groupoid endowed with a BHS $\lambda = \{\lambda^u : u \in G^{(0)}\}$. Let μ be a quasi-invariant measure for the Haar system λ. A unitary representation $(G^{(0)} * \mathscr{H}, L, \mu)$ of the groupoid G is said to be *amenable* if there exists an invariant family of states $\{M^u : u \in G^{(0)}\}$ on $L^\infty(G^{(0)}, \mathscr{H}, \mu)$, i.e.

(1) $M^u \in \mathscr{B}(\mathscr{H}(u))^*$, $M^u \geq 0$, $M^u(I) = 1$ (this means that M^u is a state on $\mathscr{H}(u)$) for μ-a.e. $u \in G^{(0)}$.
(2) $u \mapsto M^u(A(u))$ is μ-measurable for all $A \in L^\infty(G^{(0)}, \mathscr{H}, \mu)$.
(3) $M^{r(x)}(L(x)AL(x^{-1})) = M^{d(x)}(A)$ for $\mu \circ \lambda$-a.e. $x \in G$ and all $A \in \mathscr{B}(\mathscr{H}(d(x)))$.

This definition extends from groups to groupoids the notion of amenability for an arbitrary unitary representation, introduced by E.B. Bekka in [2].

In [4, Theorem 6] we characterized the amenable groupoids by amenability of all their unitary representations as follows.

Theorem 2. *Let G be a Borel groupoid endowed with a BHS $\lambda = \{\lambda^u : u \in G^{(0)}\}$. Let μ be a quasi-invariant measure for the Haar system λ. Then the following conditions are equivalent:*

(1) (G, λ, μ) *is amenable.*
(2) *Every unitary representation $(G^{(0)} * \mathscr{H}, L, \mu)$ of G is amenable.*
(3) *The left regular representation Reg_μ is amenable.*

We shall use this theorem for a semi-direct product $S * G$. If $\lambda = \{\lambda^u : u \in G^{(0)}\}$ is a BHS on G, then we endow $S * G$ with the BHS $\{\varepsilon_s \times \lambda^{r(s)} : s \in S\}$. A unitary representation $(S * \mathscr{H}, L, \mu)$ of $S * G$ is amenable if there exists an invariant family of states $\{M^s : s \in S\}$ on $L^\infty(S, \mathscr{H}, \mu)$. This means that

(1) $M^s \in \mathscr{B}(\mathscr{H}(s))^*$, $M^s \geq 0$, $M^s(I) = 1$ for μ-a.e. $s \in S$.
(2) $s \mapsto M^s(A(s))$ is μ-measurable for all $A \in L^\infty(S, \mathscr{H}, \mu)$.
(3) $M^s(L(s,x)AL(s,x)^{-1}) = M^{x^{-1}s}(A)$ for $\mu \circ \lambda$-a.e. $(s,x) \in S * G$ and all $A \in \mathscr{B}(\mathscr{H}(x^{-1}s))$.

Let us define an analog of Reg_μ for an equivariant map. Consider a G-equivariant map $\pi : G \to S$. Here S is a (left) Borel G-space and also G is a Borel G-space by left multiplication. Let $\beta = \{\beta^s : s \in S\}$ be a G-invariant π-system of measures and μ be a quasi-invariant measure for the Haar system on $S * G$. Let g_0 be a positive Borel function such that $\beta(g_0) = 1$. Let $\mathscr{H}(s) = L^2(\beta^s)$, to obtain a Hilbert bundle $S * \mathscr{H}$ on S. The Borel structure on $S * \mathscr{H}$ is given by a sequence of sections $(f_n)_n$ defined as follows: let $(h_n)_n$ be a sequence of bounded, non-negative Borel functions on G that separate points, and let us define $f_n : S \to S * \mathscr{H}$, by $f_n(s) = (s, \hat{f}_n(s))$, where $\hat{f}_n(s)(x) = h_n(x) g_0^{-1/2}(x)$, $x \in \pi^{-1}(\{s\})$. Let μ be a quasi-invariant measure for the Haar system on $S * G$. Let us define a representation $(\mu, S * \mathscr{H}, R_{\pi,\mu})$ where

$$\hat{R}_{\pi,\mu}(s,x) : L^2(\beta^{x^{-1}s}) \to L^2(\beta^s)$$

is given by the formula

$$\left(\hat{R}_{\pi,\mu}(s,x) f\right)(y) = f(x^{-1}y).$$

Let us call this representation of $S * G$ the *unitary representation associated to π induced by the G-action*, and denote it $R_{\pi,\mu}$.

Theorem 3. *Let G be a Borel groupoid endowed with a BHS $\lambda = \{\lambda^u : u \in G^{(0)}\}$. Let S be a (left) Borel G-space and let $\pi : G \to S$ be a G-equivariant map. Let $\beta = \{\beta^s : s \in S\}$ be a G-invariant π-system of measure and μ be a quasi-invariant measure for the Haar system on $S * G$. Then the following conditions are equivalent:*

(1) *$\pi : G \to S$ is an amenable map with respect to (λ, μ, β).*
(2) *Every unitary representation $(S * \mathcal{H}, L, \mu)$ of $S * G$ is amenable.*
(3) *The representation $R_{\pi,\mu}$ is amenable.*

Proof. $(1 \Rightarrow 2)$ Let $\{m^s : s \in S\}$ be an invariant measurable π-system of means with respect to (λ, μ, β) and let $(S * \mathcal{H}, L, \mu)$ be a unitary representation of $S * G$. Since $(S * \mathcal{H}, L, \mu)$ is a representation of $S * G$, there is a Borel homomorphism $L_0 : S * G \to \mathrm{Iso}(S * \mathcal{H})$ that agrees $\mu \circ \lambda_S$-a.e. with L, where $\lambda_S = \{\varepsilon_s \times \lambda^{r(s)} : s \in S\}$. We replace L by L_0. Let us choose $C \in S * \mathcal{B}(\mathcal{H})$ with $C(s) \geq 0$ and $\|C(s)\|_1 = 1$ for all $s \in S$. For each $s \in S$ and each $A \in \mathcal{B}(\mathcal{H}(s))$, define $\phi^s_A : G^s \to \mathbb{C}$ by

$$\begin{aligned}\phi^s_A(x) &= \mathrm{Tr}\left(AL(\pi(x), x)C(x^{-1}\pi(x))L(\pi(x), x)^{-1}\right)\\ &= \mathrm{Tr}\left(AL(s, x)C(x^{-1}s)L(s, x)^{-1}\right), \qquad x \in G^s.\end{aligned}$$

Then we have

$$|\phi^s_A(x)| \leq \|AL(s,x)C(x^{-1}s)L(s,x)^{-1}\|_1 \leq \|L(s,x)^{-1}AL(s,x)\|\,\|C(x^{-1}s)\|_1 \leq \|A\|.$$

Thus $\phi^s_A \in L^\infty(G, \beta^s)$. For each $s \in S$, define $M^s : \mathcal{B}(\mathcal{H}(s)) \to \mathbb{C}$ by

$$M^s(A) = m^s(\phi^s_A), \qquad A \in \mathcal{B}(\mathcal{H}(s)).$$

Let verify the invariance of the family of states $\{M^s : s \in S\}$. By definition we have

$$\begin{aligned}M^{x^{-1}s}(A) &= m^{x^{-1}s}\left(y \mapsto \mathrm{Tr}(AL(\pi(y), y)C(y^{-1}\pi(y))L(\pi(y), y)^{-1})\right)\\ &= m^s\left(y \to \mathrm{Tr}\left(AL(x^{-1}\pi(y), x^{-1}y)C(y^{-1}\pi(y))L(x^{-1}\pi(y), x^{-1}y)^{-1}\right)\right)\\ &= m^s\left(y \mapsto \mathrm{Tr}\left(AL(x^{-1}s, x^{-1})L(s, y)C(y^{-1}s)L(s, y)^{-1}L(x^{-1}s, x^{-1})^{-1}\right)\right)\\ &= m^s\left(y \mapsto \mathrm{Tr}\left(L(x^{-1}s, x^{-1})^{-1}AL(x^{-1}s, x^{-1})L(s, yt)C(y^{-1}s)L(s, y)^{-1}\right)\right)\\ &= m^s\left(\phi^s_{L(x^{-1}s,x^{-1})^{-1}AL(x^{-1}s,x^{-1})}\right)\\ &= M^s\left(L(x^{-1}s, x^{-1})^{-1}AL(x^{-1}s, x^{-1})\right)\\ &= M^s\left(L(s, x)AL(s, x)^{-1}\right)\end{aligned}$$

for $\mu \circ \lambda$-a.e. $x \in G$ and all $A \in \mathcal{B}(\mathcal{H}(d(x)))$.

The implication $(2 \Rightarrow 3)$ is trivial.

$(3 \Rightarrow 1)$ For each $s \in S$, let T^s_ϕ be the operator of multiplication by $\phi \in L^\infty(G, \beta^s)$, acting on $L^2(G, \beta^s)$. Then we have $T^s_1 = 1$, $T^s_\phi \geq 0$ if $\phi \geq 0$, and

$$R_\pi(s, x)T^{x^{-1}s}_\phi R_\pi(s, x)^{-1} = T^s_{x\phi} \qquad \text{for all } \phi \in L^\infty(G, \beta^s).$$

For each $s \in S$ define $m^s(\phi) = M^s(T^s_\phi)$, $\phi \in L^\infty(G, \beta^s)$. We have

$$m^s({}_x\phi) = M^s(T^s_{x\phi}) = M^s\left(R_\pi(s, x)T^{x^{-1}s}_\phi R_\pi(s, x)^{-1}\right) = M^{x^{-1}s}(T^{x^{-1}s}_\phi) = m^{x^{-1}s}(\phi)$$

for $\mu \circ \lambda$-a.e. $(s, x) \in S * G$ and all $\phi \in L^\infty(G, \beta^{x^{-1}s})$. Thus $\{m^s : s \in S\}$ is an invariant measurable π-system of means with respect to (β, λ, μ). □

Corollary 4. *Let G be a Borel groupoid endowed with a BHS $\lambda = \{\lambda^u : u \in G^{(0)}\}$. Let S be a (left) Borel G-space and let $\pi : G \to S$ be a G-equivariant map. Let $\beta = \{\beta^s : s \in S\}$ be a G-invariant π-system of measure and μ be a quasi-invariant measure for the Haar system $\lambda_S = \{\varepsilon_s \times \lambda^{r(s)} : s \in S\}$ on $S * G$. Then π is amenable with respect to (λ, μ, β) if and only if $(S * G, \lambda_S, \mu)$ is amenable.*

Proof. According to Theorem 2, $(S * G, \lambda_S, \mu)$ is amenable if and only if any unitary representation $(S * \mathscr{H}, L, \mu)$ of $S * G$ is amenable. But applying Theorem 3, any unitary representation $(S * \mathscr{H}, L, \mu)$ of $S * G$ is amenable if and only if $\pi : G \to S$ is an amenable map with respect to (λ, μ, β). □

3. THE G-EQUIVARIANT MAP θ

Let G be a locally compact second countable groupoid and R be its associated principal groupoid. R is the image of G under the homomorphism $\theta : G \to R$ defined by $\theta(x) = (r(x), d(x))$. We endow R with the quotient topology induced from G by the map θ. This topology consists of the sets whose inverse images by θ in G are open. In particular, G and R are Borel groupoids. Moreover, R is the image of G under θ, so it is a σ-compact groupoid. G can be viewed as a (left) G-space acting on itself by translation. Also R can be viewed as a (left) G-space taking $r : R \to G^{(0)}$, $r(u, v) = u$, and the action

$$x(d(x), v) = (r(x), v),$$

defined from $G * R = \{(x, (u, v)) : d(x) = u\}$ into R. The map θ is G-equivariant because $\theta(xy) = (r(x), d(y)) = x(r(y), d(y)) = x\theta(y)$ for all $(x, y) \in G^{(2)} = G * G$.

Let $\lambda = \{\lambda^u : u \in G^{(0)}\}$ be a BHS on G. In other words λ is a G-invariant r-system of measures ($r : G \to G^{(0)}$).

Let us recall some results on the structure of the Haar systems, as developed by Renault in [14, Section 1], and also by Ramsay and Walter in [11, Section 2]. In [14, Section 1] a BHS for the isotropy group bundle $G' = \{x \in G : r(x) = d(x)\}$ is constructed. One way to do this is to choose a continuous function F_0 with conditionally support, which is nonnegative and equal to 1 at each $u \in G^{(0)}$. For each $u \in G^{(0)}$, choose a left Haar measure β_u^u on G_u^u so that the integral of F_0 with respect to β_u^u is 1, then define $\beta_v^u = x\beta_v^v$ if $x \in G_v^u$, where $x\beta_v^v(f) = \int f(xy) d\beta_v^v(y)$ as usual. If z is another element in G_v^u, then $x^{-1}z \in G_v^v$, and since β_v^v is a left Haar measure on G_v^v, it follows that β_v^u is independent of the choice of x. A 1-cocycle δ on G such that, for every $u \in G^{(0)}$, $\delta|_{G_u^u}$ is the modular function for β_u^u, is also defined in [14]. δ and $\delta^{-1} = 1/\delta$ are bounded on compact sets in G. The system of measures $\beta = \{\beta_u^v : (u, v) \in R\}$ satisfies the following conditions:

(1) $\operatorname{supp}(\beta_v^u) = G_v^u$ for all $(u, v) \in R$.
(2) $(u, v) \mapsto \int f \, d\beta_u^v$ is Borel for every nonnegative Borel function f on G.
(3) $\int f(y) \, d\beta_v^{r(x)}(y) = \int f(xy) \, d\beta_v^{d(x)}(y)$ for all $x \in G$ and v with $(v, r(x)) \in R$.
(4) $\sup_{u,v} \beta_v^u(K) < \infty$ for all compact $K \subset G$.
(5) $\int f(y) \, d\beta_v^u(y) = \int f(y^{-1})\delta(y^{-1}) \, d\beta_u^v(y)$.

Conditions $(1-4)$ above imply that β is an invariant θ-system of measures.

With this apparatus in place, Renault describes a decomposition of the BHS $\{\lambda^u : u \in G^{(0)}\}$ for G over R, proving that there is a unique BHS α for R with the property that

$$\lambda^u = \int \beta_t^s \, d\alpha^u(s,t) \qquad \text{for all } u \in G^{(0)}.$$

In [11, Section 2] Ramsay and Walter prove that

$$\sup_u \alpha^u(\theta(K)t) < \infty \qquad \text{for all compact } K \subset G.$$

We shall call the pair of system of measures

$$\left(\{\beta_v^u\}_{(u,v)\in R}, \{\alpha^u\}_{u\in G^{(0)}}\right)$$

described above a *decomposition of the Haar system* $\{\lambda^u : u \in G^{(0)}\}$ *over the principal groupoid associated to* G. Also we shall call δ the 1*-cocycle associated to the decomposition.*

For each $u \in G^{(0)}$ the measure α^u is concentrated on $\{u\} \times [u]$. Therefore there is a measure μ^u concentrated on $[u]$ such that $\alpha^u = \varepsilon_u \times \mu^u$. Since $\{\alpha^u : u \in G^{(0)}\}$ is a Haar system, we have $\mu^u = \mu^v$ for all $(u,v) \in R$, and the function

$$u \mapsto \int f(s)\, d\mu^u(s)$$

is Borel for all $f \geq 0$ Borel on $G^{(0)}$. For each u the measure μ^u is quasi-invariant, cf. [11, Section 2]. Therefore μ^u is equivalent to $d_*(\nu^u)$ ([9, Lemma 4.5]).

Let us summarize: $\lambda = \{\lambda^u : u \in G^{(0)}\}$ is a BHS on G, $\theta : G \to R$ is a G-equivariant map, $\beta = \{\beta_u^v : (u,v) \in R\}$ is an invariant θ-system of measures, and $\alpha = \{\alpha^u : u \in G^{(0)}\}$ is a an invariant r-system of measures ($r : R \to G^{(0)}$, $r(u,v) = u$).

Let μ be a quasi-invariant measure (with respect to λ) on $G^{(0)}$. Let us note that $\mu \circ \alpha$ is a quasi-invariant measure on R. If $\theta : G \to R$ is a map which is amenable with respect to $(\lambda, \beta, \mu \circ \alpha)$, then there is an invariant mean $m : L^\infty(G) \to L^\infty(R)$. The invariance property of m can be written as

$$m(f * \varphi)(u,v) = \int f(x)\, m(\varphi)(d(x), v)\, d\lambda^u(x), \qquad (u,v) \in R,$$

for all $f \in B_b(G,\lambda)$ and $\varphi \in L^\infty(G)$. The existence of the invariant mean $m : L^\infty(G) \to L^\infty(R)$ is equivalent to the existence of an *approximate strongly invariant mean.* This is a net $(g_i)_i$ in $L^\infty(R, L^1(G,\beta))_1^+$ such that

(1) $\beta(g_i) = 1$.
(2) For all $f \in L^1(R * G)$,

$$\lim_i \int f((r(y), d(y)), x) |g_i(x^{-1}y) - g_i(y)|\, d\lambda^u(x)\, d\lambda^u(y)\, d\mu(u) = 0.$$

We may equivalently work with a sequence $(g_n)_n$ instead of a net $(g_i)_i$.

4. Topologically null functions on groupoids

In the following considerations we use the notation from the preceding section. In [3], we considered a locally compact second countable groupoid G having non-compact isotropy groups, endowed with a Haar system $\lambda = \{\lambda^u : u \in G^{(0)}\}$ and a quasi-invariant measure μ. We assumed that there is an approximate strongly invariant mean

$(g_n)_n$ (in the sense of [1, Definition 3.1.24]) for the range map $r : G \to G^{(0)}$ (or equivalently that the range map is amenable). We proved in [3, Theorem 13] that if the modular function Δ of μ is bounded on a neighborhood of the unit space $G^{(0)}$, then there is a subsequence of $(g_n)_n$, also denoted $(g_n)_n$, such that, for every compact set $K \subset G$ and every element $\varphi \in L^1(G^{(0)}, \mu)$,

$$\lim_{n\to\infty} \int 1_K(x)\varphi(r(x))g_n(x)\, d\lambda^u(x)\, d\mu(u) = 0.$$

In this section we shall prove a similar property for an approximate strongly invariant mean $(g_n)_n$ associated with the map $\theta : G \to R$, $\theta(x) = (r(x), d(x))$. The existence of an approximate strongly invariant mean $(g_n)_n$ is equivalent to the amenability of θ with respect to $(\lambda, \beta, \mu \circ \alpha)$, and is a weaker condition than the amenability of $r : G \to G^{(0)}$. We shall not need the boundedness of Δ. In the case of a locally compact non-compact group this result, as well as the result in [3], coincide with the result in 21.2 of [12].

Lemma 5. *Let G be a locally compact second countable groupoid endowed with a Haar system $\lambda = \{\lambda^u : u \in G^{(0)}\}$ and a quasi-invariant measure μ. Let*

$$\lambda^u = \int \beta^s_t\, d\alpha^u(s,t)$$

be a decomposition of the Haar system over the principal groupoid R. If $(g_n)_n$ is an approximate strongly invariant mean for the G-equivariant map $\theta : G \to R$, then for each element $f \in B_b(G,\lambda)$ and each $\varphi \in L^1(R, \mu\circ\alpha)$ we have

$$\lim_{n\to\infty} \int f(xy^{-1})\varphi(r(y), d(y))|g_n(y) - g_n(x)|\, d\lambda_{d(x)}(y)\, d\lambda^u(x)\, d\mu(u) = 0.$$

Proof. First let us note that $((u,v),x) \mapsto f(x)\varphi(u,v)$ belongs to $L^1(R*G)$. Then it follows from condition (2) fulfilled by the approximate strongly invariant mean $(g_n)_n$ that

$$\begin{aligned}
0 &= \lim_n \int f(x)\varphi(r(y), d(y))|g_n(x^{-1}y) - g_n(y)|\, d\lambda^u(x)\, d\lambda^u(y)\, d\mu(u)\\
&= \lim_n \int f(x)\varphi(r(y), d(y))|g_n((y^{-1}x)^{-1}) - g_n(y)|\, d\lambda^{r(y)}(x)\, d\lambda^u(y)\, d\mu(u)\\
&= \lim_n \int f(yx^{-1})\varphi(r(y), d(y))|g_n(x) - g_n(y)|\, d\lambda_{d(y)}(x)\, d\lambda^u(y)\, d\mu(u).
\end{aligned}$$

□

Proposition 6. *Let G be a locally compact second countable groupoid endowed with a Haar system $\lambda = \{\lambda^u : u \in G^{(0)}\}$ and a quasi-invariant measure μ. Let*

$$\lambda^u = \int \beta^s_t\, d\alpha^u(s,t)$$

be a decomposition of the Haar system over the principal groupoid R. If $(g_n)_n$ is an approximate strongly invariant mean for the G-equivariant map $\theta : G \to R$, then there exists a sequence $(h_n)_n$ of Borel positive functions on $G^{(0)}$ with the following properties:

(1) *For every compact subset $K \subset G$ and every element $\varphi \in L^1(R, \mu\circ\alpha)$ we have*

$$\lim_{n\to\infty} \int 1_K(x)\varphi(r(x), d(x))|g_n(x) - h_n(d(x))|\, d\lambda^u(x)\, d\mu(u) = 0.$$

(2) $\sup_{u\in K} h_n(u) < \infty$ *for all compact subsets* K *of* $G^{(0)}$.

Proof. Let U be a symmetric neighborhood of the unit space $G^{(0)}$, $U_1 \subset U$ a closed d-compact neighborhood of $G^{(0)}$, and $b : G \to [0,1]$ a continuous function with d-compact support contained in U such that $b|_{U_1} \equiv 1$. Then $a(x) = b(x)/\int b(y)\, d\lambda_{d(x)}$ defines a continuous function $a : G \to \mathbb{R}_+$ with d-compact support and with the property that $\int a(y)\, d\lambda_u(y) = 1$ for all $u \in G^{(0)}$. We define $h_n : G^{(0)} \to [0,\infty]$ by

$$h_n(u) = \int a(y) g_n(y)\ d\lambda_u(y).$$

We denote by L the support of the function a. Since L is a d-compact set, it follows that L^{-1} is an r-compact set and therefore, for any compact set K, the set KL^{-1} is compact. We fix a compact set $K \subset G$. If $f(x) = 1_{KL^{-1}}(x)$, then $f \in B_b(G,\lambda)$ because

$$\int |f(x)|\ d\lambda^u(x)\ \le\ \sup_u \lambda^u(KL^{-1}) < \infty.$$

Applying Lemma 5 to the functions f and φ, we have

$$\lim_{n\to\infty}\int \varphi(r(x),d(x)) 1_{KL^{-1}}(xy^{-1})|g_n(y)-g_n(x)| d\lambda_{d(x)}(y) d\lambda^u(x) d\mu(u) = 0.$$

If $M_0 = \sup_{x\in L\cap d^{-1}(d(K))} a(x)$, then for all $(x,y) \in \bigcup_{u\in G^{(0)}} (G_u \times G_u)$ we have

$$\begin{aligned}
M_0 1_{KL^{-1}}(xy^{-1})|\varphi(r(x),d(x))|\,|g_n(y)-g_n(x)| \\
&\ge M_0 1_K(x) 1_{L^{-1}}(y^{-1})|\varphi(r(x),d(x))|\,|g_n(y)-g_n(x)| \\
&\ge M_0 1_K(x) 1_L(y)|\varphi(r(x),d(x))|\,|g_n(y)-g_n(x)| \\
&\ge a(y) 1_K(x)|\varphi(r(x),d(x))|\,|g_n(y)-g_n(x)|.
\end{aligned}$$

Hence we obtain

$$\lim_{n\to\infty}\int 1_K(x)\varphi(r(x),d(x)) a(y)|g_n(y)-g_n(x)|\ d\lambda_{d(x)}(y)\ d\lambda^u(x)\ d\mu(u) = 0.$$

On the other hand

$$\begin{aligned}
&\left|\int 1_K(x)\varphi(r(x),d(x))|g_n(x)-h_n(d(x))|\ d\lambda^u(x)\ d\mu(u)\right| \\
&\le \int 1_K(x)|\varphi(r(x),d(x))| \\
&\qquad \cdot \left|\int a(y) g_n(x)\ d\lambda_{d(x)}(y) - \int a(y) g_n(y)\ d\lambda_{d(x)}(y)\right|\ d\lambda^u(x)\ d\mu(u) \\
&\le \int 1_K(x)|\varphi(r(x),d(x))| \int a(y)|g_n(x)-g_n(y)|\ d\lambda_{d(x)}(y)\ d\lambda^u(x)\ d\mu(u)\ \xrightarrow{n\to\infty}\ 0.
\end{aligned}$$

Let us prove that $(h_n)_n$ is uniformly bounded on the compact subsets of $G^{(0)}$. Let δ be the 1-cocycle associated to the decomposition over the principal groupoid associated with G. First let us note that

$$\begin{aligned} h_n(u) &= \int a(y^{-1})g_n(y^{-1})\ d\lambda^u(y) = \int a(y^{-1})g_n(y^{-1})\ d\beta_v^w(y)\ d\alpha^u(w,v) \\ &= \int a(y)g_n(y)\delta(y^{-1})\ d\beta_w^v(y)\ d\alpha^u(w,v) \\ &= \int \frac{b(y)}{\int b(x)\,d\lambda_{d(y)}(x)}\ g_n(y)\delta(y^{-1})\ d\beta_w^v(y)\ d\alpha^u(w,v) \\ &= \frac{1}{\int b(x)\,d\lambda_u(x)} \int b(y)g_n(y)\delta(y^{-1})\ d\beta_w^v(y)\ d\alpha^u(w,v). \end{aligned}$$

Let K be a compact subset of $G^{(0)}$. If we set

$$M = \sup_{y\in L\cap d^{-1}(K)} \delta(y^{-1}) \sup_{u\in K} \frac{1}{\int b(x)\,d\lambda_u(x)} \sup_u \alpha^u\big(\theta(L\cap d^{-1}(K))\big),$$

then we have $\sup_{u\in K} h_n(u) \le M$. □

Theorem 7. *Let G be a locally compact second countable groupoid with non-compact isotropy groups, endowed with a Haar system $\lambda = \{\lambda^u : u \in G^{(0)}\}$ and a quasi-invariant measure μ. Let*

$$\lambda^u = \int \beta_t^s\ d\alpha^u(s,t)$$

be the decomposition of the Haar system over the principal groupoid R. If $(g_n)_n$ is an approximate strongly invariant mean for the G-equivariant map $\theta : G \to R$, then there is a subsequence of $(g_n)_n$, also denoted $(g_n)_n$, such that for every compact set $K \subset G$ and every element $\varphi \in L^1(R, \mu\circ\alpha)$ we have

$$\lim_{n\to\infty} \int 1_K(x)\varphi(r(x),d(x))g_n(x)\ d\beta_v^u(x) = 0$$

for $\mu\circ\alpha$-a.e. $(u,v) \in R$.

Proof. From the preceding proposition there exists a sequence $(h_n)_n$ of Borel positive functions on $G^{(0)}$ such that, for every compact subset $K \subset G$ and every $\varphi \in L^1(R,\mu\circ\alpha)$, we have

$$\lim_{n\to\infty} \int 1_K(x)\varphi(r(x),d(x))|g_n(x) - h_n(d(x))|\ d\lambda^u(x)\ d\mu(u) = 0,$$

$$\lim_{n\to\infty} \int \varphi(w,v)1_K(x)|g_n(x) - h_n(d(x))|\beta_v^w(x)\ d\alpha^u(w,v)\ d\mu(u) = 0.$$

For each compact set K take $\varphi = 1_{\theta(K)}$. Then there exists a subsequence such that

$$\lim_{i\to\infty} \int 1_K(x)|g_{n_i}(x) - h_{n_i}(d(x))|\ d\beta_v^u(x) = 0, \qquad \mu\circ\alpha\text{-a.e.}$$

Let $(K_n)_n$ be an increasing sequence of compact sets with $\bigcup_n K_n = G$. For every natural number $m > 0$ there is n_m such that

$$\int 1_{K_m}(x)|g_{n_m}(x) - h_{n_m}(d(x))|\ d\beta_v^u(x) < \frac{1}{m}, \qquad \mu\circ\alpha\text{-a.e.}$$

Since for every compact set K there exists K_m with $K \subset K_m$, this gives

$$\lim_{m\to\infty}\int 1_K(x)|g_{n_m}(x)-h_{n_m}(d(x))|\ d\beta_v^u(x)=0, \qquad \mu\circ\alpha\text{-a.e.}$$

We shall denote the preceding subsequences also by $(g_n)_n$ and $(h_n)_n$. For μ-a.e. $u\in G^{(0)}$, there is $v\in G^{(0)}$ such that $(u,v)\in R$ and

$$\lim_{m\to\infty}\int 1_K(x)|g(x)-h_n(d(x))|\ d\beta_v^u(xt)=0$$

for all compact subsets K of $G^{(0)}$. The sequence $(h_n(u))_n$ is bounded, therefore it has a convergent subsequence, also denoted by $(h_n(u))_n$. Let us set $L=\lim_{n\to\infty}h_n(u)$. Using the decomposition $\lambda^u=\int\beta_v^u\,d\mu^{\dot u}(v)$, we obtain

$$\begin{aligned}
h_n(u)\int 1_K(x)\ d\beta_u^v(x) &= \int 1_K(x)h_n(d(x))\ d\beta_u^v(x)\\
&\le \int 1_K(x)|g_n(x)-h_n(d(x))|\ d\beta_u^v(x)+\int 1_K(x)g_n(x)\ d\beta_u^v(x)\\
&\le \int 1_K(x)|g_n(x)-h_n(d(x))\ |d\beta_u^v(x)+\int g_n(x)\ d\beta_u^v(x)\\
&\le \int 1_K(x)|g_n(x)-h_n(d(x))|\ d\beta_u^v(x)+1.
\end{aligned}$$

Passing to limit we obtain

$$L\int 1_K(x)\ d\beta_u^v(x)\le 1.$$

If $L\neq 0$, then for all compact K, $\int 1_K(x)\,d\beta_u^v(x)\le 1/L$, and hence $\int 1\,d\beta_u^v(x)\le 1/L$. If we take $z\in G$ such that $r(z)=u$ and $d(z)=v$, then

$$\int 1\ d\beta_u^u(x)=\int 1(zx)\ d\beta_u^v(x)=\int 1\ d\beta_u^v(x)<\infty.$$

It follows that the isotropy group G_u^u is compact, which contradicts the hypotheses. Thus $L=0$. We have proved that every limit point for $(h_n(u))_n$ is equal to 0. Thus $\lim_{n\to\infty}h_n(u)=0$. By the Dominated Convergence Theorem we have

$$\lim_{n\to\infty}\int 1_K(x)\varphi(r(x),d(x))h_n(d(x))\ d\lambda^u(x)\ d\mu(u)=0.$$

Passing to limit in the inequality

$$\begin{aligned}
&\left|\int 1_K(x)\varphi(r(x),d(x))g_n(x)\ d\lambda^u(x)\ d\mu(u)\right|\\
&\quad\le \int 1_K(x)|\varphi(r(x),d(x))|\ |g_n(x)-h_n(d(x))|\ d\lambda^u(x)\ d\mu(u)\\
&\qquad+\int 1_K(x)|\varphi(r(x),d(x))|h_n(d(x))\ d\lambda^u(x)\ d\mu(u),
\end{aligned}$$

we obtain the desired result. □

Remark 8. An *invariant measurable* θ-system of means with respect to (β,λ,μ) [1, Definition 3.1.26] is a family $\{m_v^u:(u,v)\in R\}$ of states m_u^v on $L^\infty(G,\beta_v^u)$ such that, for every $\varphi\in L^\infty(G)$,

(1) $(u,v)\mapsto m_v^u(\varphi)$ is $\mu\circ\alpha$-measurable;

(2) $m_v^{d(x)}(\varphi(x\cdot)) = m_v^{r(x)}(\varphi)$ for $\mu\circ\alpha\circ\lambda$-a.e. $((r(x),v),x)\in R*G$.

The existence of an approximate strongly invariant mean $(g_n)_n$ is equivalent to the existence of an invariant measurable θ-system of means (see 3.2 of [1]). In the proof of [1, Proposition 3.2.5], the following construction of an invariant measurable θ-system of means using an approximate strongly invariant mean $(g_n)_n$ is being employed. Let LIM be a "medial limit" as defined in [7], and set

$$m_v^u(\varphi) = \mathrm{LIM}\left(\int \varphi(x)g_n(x)\,d\beta_v^u(x)\right)$$

for all Borel bounded functions φ on G and all $(u,v)\in R$. Then $\{m_v^u : (u,v)\in R\}$ is an invariant measurable θ-system of means.

Theorem 9. *Let G be a locally compact second countable groupoid with non-compact isotropy groups, endowed with a Haar system $\lambda=\{\lambda^u : u\in G^{(0)}\}$ and with a quasi-invariant measure μ. Let*

$$\lambda^u = \int \beta_t^s\,d\alpha^u(s,t)$$

be a decomposition of the Haar system over the principal groupoid R. Assume that the G-equivariant map $\theta: G\to R$ is amenable, and that $(g_n)_n$ is an approximate strongly invariant mean for θ. If $\{m_v^u : (u,v)\in R\}$ is the invariant measurable θ-system of means induced by $(g_n)_n$, then

$$m_v^u(\varphi)=0$$

for any Borel bounded function φ on G with compact support and for $\mu\circ\alpha$-a.e. $(u,v)\in R$.

Proof. Theorem 7 implies that there is a subsequence of $(g_n)_n$, also denoted $(g_n)_n$, such that for every compact set $K\subset G$ we have

$$\lim_{n\to\infty}\int 1_K(x)g_n(x)\,d\beta_v^u(x) = \lim_{n\to\infty}\int 1_K(x)1_{\theta(K)}(r(x),d(x))g_n(x)\,d\beta_v^u(x)=0$$

for $\mu\circ\alpha$-a.e. $(u,v)\in R$. Hence

$$\lim_{n\to\infty}\int \varphi(x)g_n(x)\,d\beta_v^u(x)=0$$

for any Borel bounded function φ on G with compact support and $\mu\circ\alpha$-a.e. $(u,v)\in R$. Therefore

$$m_v^u(\varphi) = \mathrm{LIM}\left(\int \varphi(x)g_n(x)\,d\beta_v^u(x)\right)=0$$

for any φ as above and $\mu\circ\alpha$-a.e. $(u,v)\in R$. □

Remark 10. An invariant mean for θ is a mean $m: L^\infty(G)\to L^\infty(R)$ (that is a bounded $L^\infty(R)$-linear map from $L^\infty(G)$ into $L^\infty(R)$) which satisfies the invariance property

$$m(f*\varphi) = f*m(\varphi)$$

for all $f\in B_b(G,\lambda)$ and $\varphi\in L^\infty(G)$. The invariance property of m can be written as

$$m(f*\varphi)(u,v) = \int f(x)m(\varphi)(d(x),v)\,d\lambda^u(x), \qquad (u,v)\in R,$$

for all $f\in B_b(G,\lambda)$ and $\varphi\in L^\infty(G)$.

The G-equivariant map $\theta : G \to R$ is called *amenable with respect to* $(\lambda, \beta, \mu \circ \alpha)$ if there is an invariant mean $m : L^\infty(G) \to L^\infty(R)$. An *approximate weakly invariant mean* is a net $(g_n)_n$ in $L^\infty(R, L^1(G, \beta))_1^+$ satisfying the following conditions:

(1) $\beta(g_n) = 1$.
(2) For all $f \in L^1(R * G)$,

$$\lim_n \int f((r(y), d(y)), x)(g_n(x^{-1}y) - g_n(y)) \, d\lambda^u(x) \, d\lambda^u(y) \, d\mu(u) = 0.$$

Let $B_{L^\infty(R)}(L^\infty(G), L^\infty(R))$ be the Banach space of all bounded $L^\infty(R)$-linear maps from $L^\infty(G)$ into $L^\infty(R)$. According to [1, Proposition 3.1.7], $m : L^\infty(G) \to L^\infty(R)$ is an invariant mean if and only if there is an approximate weakly invariant mean $(g_i)_i$ in $L^\infty(R, L^1(G, \beta))_1^+$ such that $\lim\limits_{n\to\infty} m_{g_n} = m$ in $B_{L^\infty(R)}(L^\infty(G), L^\infty(R))$ endowed with the weak* topology, where m_{g_n} is defined by

$$m_{g_n}(\varphi)(u, v) = \int g_n(x)\varphi(x) \, d\beta_v^u(x), \qquad (u, v) \in R, \ \varphi \in L^\infty(G).$$

Obviously, any approximate strongly invariant mean $(g_i)_i$ is an approximate weakly invariant mean. Let $m = \lim_{n\to\infty} m_{g_n}$ be the invariant mean induced by an approximate strongly invariant mean $(g_n)_n$ for the G-equivariant map $\theta : G \to R$. Applying Theorem 7, it follows that

$$m(\varphi) = 0$$

for any compactly supported bounded Borel function φ on G.

Theorem 11. *Let G be a locally compact second countable groupoid with non-compact isotropy groups, endowed with a Haar system $\lambda = \{\lambda^u : u \in G^{(0)}\}$ and a quasi-invariant measure μ. Let*

$$\lambda^u = \int \beta_t^s \, d\alpha^u(s, t)$$

be a decomposition of the Haar system over the principal groupoid R. Assume that the G-equivariant map $\theta : G \to R$ is amenable with respect to $(\lambda, \mu \circ \alpha, \beta)$. Let $(R * \mathcal{H}, L, \mu \circ \alpha)$ be a unitary representation of the groupoid $R * G$ such that for $\mu \circ \alpha$-a.e. $(u, v) \in R$ there is no finite-dimensional subspace $\mathcal{K}(u, v)$ of $\mathcal{H}(u, v)$ with the property

$$L((u, v), x)\mathcal{K}(u, v) \subset \mathcal{K}(u, v) \qquad \text{for } \lambda^u \text{-a.e. } x \in G_u^u.$$

Let $\{M_v^u : (u, v) \in R\}$ be an invariant family of states on $L^\infty(S, \mathcal{H}, \mu \circ \alpha)$. Then for $\mu \circ \alpha$-a.e. $(u, v) \in R$ we have

$$M_v^u(A) = 0$$

for every compact operator $A \in \mathcal{B}(\mathcal{H}(u, v))$.

Proof. By Theorem 3, if $\theta : G \to R$ is amenable with respect to $(\lambda, \mu \circ \alpha, \beta)$, then any representation $(R * \mathcal{H}, L, \mu \circ \alpha)$ is amenable. Since for $\mu \circ \alpha$-a.e. $(u, v) \in R$ there is no finite-dimensional subspace $\mathcal{K}(u, v)$ of $\mathcal{H}(u, v)$ with the property that

$$L((u, v), x)\mathcal{K}(u, v) \subset \mathcal{K}(u, v) \qquad \text{for } \lambda^u\text{-a.e. } x \in G_u^u,$$

it follows from [4, Theorem 4] that for $\mu \circ \alpha$-a.e. $(u, v) \in R$, $M_v^u(A) = 0$ for every compact operator $A \in \mathcal{B}(\mathcal{H}(u, v))$. □

Remark 12. The above theorem may be viewed as a non-commutative analog of Theorem 9.

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