

# Morphisms which are continuous on a neighborhood of the base of a groupoid\*

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## Abstract

Kirill Mackenzie raised in [3] (p. 31) the following question: given a morphism  $F : \Omega \rightarrow \Omega'$ , where  $\Omega$  and  $\Omega'$  are topological groupoids and  $F$  is continuous on a neighborhood of the base in  $\Omega$ , is it true that  $\Omega$  is continuous everywhere?

This paper gives a negative answer to that question. Moreover, we shall prove that for a locally compact groupoid  $\Omega$  with non-singleton orbits and having open target projection, if we assume that the continuity of every morphism  $F$  on the a neighborhood of the base in  $\Omega$  implies the continuity of  $F$  everywhere, then the groupoid  $\Omega$  must be locally transitive.

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## 1 Introduction

In order to establish notation, we give some definitions that can be found (in equivalent form) in several places: [3],[5],[4]. To define the term groupoid we begin with two sets,  $\Omega$  and  $B$ , called respectively the *groupoid* and *base*, together with two maps  $\alpha$  and  $\beta$ , called respectively the *source* and *target projections*, a map  $\varepsilon : u \rightarrow \tilde{u}$  from  $B$  to  $\Omega$  called the *object inclusion map*, and a partial multiplication  $(x, y) \rightarrow xy$  in  $\Omega$  defined on the set

$$\Omega * \Omega = \{(x, y) \in \Omega \times \Omega : \alpha(x) = \beta(y)\} \text{ (the set of } \textit{composable pairs}),$$

all subject to the following conditions:

1.  $\alpha(xy) = \alpha(y)$  and  $\beta(xy) = \beta(x)$  for all  $(x, y) \in \Omega * \Omega$ .
2.  $x(yz) = (xy)z$  for all  $x, y, z \in \Omega$  such that  $\alpha(x) = \beta(y)$  and  $\alpha(y) = \beta(z)$ .

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3.  $\alpha(\tilde{u}) = \beta(\tilde{u}) = u$  for all  $u \in B$ .
4.  $x\alpha(\tilde{x}) = x$  and  $\beta(\tilde{x})x = x$  for all  $x \in \Omega$ .
5. Each  $x \in \Omega$  has a (two-sided) inverse  $x^{-1}$  such that  $\alpha(x^{-1}) = \beta(x)$ ,  $\beta(x^{-1}) = \alpha(x)$  and  $x^{-1}x = \alpha(\tilde{x})$ ,  $xx^{-1} = \beta(\tilde{x})$ .

The elements of the base  $B$  will usually be denoted by letters as  $u, v, w$  while arbitrary elements of the groupoid  $\Omega$  will be denoted by  $x, y, z$ .

The fibres of the target and the source projections will be denoted  $\Omega^u = \beta^{-1}(\{u\})$  and  $\Omega_v = \alpha^{-1}(\{v\})$ , respectively. More generally, given the subsets  $U, V \subseteq B$ , we define  $\Omega^U = \beta^{-1}(U)$ ,  $\Omega_V = \alpha^{-1}(V)$  and  $\Omega_V^U = \beta^{-1}(U) \cap \alpha^{-1}(V)$ . The relation  $u \sim v$  iff  $\Omega_v^u \neq \emptyset$  is an equivalence relation on  $B$ . Its equivalence classes are called *orbits* and the orbit of a unit  $u$  is denoted  $[u]$ . A groupoid is called *transitive* iff it has a single orbit. The set  $\Omega_u^u$ , obviously a group under the restriction of the partial multiplication in  $\Omega$ , is called *isotropy group* at  $u$ .

A *subgroupoid* of  $\Omega$  is a pair of subsets  $\Omega' \subseteq \Omega$ ,  $B' \subseteq B$ , such that  $\alpha(\Omega') \subseteq B'$ ,  $\beta(\Omega') \subseteq B'$ ,  $\tilde{u} \in \Omega'$  for all  $u \in B'$  and  $\Omega'$  is closed under the partial multiplication and inversion in  $\Omega$ . The *inner subgroupoid* (the *isotropy group bundle*) of  $\Omega$  is the subgroupoid  $G\Omega = \cup_u \Omega_u^u$ . The *base subgroupoid* of  $\Omega$  is the subgroupoid  $\tilde{B} = \{\tilde{u} : u \in B\}$ . An element  $\tilde{u}$  of  $\tilde{B}$  is called the *unity* corresponding to  $u$ .

A function  $F : \Omega \rightarrow \Omega'$ , where  $\Omega$  and  $\Omega'$  are groupoids, is called *morphism* if for all composable pairs  $(x, y) \in \Omega * \Omega$ ,  $(F(x), F(y))$  is a composable pair in  $\Omega'$  and  $F(xy) = F(x)F(y)$ . It is useful to note that  $F(x^{-1}) = F(x)^{-1}$  for all  $x \in \Omega$  and that  $F(\tilde{u})$  belongs to the base subgroupoid of  $\Omega'$  for all  $\tilde{u}$  in the base subgroupoid of  $\Omega$ .

Groupoids are generalizations of groups, but there are many other structures which fit naturally into the study of groupoids. We give some basic examples that will be needed in the subsequent considerations.

*Examples of groupoids:*

1. **Groups:** A group  $G$  is a groupoid with base  $B = \{e\}$  (the unit element) and  $G * G = G \times G$ .
2. **Spaces.** A space  $X$  may be regarded as a groupoid on itself with  $\alpha = \beta = id_X$  and every element is a unity.
3. **Equivalence relations.** Let  $\mathcal{E} \subseteq X \times X$  be an equivalence relation on the set  $X$ . We give  $\mathcal{E}$  the structure of a groupoid on  $X$  in the following way:  $\alpha$  is the projection onto the second factor of  $X \times X$  and  $\beta$  is the projection onto the first factor; the object inclusion map is  $x \rightarrow (x, x)$  and the partial multiplication is  $(x, y)(y', z) = (x, z)$  defined if and only if  $y = y'$ . The inverse of  $(x, y)$  is  $(y, x)$ . Two extreme cases deserve to be single out. If  $\mathcal{E} = X \times X$ , then  $\mathcal{E}$  is called the trivial groupoid on  $X$ , while if  $\mathcal{E} = \text{diag}(X)$ , then  $\mathcal{E}$  is called the co-trivial groupoid on  $X$  (and may be identified with the groupoid in example 2).

4. **Transformation groups.** Let  $G$  be a group acting on a set  $X$  such that for  $x \in X$  and  $g \in G$ ,  $gx$  denotes the transform of  $x$  by  $g$ . Let  $e$  be the unit element of  $G$ . Then  $G \times X$  becomes a groupoid on  $X$  in the following way:  $\alpha$  is the projection onto the second factor of  $G \times X$  and  $\beta$  is the action  $G \times X \rightarrow X$  itself; the object inclusion map is  $x \rightarrow (e, x)$  and the partial multiplication is  $(g, x)(g', y) = (gg', y)$  defined if and only if  $x = g'y$ . The inverse of  $(x, g)$  is  $(g^{-1}, gx)$ .

A *topological groupoid* consists of a groupoid  $\Omega$  on  $B$  together with topologies on  $\Omega$  and  $B$  such that the maps which define the groupoid structure are continuous: the projections  $\alpha, \beta : \Omega \rightarrow B$ , the object inclusion map  $\varepsilon : B \rightarrow \Omega$ ,  $\varepsilon(u) = \tilde{u}$ , the inversion map  $x \rightarrow x^{-1}$  from  $\Omega$  to  $\Omega$ , and the partial multiplication from  $\Omega * \Omega$  to  $\Omega$ , where  $\Omega * \Omega$  has the subspace topology from  $\Omega \times \Omega$ , are continuous.

As a consequence of the definition of a topological groupoid, the inversion  $x \rightarrow x^{-1}$  is a homeomorphism, the object inclusion map is a homeomorphism onto the base subgroupoid  $\tilde{B}$ , the projections are identification maps. If  $\Omega$  is Hausdorff, then  $\tilde{B}$  is closed in  $\Omega$ . If  $B$  is Hausdorff, then  $\Omega * \Omega$  is closed in  $\Omega \times \Omega$ .

Kirill Mackenzie raised in [3] (p. 31) the following question: given a morphism  $F : \Omega \rightarrow \Omega'$ , where  $\Omega$  and  $\Omega'$  are topological groupoids and  $F$  is continuous on a neighborhood of the base in  $\Omega$ , is it true that  $F$  is continuous everywhere? Obviously, the answer is affirmative if the groupoid  $\Omega$  is a group or if  $\Omega$  is a co-trivial groupoid. Also if  $\Omega$  is a "principal" topological groupoid, Mackenzie has proved that the answer is affirmative (Proposition 1.21/p. 30 [3]) A "*principal*" groupoid (Definition 1.18/p. 28 [3]) is a topological *transitive* groupoid  $\Omega$  on  $B$  which satisfies the following conditions:

1. For each  $v \in B$ , the map  $\beta_v : \Omega_v \rightarrow B$ , defined by  $\beta_v(x) = \beta(x)$  for all  $x \in \Omega_v$ , is open.
2. For each  $v \in B$ , the map  $\delta_v : \Omega_v \times \Omega_v \rightarrow \Omega$ , defined by  $\delta_v(x, y) = xy^{-1}$  for all  $(x, y) \in \Omega_v \times \Omega_v$ , is open.

We shall prove that the result remains true for a locally transitive groupoid  $\Omega$ . By a locally transitive groupoid we shall mean a topological groupoid  $\Omega$  on  $B$  for which the map  $\beta_v : \Omega_v \rightarrow B$  is open, for each  $v \in B$ . Also we shall prove that in general the answer to the question raised by Mackenzie is negative. Moreover, we shall show that for a locally compact groupoid  $\Omega$  with non-singleton orbits and having open target projection, if we assume that the continuity of every morphism  $F$  at all  $\tilde{u} \in \tilde{B}$  implies the continuity of  $F$  everywhere, then the groupoid  $\Omega$  must be locally transitive. Furthermore, if we suppose that  $\tilde{B}$  is an open subset of  $\Omega$  and the answer to the question of Mackenzie is affirmative, then  $B$  must be a discrete space. In particular, if  $\Omega$  is the transformation group groupoid associated with the action of a discrete group on a locally compact second countable Hausdorff space  $B$ , then  $\Omega$  has the property that every morphism which is continuous on a neighborhood of the base must be globally continuous if and only if  $B$  is a discrete space.

## 2 An example of a groupoid morphism which is continuous on a neighborhood of the base but which is not continuous everywhere

We shall construct a morphism (in the algebraic sense)  $F : \Omega \rightarrow \Omega'$  of topological groupoids with the property that the continuity of  $F$  on a neighborhood of the base in  $\Omega$  does not imply the continuity of  $F$  everywhere on  $\Omega$ . Thus we give a negative answer to a question raised by K. Mackenzie in [3] (p. 31).

Let  $\mathbf{R}$  be the set of real numbers and  $\mathbf{Z}$  the set of integer numbers. Let  $\Omega$  be the groupoid on  $\mathbf{R}$  associated with the following equivalence relation

$$\{(x, y) \in \mathbf{R}^2 : x - y \in \mathbf{Z}\} \subset \mathbf{R} \times \mathbf{R}.$$

If  $\Omega$  is endowed with the subspace topology from  $\mathbf{R} \times \mathbf{R}$ ,  $\Omega$  is a locally compact Hausdorff space. The base subgroupoid of  $\Omega$

$$\tilde{B} = \{(x, x) : x \in \mathbf{R}\}$$

is an open subset in  $\Omega$ . Let  $\Omega'$  be the group of real numbers under addition. Let  $\phi : \mathbf{R} \rightarrow \mathbf{R}$  be defined by  $\phi(x) = 0$  for all  $x \leq \frac{1}{2}$  and  $\phi(x) = 1$  otherwise. Let  $F : \Omega \rightarrow \mathbf{R}$  be defined by

$$F(x, y) = \phi(y) - \phi(x).$$

For all  $((x, y), (y', z)) \in \Omega * \Omega$  we have

$$\begin{aligned} F((x, y), (y', z)) &= F(x, z) = \phi(z) - \phi(x) \\ &= \phi(z) - \phi(y) + \phi(y) - \phi(x) \\ &= F(y, z) + F(x, y) \end{aligned}$$

Therefore  $F$  is a morphism. Obviously,  $F$  is continuous at every  $(x, x)$  in  $\Omega$ , but it is not continuous at  $(\frac{1}{2}, \frac{3}{2})$ .

**Remark 1** *The groupoid  $\Omega$  is isomorphic and homeomorphic to the transformation group groupoid obtained by letting  $\mathbf{Z}$  act on  $\mathbf{R}$ . In this setting the morphism (cocycle)  $F$  is a coboundary determined by the discontinuous function  $\phi$ .*

## 3 Sufficient conditions for the continuity of a groupoid morphism

**Theorem 2** *Let  $\Omega$  be a topological groupoid on  $B$ . Let us assume that for each  $v \in B$ , the map  $\beta_v : \Omega_v \rightarrow B$ , defined by  $\beta_v(x) = \beta(x)$  for all  $x \in \Omega_v$ , is open. Let  $\Omega'$  be a topological groupoid and  $F : \Omega \rightarrow \Omega'$  be a morphism (in the algebraic sense) which is continuous at every  $\tilde{u} \in \tilde{B}$ . Then  $F$  is continuous everywhere on  $\Omega$ .*

**Proof.** Let  $(x_i)_i$  be a net converging to  $x$  in  $\Omega$ . Let  $v = \alpha(x)$  and let us note that  $(\beta(x_i))_i$  converges to  $\beta(x) = \beta_v(x)$  in  $B$ . Since  $\beta_v : \Omega_v \rightarrow B$  is open, we may pass to a subnet and assume that there exists a net  $(y_i)_i$  in  $\Omega_v$  converging to  $x$  such that  $\beta_v(y_i) = \beta(x_i)$  for all  $i$ . The net  $(y_i x^{-1})_i$  converges to  $x x^{-1} = \widetilde{\beta(x)}$ . From the continuity of  $F$  at  $\widetilde{\beta(x)}$  it follows that  $\lim_i F(y_i x^{-1}) = F(\widetilde{\beta(x)})$ , or equivalently  $\lim_i F(y_i) = F(x)$ . On the other hand, the net  $(y_i^{-1} x_i)_i$  converges to  $x^{-1} x = \widetilde{\alpha(x)}$ . Now from the continuity of  $F$  at  $\widetilde{\alpha(x)}$  it follows that  $\lim_i F(y_i^{-1} x_i) = F(\widetilde{\alpha(x)})$ . Thus

$$\begin{aligned} \lim_i F(x_i) &= \lim_i F((y_i y_i^{-1}) x_i) = \lim_i F(y_i (y_i^{-1} x_i)) \\ &= \lim_i F(y_i) F(y_i^{-1} x_i) = F(x) F(\widetilde{\alpha(x)}) \\ &= F(x). \end{aligned}$$

■

## 4 Necessary conditions for the continuity of a groupoid morphism

Let  $\Omega$  on  $B$  be a locally compact (Hausdorff) groupoid. This means that  $\Omega$  is a topological groupoid whose topology is locally compact (Hausdorff). A subset  $A$  of  $\Omega$  is called  $\alpha$ -relatively compact if  $A \cap \alpha^{-1}(K)$  is relatively compact for any compact subset  $K$  of  $B$ . Jean Renault proved that if  $\Omega$  is a locally compact groupoid whose base  $B$  is paracompact, then the base subgroupoid  $\tilde{B}$  admits a fundamental system of  $\alpha$ -relatively compact neighborhoods (Proposition II.1.9/p. 56 [5]).

Throughout this section we shall fix a system of positive Radon measures indexed by  $B \times B$

$$\{\lambda_v^u, (u, v) \in B \times B\}$$

satisfying the following conditions:

1.  $\text{supp}(\lambda_v^u) = \Omega_v^u$  for all  $u \sim v$ .
2.  $\lambda_v^u = 0$  if  $\Omega_v^u = \emptyset$ .
3.  $\sup_{u,v} \lambda_v^u(K) < \infty$  for all compact  $K \subset \Omega$ .
4.  $\int f(y) d\lambda_v^{\beta(x)}(y) = \int f(xy) d\lambda_v^{\alpha(x)}(y)$  for all  $x \in \Omega$  and  $v \in [\beta(x)]$ .

A construction of such system of measures can be found in [6]. In Section 1 of [6] Jean Renault constructs a Borel Haar system  $\{\lambda_u^u, u \in B\}$  for the inner subgroupoid  $G\Omega$ . One way to do this is to choose a function  $F_0$  continuous

with  $\alpha$ -relatively compact support which is nonnegative and equal to 1 at each  $\tilde{u} \in \tilde{B}$ . Then for each  $u \in B$  choose a left Haar measure  $\lambda_u^u$  on  $\Omega_u^u$  so the integral of  $F_0$  with respect to  $\lambda_u^u$  is 1.

Renault defines  $\lambda_v^u = x\lambda_v^v$  if  $x \in \Omega_v^u$  (where  $x\lambda_v^v(f) = \int f(xy) d\lambda_v^v(y)$ , as usual). If  $z$  is another element in  $\Omega_v^u$ , then  $x^{-1}z \in \Omega_v^v$ , and since  $\lambda_v^v$  is a left Haar measure on  $\Omega_v^v$ , it follows that  $\lambda_v^u$  is independent of the choice of  $x$ . If  $K$  is a compact subset of  $\Omega$ , then  $\sup_{u,v} \lambda_v^u(K) < \infty$ .

**Lemma 3** *Let  $\Omega$  be a locally compact groupoid on  $B$ , whose base subgroupoid  $\tilde{B}$  has a fundamental system of  $\alpha$ -relatively compact neighborhoods. Let  $v \in B$  and  $f : G \rightarrow \mathbf{R}$  be a continuous function with compact support. Then for each  $\varepsilon > 0$  there is a symmetric  $\alpha$ -relatively compact neighborhood  $W_\varepsilon$  of  $\tilde{B}$ , such that:*

$$\left| \int f(y) d\lambda_v^{\alpha(x)}(y) - \int f(y) d\lambda_v^{\beta(x)}(y) \right| < \varepsilon \lambda_v^{\alpha(x)}(UL) \text{ for all } x \in W_\varepsilon$$

where  $L$  is the support of  $f$  and  $U$  is a symmetric  $\alpha$ -compact neighborhood of  $\tilde{B}$ .

**Proof.** Since  $\Omega$  is a locally compact groupoid whose base subgroupoid  $\tilde{B}$  has a fundamental system of  $\alpha$ -relatively compact neighborhoods, by Proposition 3.4/p. 40 [1], it results that  $f$  is left "uniformly continuous". Hence for each  $\varepsilon > 0$  there is a symmetric  $\alpha$ -relatively compact neighborhood  $W_\varepsilon$  of  $\tilde{B}$ , such that

$$x \in W_\varepsilon \Rightarrow |f(xy) - f(y)| < \varepsilon \text{ for all } y \in \Omega^{\alpha(x)}$$

Let  $U$  be a symmetric  $\alpha$ -compact neighborhood of the base subgroupoid  $\tilde{B}$  and assume, without loss of generality, that  $W_\varepsilon \subset U$ . Then

$$|f(xy) - f(y)| < \varepsilon 1_{UL}(y),$$

for all  $x \in W_\varepsilon$ ,  $y \in \Omega^{\alpha(x)}$ , where  $1_{UL}$  denotes the indicator function of  $UL$ . It follows that

$$\begin{aligned} \left| \int f(xy) d\lambda_v^{\alpha(x)}(y) - \int f(y) d\lambda_v^{\alpha(x)}(y) \right| &\leq \int |f(xy) - f(y)| d\lambda_v^{\alpha(x)}(y) \\ &< \int \varepsilon 1_{UL}(y) d\lambda_v^{\alpha(x)}(y) = \varepsilon \lambda_v^{\alpha(x)}(UL) \end{aligned}$$

for all  $x \in W_\varepsilon$ . ■

**Proposition 4** *Let  $\Omega$  be a locally compact groupoid on  $B$ , whose base subgroupoid  $\tilde{B}$  has a fundamental system of  $\alpha$ -relatively compact neighborhoods.*

For each  $v \in B$  and each continuous function with compact support,  $f : \Omega \rightarrow \mathbf{R}$ , let us denote by  $F_f^v : \Omega \rightarrow \mathbf{R}$  the map defined by

$$F_f^v(x) = \int f(y) d\lambda_v^{\alpha(x)}(y) - \int f(y) d\lambda_v^{\beta(x)}(y) \text{ for all } x \in \Omega.$$

Then

1.  $F_f^v$  is a morphism of groupoids.
2. For each  $\varepsilon > 0$  there is a symmetric neighborhood  $W_\varepsilon$  of the base subgroupoid  $\tilde{B}$  such that:

$$|F_f^v(x) - F_f^v(\tilde{u})| = |F_f^v(x)| < \varepsilon \text{ for all } x \in W_\varepsilon.$$

Consequently,  $F_f^v$  is a morphism of groupoids which is continuous at every unity  $\tilde{u} \in \tilde{B}$ .

**Proof.** For all  $(x, y) \in \Omega * \Omega$  we have

$$\begin{aligned} F_f^v(xy) &= \int f(z) d\lambda_v^{\alpha(y)}(z) - \int f(z) d\lambda_v^{\beta(x)}(z) \\ &= \int f(z) d\lambda_v^{\alpha(y)}(z) - \int f(z) d\lambda_v^{\beta(y)}(z) + \\ &\quad + \int f(z) d\lambda_v^{\alpha(x)}(z) - \int f(z) d\lambda_v^{\beta(x)}(z) \\ &= F_f^v(y) + F_f^v(x) \end{aligned}$$

Therefore  $F_f^v$  is a morphism. Let  $L$  be the support of  $f$ .

According to the preceding lemma, it follows that for each  $\varepsilon > 0$  there is a symmetric neighborhood  $W_\varepsilon$  of the base subgroupoid  $\tilde{B}$  such that:

$$|F_f^v(x) - F_f^v(\tilde{u})| < \varepsilon \lambda_v^{\alpha(x)}(UL) \text{ for all } x \in W_\varepsilon$$

$$|F_f^v(x) - F_f^v(\tilde{u})| < \varepsilon \sup_{u \in B} \lambda_v^u(UL) \text{ for all } x \in W_\varepsilon$$

Hence  $F_f^v$  is continuous at every  $\tilde{u} \in \tilde{B}$ . ■

**Proposition 5** *Let  $\Omega$  be a locally compact groupoid on  $B$ , whose base subgroupoid  $\tilde{B}$  has a fundamental system of  $\alpha$ -relatively compact neighborhoods. Let  $v \in B$ . Then the following conditions are equivalent:*

1. The map  $\beta_v : \Omega_v \rightarrow B$ ,  $\beta_v(x) = \beta(x)$  is open (or equivalently, the map  $\alpha_v : \Omega^v \rightarrow B$ ,  $\alpha_v(x) = \alpha(x)$  is open).

2. For each continuous function with compact support,  $f : \Omega \rightarrow \mathbf{R}$ , the map

$$u \xrightarrow{\phi_f^v} \int f(y) d\lambda_v^u(y) [ : B \rightarrow \mathbf{R} ]$$

is continuous at every  $u \in [v]$ .

**Proof.** 1  $\Rightarrow$  2. Let  $f : \Omega \rightarrow \mathbf{R}$  be a continuous function with compact support. Let us denote by  $F_f^v : \Omega \rightarrow \mathbf{R}$  the map defined by

$$F_f^v(x) = \int f(y) d\lambda_v^{\alpha(x)}(y) - \int f(y) d\lambda_v^{\beta(x)}(y) \text{ for all } x \in \Omega.$$

Let  $u \in [v]$  and let  $(u_i)_i$  be a net in  $B$  converging to  $u$ . Let  $x \in \Omega_v$  be such that  $\beta_v(x) = u$ . Since  $\beta_v : \Omega_v \rightarrow B$  is open, we may pass to a subnet and assume that there exists a net  $(x_i)_i$  in  $\Omega_v$  converging to  $x$  such that  $\beta_v(x_i) = u_i$  for all  $i$ . Thus, we obtain a net  $(x_i x^{-1})_i$  converging to  $\tilde{u}$  in  $\Omega$ . According to Proposition 4,  $F_f^v$  is continuous at  $\tilde{u}$ . Hence

$$\lim_i F_f^v(x_i x^{-1}) = 0$$

that is

$$\lim_i F_f^v(x_i) - F_f^v(x) = 0.$$

Since  $F_f^v(x_i) = \phi_f^v(\alpha(x_i)) - \phi_f^v(\beta(x_i)) = \phi_f^v(v) - \phi_f^v(u_i)$  and  $F_f^v(x) = \phi_f^v(v) - \phi_f^v(u)$ , it follows that  $\lim_i \phi_f^v(u_i) = \phi_f^v(u)$ .

2  $\Rightarrow$  1. Let  $U$  be an open set in  $\Omega$  and  $u_0 \in \beta_v(U \cap \Omega_v)$ . Let  $x_0 \in U$  be such that  $u_0 = \beta(x_0)$ . Let us choose a compactly supported nonnegative continuous function,  $f : \Omega \rightarrow [0, 1]$ , such that  $f(x_0) > 0$  and  $f$  vanishes outside  $U$ . Since the map

$$u \mapsto \int f(y) d\lambda_v^u(y) [ : B \rightarrow \mathbf{R} ]$$

is continuous at  $u_0$ , it follows that

$$W = \left\{ u : \int f(y) d\lambda_v^u(y) > \frac{1}{2} \int f(y) d\lambda_v^{u_0}(y) \right\}$$

is a neighborhood of  $u_0$ . It is easy to see that  $W$  is contained in the set

$$\left\{ u : \int f(y) d\lambda_v^u(y) \neq 0 \right\}$$

which is contained in  $\beta_v(U \cap \Omega_v)$ . Therefore  $\beta_v(U \cap \Omega_v)$  is a neighborhood of  $u_0$ . ■



**Theorem 6** Let  $\Omega$  be a locally compact groupoid on  $B$ , whose base subgroupoid  $\tilde{B}$  has a fundamental system of  $\alpha$ -relatively compact neighborhoods. Suppose that  $\beta$ , and hence  $\alpha$ , are open maps. For each  $v \in B$  and each continuous function with compact support,  $f : \Omega \rightarrow \mathbf{R}$ , let us denote by  $F_f^v : \Omega \rightarrow \mathbf{R}$  the map defined by

$$F_f^v(x) = \int f(y) d\lambda_v^{\alpha(x)}(y) - \int f(y) d\lambda_v^{\beta(x)}(y), \quad x \in \Omega$$

and let us denote by  $\phi_f^v : B \rightarrow \mathbf{R}$  the map defined by

$$\phi_f^v(u) = \int f(y) d\lambda_v^u(y), \quad u \in B.$$

For each continuous function with compact support,  $f : \Omega \rightarrow \mathbf{R}$ , assume that  $F_f^v$  is continuous on  $\Omega$ . If  $[v]$  is not singleton, then  $\phi_f^v$  is continuous at every  $u \in [v]$ .

**Proof.** Let  $u \in B \cap [v]$ . Because  $[v]$  contains at least two elements, there is  $v_0 \in [v]$  such that  $v_0 \neq u$ . Since  $u \neq v_0$ , there are two open subsets of  $B$ ,  $u \in U_0$  and  $v_0 \in V_0$  such that  $U_0 \cap V_0 = \emptyset$ . Let  $g : B \rightarrow [0, 1]$  be a continuous function with the following properties:

1.  $g(w) = 1$  for all  $w$  in a compact neighborhood  $U$  of  $u$  contained in  $U_0$
2.  $g(w) = 0$  for all  $w \notin U_0$ .

For each continuous function with compact support,  $f : \Omega_v \rightarrow \mathbf{R}$ , let us set  $f_U = f \circ g \circ \beta$ . For each  $w \in B$ , we have

$$\begin{aligned} \phi_{f_U}^v(w) &= \int f_U(y) d\lambda_v^w(y) = \int f(y) g(\beta(y)) d\lambda_v^w(y) = \\ &= g(w) \int f(y) d\lambda_v^w(y) = g(w) \phi_f^v(w) \end{aligned}$$

Therefore  $\phi_f^v(w) = \phi_{f_U}^v(w)$  for all  $w \in U$  and  $\phi_f^v(w) = 0$  for all  $w \in V_0$ .

Let  $(u_i)_i$  be a net converging to  $u$ . Without loss of generality we may assume that  $u_i \in U$  for all  $i$ , and hence,  $\phi_f(u_i) = \phi_{f_U}(u_i)$  for all  $i$ . So to prove that  $\phi_f^v$  is continuous in  $u$  it suffices to show that  $\lim_i \phi_{f_U}^v(u_i) = \phi_{f_U}^v(u)$ . Let us choose  $x$  such that  $\beta(x) = u$  and  $\alpha(x) = v_0$ . Because  $\beta : \Omega \rightarrow B$  is open, we may pass to a subnet and assume that there exists a net  $(x_i)_i$  in  $\Omega$  converging to  $x$  such that  $\beta(x_i) = u_i$  for all  $i$ . Since  $F_{f_U}^v$  is continuous on  $\Omega$ ,

$$\lim_i F_{f_U}^v(x_i) = F_{f_U}^v(x) = \phi_{f_U}^v(v) - \phi_{f_U}^v(u) = -\phi_{f_U}^v(u). \quad (1)$$

On the other hand,  $\lim_i \alpha(x_i) = \alpha(x) = v_0$  and for large  $i$  we may assume that  $\alpha(x_i) \in V_0$ . Hence,  $\phi_{f_U}^v(\alpha(x_i)) = 0$  and from  $F_{f_U}^v(x_i) = \phi_{f_U}^v(\alpha(x_i)) - \phi_{f_U}^v(\beta(x_i))$ , it follows that

$$F_{f_U}^v(x_i) = -\phi_{f_U}^v(\beta(x_i)). \quad (2)$$

By 1 and 2, it results that  $\lim_i \phi_{f_U}^v(u_i) = \phi_{f_U}^v(u)$ , as required. ■

**Corollary 7** *Let  $\Omega$  be a locally compact groupoid on  $B$ , whose base subgroupoid  $\tilde{B}$  has a fundamental system of  $\alpha$ -relatively compact neighborhoods. Suppose that  $\beta$ , and hence  $\alpha$ , are open maps. For each  $v \in B$  and each continuous function with compact support,  $f : \Omega \rightarrow \mathbf{R}$ , let us denote by  $F_f^v : \Omega \rightarrow \mathbf{R}$  the map defined by*

$$F_f^v(x) = \int f(y) d\lambda_v^{\alpha(x)}(y) - \int f(y) d\lambda_v^{\beta(x)}(y), \quad x \in \Omega.$$

*Then each  $F_f^v$  is a morphism of groupoids which is continuous at every unity  $\tilde{u} \in \tilde{B}$ . If all morphisms  $F_f^v$  are continuous on  $\Omega$ , then for each  $v \in B$  with non-singleton orbit, the map  $\beta_v : \Omega_v \rightarrow B$ ,  $\beta_v(x) = \beta(x)$  is open.*

**Proof.** According to the preceding theorem, for each  $v$  with  $[v] \neq \{v\}$  and for each continuous function with compact support,  $f : \Omega \rightarrow \mathbf{R}$ , the map  $\phi_f^v : B \rightarrow \mathbf{R}$  defined by

$$\phi_f^v(u) = \int f(y) d\lambda_v^u(y), \quad u \in B$$

is continuous at every  $u \in [v]$ . Applying Proposition 5, it follows that the map  $\beta_v : \Omega_v \rightarrow B$ ,  $\beta_v(x) = \beta(x)$  is open for all  $v \in B$  with  $[v] \neq \{v\}$ . ■

**Corollary 8** *Let  $\Omega$  be a locally compact groupoid on  $B$ , for which the target projection  $\beta$  is an open map and whose base subgroupoid  $\tilde{B}$  is paracompact. Let us assume that for any morphism  $F : \Omega \rightarrow \mathbf{R}$  the continuity of  $F$  at all  $\tilde{u} \in \tilde{B}$  implies the continuity of  $F$  everywhere on  $\Omega$ . Then for each  $v \in B$  with non-singleton orbit, the map  $\beta_v : \Omega_v \rightarrow B$ ,  $\beta_v(x) = \beta(x)$  is open.*

**Proof.** If the base subgroupoid  $\tilde{B}$  is paracompact, then  $\tilde{B}$  has a fundamental system of  $\alpha$ -relatively compact neighborhoods (see the proof of Proposition II.1.9/p. 56 [5]). Therefore the hypotheses of the preceding corollary are satisfied. ■

**Definition 9** *A locally compact groupoid is  $\beta$ -discrete if its base subgroupoid is an open subset (Definition I.2.6/p. 18 [5]).*

The  $\beta$ -discrete groupoids are generalizations of transformation groups  $G \times B \rightarrow B$ , where the group  $G$  is assumed to be discrete. Any locally compact groupoid for which  $\beta$  (and hence  $\alpha$ ) is a local homeomorphism is  $\beta$ -discrete. Such groupoids are called *etale groupoids* ([4] p. 46) or *sheaf groupoids* ([2] p. 209).

**Corollary 10** *Let  $\Omega$  be a locally compact  $\beta$ -discrete groupoid, for which the target projection  $\beta$  is an open map and whose base subgroupoid  $\tilde{B}$  is paracompact. Let us assume that for any morphism  $F : \Omega \rightarrow \mathbf{R}$  the continuity of  $F$  on a neighborhood of  $\tilde{B}$  implies the continuity of  $F$  everywhere on  $\Omega$ . If for each  $v \in B$  the orbit  $[v]$  contains at least two elements, then  $B$  is a discrete space.*

**Proof.** Applying Lemma I.2.7/p. 18 [5], it follows that for each  $v \in B$ ,  $\Omega_v$  is a discrete space. According to the preceding corollary, for each  $v \in B$ , the map  $\beta_v : \Omega_v \rightarrow B$ ,  $\beta_v(x) = \beta(x)$  is open. Since  $\beta_v(\tilde{v}) = v$ , it follows that  $\{v\}$  is open in  $B$ . ■

**Remark 11** *We have proved that for general topological groupoids the continuity of a morphism on a neighborhood of the base does not necessarily imply the continuity everywhere. Perhaps, the question of Mackenzie should be reformulate: For what kind of groupoid morphism does the continuity on a neighborhood of the base imply the continuity everywhere?*

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