

# An application of Mackey's selection lemma

Mădălina Roxana Buneci

ABSTRACT. Let  $G$  be a locally compact second countable groupoid. Let  $F$  be a subset of  $G^{(0)}$  meeting each orbit exactly once. Let us denote by  $d_F$  the restriction of the domain map to  $G^F$  and by  $r'$  the restriction of the range map to the isotropy group bundle of  $G$ . We shall prove that if  $d_F$  is open, then  $r'$  is open and  $d_F$  has a regular Borel cross section. Conversely, we shall prove that if  $r'$  is open and  $d_F$  admits a regular cross section (a right inverse which carries each compact subset of  $G^{(0)}$  into a relatively compact subset of  $G^F$ ), then  $d_F$  is open. We shall also prove that, if  $d_F$  is open, then  $F$  is a closed subset of  $G^{(0)}$ , and the orbit space  $G^{(0)}/G$  is a proper space. If  $F$  is closed and regular (the intersection of  $F$  with the saturated of any compact subset of  $G^{(0)}$  is relatively compact) and  $G^{(0)}/G$  is proper, then  $d_F$  is open.

AMS 2000 Subject Classification: 22A22, 43A05, 54D45, 37A55

Key words: locally compact groupoid, regular cross section, orbit space

to appear in *Studia Universitatis Babes-Bolyai, Matematica* vol. L(4) (2005), 23-32.

## 1. Introduction

We shall consider a locally compact groupoid  $G$  and a set  $F$  containing exactly one element from each orbit of  $G$ . We shall study the connection between the openness of  $d_F$ , the restriction of the domain map to  $G^F$ , and the existence of a regular cross section of  $d_F$  (a right inverse which carries each compact subset of  $G^{(0)}$  into a relatively compact subset of  $G^F$ ). The motivation for studying the map  $d_F$  comes from the fact that if  $F$  is closed and  $d_F$  is open, then  $G$  and  $G_F^F$  are (Morita) equivalent locally compact groupoids (in the sense of Definition 2.1/p. 6 [4]). A result of Paul Muhly, Jean Renault and Dana Williams states that the  $C^*$ -algebras associated to (Morita) equivalent locally compact second countable groupoids are strongly Morita equivalent (Theorem 2.8/p. 10 [4]). Also the notion of topological amenability is invariant under the equivalence of groupoids (Theorem 2.2.7/p. 50 [1]). Consequently, if  $F$  is closed and  $d_F$  is open, then  $G$  and the bundle group  $G_F^F$  have strongly Morita equivalent  $C^*$ -algebras. Also, the equivalence of  $G$  and  $G_F^F$  implies that  $G$  is amenable if and only if each isotropy group  $G_u^u$  is amenable.

For establishing notation, we include some definitions that can be found in several places (e.g. [5]). A groupoid is a set  $G$ , together with a distinguished

---

This work is supported by the MEC-CNCSIS grant At 127/2004 (33346/29.06.2004).

subset  $G^{(2)} \subset G \times G$  (called the set of composable pairs), and two maps:

$$\begin{aligned} (x, y) &\rightarrow xy \left[ : G^{(2)} \rightarrow G \right] \text{ (product map)} \\ x &\rightarrow x^{-1} \left[ : G \rightarrow G \right] \text{ (inverse map)} \end{aligned}$$

such that the following relations are satisfied:

(1) If  $(x, y) \in G^{(2)}$  and  $(y, z) \in G^{(2)}$ , then  $(xy, z) \in G^{(2)}$ ,  $(x, yz) \in G^{(2)}$  and  $(xy)z = x(yz)$ .

(2)  $(x^{-1})^{-1} = x$  for all  $x \in G$ .

(3) For all  $x \in G$ ,  $(x, x^{-1}) \in G^{(2)}$ , and if  $(z, x) \in G^{(2)}$ , then  $(zx)x^{-1} = z$ .

(4) For all  $x \in G$ ,  $(x^{-1}, x) \in G^{(2)}$ , and if  $(x, y) \in G^{(2)}$ , then  $x^{-1}(xy) = y$ .

The maps  $r$  and  $d$  on  $G$ , defined by the formulae  $r(x) = xx^{-1}$  and  $d(x) = x^{-1}x$ , are called the range and the source maps. It follows easily from the definition that they have a common image called the unit space of  $G$ , which is denoted  $G^{(0)}$ . Its elements are units in the sense that  $xd(x) = r(x)x = x$ . It is useful to note that a pair  $(x, y)$  lies in  $G^{(2)}$  precisely when  $d(x) = r(y)$ , and that the cancellation laws hold (e.g.  $xy = xz$  iff  $y = z$ ). The fibers of the range and the source maps are denoted  $G^u = r^{-1}(\{u\})$  and  $G_v = d^{-1}(\{v\})$ , respectively. More generally, given the subsets  $A, B \subset G^{(0)}$ , we define  $G^A = r^{-1}(A)$ ,  $G_B = d^{-1}(B)$  and  $G_B^A = r^{-1}(A) \cap d^{-1}(B)$ .  $G_B^A$  becomes a groupoid (called the reduction of  $G$  to  $A$ ) with the unit space  $A$ , if we define  $(G_B^A)^{(2)} = G^{(2)} \cap (G_B^A \times G_B^A)$ . For each unit  $u$ ,  $G_u^u$  is a group, called isotropy group at  $u$ . The group bundle

$$\{x \in G : r(x) = d(x)\}$$

is denoted  $G'$ , and is called the isotropy group bundle of  $G$ . The relation  $u \sim v$  iff  $G_v^u \neq \emptyset$  is an equivalence relation on  $G^{(0)}$ . Its equivalence classes are called orbits and the orbit of a unit  $u$  is denoted  $[u]$ . Let

$$R = (r, d)(G) = \{(r(x), d(x)), x \in G\}$$

be the graph of the equivalence relation induced on  $G^{(0)}$ . The quotient space for this equivalence relation is called the orbit space of  $G$  and denoted  $G^{(0)}/G$ .

A topological groupoid consists of a groupoid  $G$  and a topology compatible with the groupoid structure. We are exclusively concerned with topological groupoids which are locally compact Hausdorff. The Borel sets of a topological space are taken to be the  $\sigma$ -algebra generated by the open sets.

## 2. Necessary and sufficient conditions for the openness of $d_F$ .

**DEFINITION 1.** *Let  $X, Y$  be two topological spaces. A cross section of a map  $f : X \rightarrow Y$  is a function  $\sigma : Y \rightarrow X$  such that  $f(\sigma(y)) = y$  for all  $y \in Y$ . We shall say that the cross section  $\sigma$  is regular if  $\sigma(K)$  has compact closure in  $X$  for each compact set  $K$  in  $Y$ .*

We shall need the following lemma proved by Mackey (Lemma 1.1/p. 102 [3]):

**LEMMA 1.** *If  $X$  and  $Y$  are second countable, locally compact spaces, and  $f : X \rightarrow Y$  is a continuous open function onto  $Y$ , then  $f$  has a Borel regular cross section.*

PROPOSITION 1. *Let  $G$  be a locally compact groupoid. Let  $F$  be a subset of  $G^{(0)}$  meeting each orbit exactly once. Let us define the function  $e : G^{(0)} \rightarrow G^{(0)}$  by*

$$e(u) = F \cap [u], u \in G^{(0)}$$

*If the map  $d_F : G^F \rightarrow G^{(0)}$ ,  $d_F(x) = d(x)$ , is open, then the function  $e$  is continuous and  $F$  is a closed subset of  $G^{(0)}$ .*

PROOF. Let  $(u_i)_i$  be a net converging to  $u$  in  $G^{(0)}$ . Let  $x \in G$  be such that  $r(x) = e(u)$  and  $d(x) = u$ . Since  $(u_i)_i$  converges to  $d_F(x)$  and  $d_F$  is an open map, we may pass to a subnet and assume that there is a net  $(x_i)_i$  converging to  $x$  in  $G^F$  such that  $d_F(x_i) = u_i$ . It is easy to see that  $r(x_i) = e(u_i)$  ( $r(x_i) \in F$  and  $r(x_i) \in [d(x_i)] = [u_i]$ ). Thus  $e(u_i) = r(x_i)$  converges to  $r(x) = e(u)$ . Since  $G^{(0)}$  is Hausdorff,  $F$  is closed in  $G^{(0)}$ , being the image of the map  $e$  whose square is itself.  $\square$

PROPOSITION 2. *Let  $G$  be a locally compact groupoid. Let  $F$  be a subset of  $G^{(0)}$  meeting each orbit exactly once. If the map  $d_F : G^F \rightarrow G^{(0)}$ ,  $d_F(x) = d(x)$ , is open, then graph*

$$R = \{(r(x), d(x)), x \in G\}$$

*of the equivalence relation induced on  $G^{(0)}$  is closed in  $G^{(0)} \times G^{(0)}$ , and the map  $(r, d) : G \rightarrow R$ ,  $(r, d)(x) = (r(x), d(x))$  is open, where  $R$  is endowed with the product topology induced from  $G^{(0)} \times G^{(0)}$ .*

PROOF. Let us define the function  $e : G^{(0)} \rightarrow G^{(0)}$  by

$$e(u) = F \cap [u], u \in G^{(0)}.$$

By Proposition 1, the function  $e$  is continuous. Let  $((u_i, v_i))_i$  be a net in  $R$  which converges to  $(u, v)$  in  $G^{(0)} \times G^{(0)}$  (with respect to with the product topology). Then  $(u_i)_i$  converges to  $u$ ,  $(v_i)_i$  converges to  $v$ , and  $u_i \sim v_i$  for all  $i$ . We have

$$\begin{aligned} \lim_i e(u_i) &= e(u) \\ \lim_i e(v_i) &= e(v) \end{aligned}$$

because  $e$  is continuous. On the other hand, the fact that  $u_i \sim v_i$  for all  $i$  implies that  $e(u_i) = e(v_i)$  for all  $i$ . Hence  $e(u) = e(v)$ , or equivalently,  $u \sim v$ . Therefore  $(u, v) \in R$ .

Let us prove that the map  $(r, d) : G \rightarrow R$ ,  $(r, d)(x) = (r(x), d(x))$  is open, where  $R$  is endowed with the product topology induced from  $G^{(0)} \times G^{(0)}$ . Let  $x \in G$ , and let  $((u_i, v_i))_i$  be a net in  $R$  converging to  $(r, d)(x)$ . Then  $(u_i)_i$  converges to  $r(x)$ ,  $(v_i)_i$  converges to  $d(x)$ , and  $u_i \sim v_i$  for all  $i$ . Let  $s \in G$  be such that  $r(s) = e(r(x))$  and  $d(s) = r(x)$  and let  $t = sx$ . Obviously,  $s, t \in G^F$  and

$$\begin{aligned} \lim_i u_i &= r(x) = d(s) \\ \lim_i v_i &= d(x) = d(sx) = d(t). \end{aligned}$$

Since  $d_F$  is an open map, we may pass to subnets and assume that there is a net  $(s_i)_i$  converging to  $s$  in  $G^F$  and there is a net  $(t_i)_i$  converging to  $t$  in  $G^F$  such

that  $d_F(s_i) = u_i$  and  $d_F(t_i) = v_i$ . The fact that  $e(u_i)$  is the only element of  $F$ , which is equivalent to  $u_i \sim v_i$ , implies that  $r(s_i) = e(u_i) = e(v_i) = r(t_i)$ . We have

$$\begin{aligned} \lim_i s_i^{-1} t_i &= s^{-1} t = x \\ r(s_i^{-1} t_i) &= d(s_i) = u_i, d(s_i^{-1} t_i) = d(t_i) = v_i \end{aligned}$$

Therefore the map  $(r, d)$  is open  $\square$

**COROLLARY 1.** *Let  $G$  be a locally compact groupoid having open range map. Let  $F$  be a subset of  $G^{(0)}$  meeting each orbit exactly once. If the map  $d_F : G^F \rightarrow G^{(0)}$ ,  $d_F(x) = d(x)$ , is open, then the orbit space  $G^{(0)}/G$  is proper.*

**PROOF.** The fact that  $G^{(0)}/G$  is a proper space means that  $G^{(0)}/G$  is Hausdorff and the map  $(r, d) : G \rightarrow R$ ,  $(r, d)(x) = (r(x), d(x))$  is open, where  $R$  is endowed with the product topology induced from  $G^{(0)} \times G^{(0)}$ . Let us note that the quotient map  $\pi : G^{(0)} \rightarrow G^{(0)}/G$  is open (because the range map of  $G$  is open). Since the graph  $R$  of the equivalence relation is closed in  $G^{(0)} \times G^{(0)}$ , it follows that  $G^{(0)}/G$  is Hausdorff.  $\square$

**LEMMA 2.** *Let  $G$  be a locally compact groupoid having open range map. Let  $F$  be a subset of  $G^{(0)}$  meeting each orbit exactly once. If the map  $d_F : G^F \rightarrow G^{(0)}$ ,  $d_F(x) = d(x)$ , is open, then  $F$  and  $G^{(0)}/G$  are homeomorphic spaces.*

**PROOF.** Let  $\pi : G^{(0)} \rightarrow G^{(0)}/G$  be the quotient map. We prove that the map  $\pi_F : F \rightarrow G^{(0)}/G$ ,  $\pi_F(x) = \pi(x)$  is a homeomorphism. It suffices to prove that  $\pi_F$  is an open map (because  $\pi_F$  is one-to-one from  $F$  onto  $G^{(0)}/G$ ). Let  $u \in F$  and  $(\dot{u}_i)_i$  be a net converging to  $\pi(u)$  in  $G^{(0)}/G$ . Since  $\pi \circ d_F$  is open, we may pass to a subnet and assume that there is a net  $(x_i)_i$  converging to  $u$  in  $G^F$  such that  $\pi(d_F(x_i)) = \dot{u}_i$ . Then  $(r(x_i))_i$  is a net in  $F$  which converges to  $u$ .  $\square$

**REMARK 1.** *Let  $G$  be a locally compact groupoid. If the map  $(r, d) : G \rightarrow R$  is open (where  $R = \{(r(x), d(x)), x \in G\}$  is endowed with the product topology induced from  $G^{(0)} \times G^{(0)}$ ), then the map  $r' : G' \rightarrow G^{(0)}$ ,  $r'(x) = r(x)$ , is open, where*

$$G' = \{x \in G : r(x) = d(x)\},$$

*is the isotropy group bundle of  $G$ .*

**PROPOSITION 3.** *Let  $G$  be a locally compact second countable groupoid. Let  $F$  be a subset of  $G^{(0)}$  meeting each orbit exactly once. If the map  $d_F : G^F \rightarrow G^{(0)}$ ,  $d_F(x) = d(x)$ , is open, then  $d_F$  has Borel regular cross section.*

**PROOF.** If  $d_F$  is open, then according to Proposition 1,  $F$  is a closed subset of  $G^{(0)}$ . Therefore  $G^F$  is a locally compact space, and we may apply Lemma 1.  $\square$

We shall need a system of measures

$$\{\beta_v^u, (u, v) \in (r, d)(G)\}$$

satisfying the following conditions:

1.  $\text{supp}(\beta_v^u) = G_v^u$  for all  $u \sim v$ .
2.  $\sup_{u, v} \beta_v^u(K) < \infty$  for all compact  $K \subset G$ .
3.  $\int f(y) d\beta_v^{r(x)}(y) = \int f(xy) d\beta_v^{d(x)}(y)$  for all  $x \in G$  and  $v \sim r(x)$ .

In Section 1 of [6] Jean Renault constructs a Borel Haar system for  $G'$ . One way to do this is to choose a function  $F_0$  continuous with conditionally support, which is nonnegative and equal to 1 at each  $u \in G^{(0)}$ . Then for each  $u \in G^{(0)}$  choose a left Haar measure  $\beta_u^u$  on  $G_u^u$  so the integral of  $F_0$  with respect to  $\beta_u^u$  is 1. Renault defines  $\beta_v^u = x\beta_v^v$  if  $x \in G_v^u$  (where  $x\beta_v^v(f) = \int f(xy) d\beta_v^v(y)$  as usual). If  $z$  is another element in  $G_v^u$ , then  $x^{-1}z \in G_v^v$ , and since  $\beta_v^v$  is a left Haar measure on  $G_v^v$ , it follows that  $\beta_v^u$  is independent of the choice of  $x$ . If  $K$  is a compact subset of  $G$ , then  $\sup_{u,v} \beta_v^u(K) < \infty$ . We obtain another construction of a system of measures with the above properties if in the proof of Theorem 8/p. 331[2] we replace the regular cross section of  $G^u \xrightarrow{d} G^{(0)}$  (in the transitive case) with a regular cross section of  $G^F \xrightarrow{d} G^{(0)}$ , where  $F$  is a subset of  $G^{(0)}$  meeting each orbit exactly once.

LEMMA 3. *Let  $G$  be a locally compact groupoid. Let  $F$  be a subset of  $G^{(0)}$  meeting each orbit exactly once and let us denote  $e(u)$  the unique element of  $F$  equivalent to  $u$ . If the map*

$$u \mapsto \int f(y) d\beta_u^{e(u)}(y) \left[ : G^{(0)} \rightarrow \mathbf{C} \right]$$

*is continuous for any continuous function with compact support,  $f : G \rightarrow \mathbf{C}$ , then the map*

$$d_F : G^F \rightarrow G^{(0)}, d_F(x) = d(x).$$

*is open.*

PROOF. Let  $x_0 \in G^F$  and let  $U$  be a nonempty compact neighborhood of  $x_0$ . Choose a nonnegative continuous function,  $f$  on  $G$ , with  $f(x_0) > 0$  and  $\text{supp}(f) \subset U$ . Let  $W$  be the set of units  $u$  with the property that  $\beta_u^{e(u)}(f) > 0$ . Then  $W$  is an open neighborhood of  $u_0 = d(x_0)$  contained in  $d_F(U)$ .  $\square$

PROPOSITION 4. *Let  $G$  be a locally compact second countable groupoid. Let  $F$  be a subset of  $G^{(0)}$  containing exactly one element from each orbit of  $G$ , and let us denote  $e(u)$  the unique element of  $F$  equivalent to  $u$ . Let us assume that the map  $r' : G' \rightarrow G^{(0)}$ ,  $r'(x) = r(x)$  is open, where  $G'$  is the isotropy group bundle of  $G$ . If the map  $d_F : G^F \rightarrow G^{(0)}$ ,  $d_F(x) = d(x)$ , has a regular cross section  $\sigma$ , then for each continuous with compact support function  $f : G \rightarrow \mathbf{C}$ , the map*

$$u \rightarrow \int f(y) d\beta_u^{e(u)}(y)$$

*is continuous on  $G$ .*

PROOF. By Lemma 1.3/p. 6 [6], for each  $f : G \rightarrow \mathbf{C}$  continuous with compact support, the function  $u \rightarrow \int f(y) d\beta_u^{e(u)}(y) \left[ : G^{(0)} \rightarrow \mathbf{C} \right]$  is continuous. Let  $(u_i)_i$  be a sequence in  $G^{(0)}$  converging to  $u$ . Let  $x_i = \sigma(u_i)^{-1}$ . Since  $\sigma$  is regular, it follows that  $(x_i)_i$  has a convergent subsequence in  $G^F$ . Let  $x$  be the limit of this subsequence. Let  $f : G \rightarrow \mathbf{C}$  be a continuous function with compact support and let  $g$  be a continuous extension on  $G$  of  $y \rightarrow f(xy) \left[ : G^{d(x)} \rightarrow \mathbf{C} \right]$ . Let  $K$  be the compact set  $\left( \{x, x_i, i = 1, 2, \dots\}^{-1} \text{supp}(f) \cup \text{supp}(g) \right) \cap r^{-1}(\{d(x), d(x_i), i = 1, 2, \dots\})$ . We

have

$$\begin{aligned}
& \left| \int f(y) d\beta_u^{e(u)}(y) - \int f(y) d\beta_{u_i}^{e(u_i)}(y) \right| \\
&= \left| \int f(xy) d\beta_{d(x)}^{d(x)}(y) - \int f(x_i y) d\beta_{d(x_i)}^{d(x_i)}(y) \right| \\
&= \left| \int g(y) d\beta_{d(x)}^{d(x)}(y) - \int f(x_i y) d\beta_{d(x_i)}^{d(x_i)}(y) \right| \\
&\leq \left| \int g(y) d\beta_{d(x)}^{d(x)}(y) - \int g(y) d\beta_{d(x_i)}^{d(x_i)}(y) \right| + \\
&\quad + \left| \int g(y) d\beta_{d(x_i)}^{d(x_i)}(y) - \int f(x_i y) d\beta_{d(x_i)}^{d(x_i)}(y) \right| \\
&\leq \left| \int g(y) d\beta_u^u(y) - \int g(y) d\beta_{u_i}^{u_i}(y) \right| + \\
&\quad + \sup_{y \in G_{u_i}^{u_i}} |g(y) - f(x_i y)| \beta_{u_i}^{u_i}(K)
\end{aligned}$$

A compactness argument shows that  $\sup_{y \in G_{u_i}^{u_i}} |g(y) - f(x_i y)|$  converges to 0. Also  $\left| \int g(y) d\beta_{d(x_i)}^{d(x_i)}(y) - \int f(x_i y) d\beta_{d(x_i)}^{d(x_i)}(y) \right|$  converges to 0, because the function  $u \rightarrow \int f(y) d\beta_u^u(y)$  is continuous on  $G^{(0)}$ . Hence

$$\left| \int f(y) d\beta_u^{e(u)}(y) - \int f(y) d\beta_{u_i}^{e(u_i)}(y) \right|$$

converges to 0.  $\square$

**COROLLARY 2.** *Let  $G$  be a locally compact second countable groupoid. Let  $F$  be a subset of  $G^{(0)}$  meeting each orbit exactly once. If the restriction  $r'$  of the range map to the isotropy group bundle  $G'$  of  $G$  is open, and if the map  $d_F : G^F \rightarrow G^{(0)}$ ,  $d_F(x) = d(x)$ , has a regular cross section, then  $d_F$  is an open map.*

**THEOREM 1.** *Let  $G$  be a locally compact second countable groupoid. Let  $F$  be a subset of  $G^{(0)}$  meeting each orbit exactly once, and let  $d_F : G^F \rightarrow G^{(0)}$  be the map defined by  $d_F(x) = d(x)$  for all  $x \in G^F$ . If  $d_F$  is open then  $d_F$  admits a Borel regular cross section. If the restriction  $r'$  of the range map to the isotropy group bundle  $G'$  of  $G$  is open and if  $d_F$  admits a regular cross section, then  $d_F$  is an open map.*

**PROOF.** If  $d_F$  is an open map, then, according Proposition 3,  $d_F$  has a regular cross section. Conversely, if  $d_F$  admits a regular cross section, then applying Proposition 4 and Lemma 3, it follows that  $d_F$  is open.  $\square$

**REMARK 2.** *Let us assume that  $G^{(0)}/G$  is **proper**. There is a regular Borel cross section  $\sigma_0$  of the quotient map  $\pi : G^{(0)} \rightarrow G^{(0)}/G$ . Let us assume that  $F = \sigma_0(G^{(0)}/G)$  is **closed** in  $G^{(0)}$ . Then the function  $e : G^{(0)} \rightarrow G^{(0)}$  defined by  $e(u) = F \cap [u]$  is continuous. If  $\sigma_1 : R \rightarrow G$  is regular Borel cross section of  $(r, d)$ , then  $\sigma : G^{(0)} \rightarrow G^F$ ,  $\sigma(u) = \sigma_1(e(u), u)$  is a Borel regular cross section of  $d_F$ . Therefore is that case  $d_F$  is open.*

### References

- [1] C. Anantharaman-Delaroche, J. Renault, *Amenable groupoids*, Monographie de L'Enseignement Mathematique No **36**, Geneve, 2000.
- [2] M. Buneci, *Consequences of Hahn structure theorem for the Haar measure*, Math. Reports, **4(54)** (2002), no. 4, 321-334 .
- [3] G. Mackey, *Induced representations of locally compact groups. I*, Ann. of Math., **55**(1952), 101-139.
- [4] P. Muhly, J. Renault and D. Williams, *Equivalence and isomorphism for groupoid  $C^*$ -algebras*, J. Operator Theory **17**(1987), 3-22.
- [5] J. Renault, *A groupoid approach to  $C^*$ - algebras*, Lecture Notes in Math., Springer-Verlag, **793**, 1980.
- [6] J. Renault, *The ideal structure of groupoid crossed product algebras*, J. Operator Theory, **25**(1991),3-36.

University Constantin Brâncuși of Târgu-Jiu,  
Bulevardul Republicii, Nr. 1, 210152 Târgu-Jiu, Romania.  
e-mail: ada@utgjiu.ro