C^*-algebras associated to groupoids with proper orbit space

Mădălina Roxana Buneci
University Constantin Brâncuşi, Târgu-Jiu

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Abstract

We shall consider a locally compact groupoid endowed with a Haar system ν and having proper orbit space. We shall associated to each appropriated cross section $\sigma : G^{(0)} \to G^F$ for $d_F : G^F \to G^{(0)}$ (where F is a Borel subset $G^{(0)}$ meeting each orbit exactly once) a C^*-algebra $M^*_\sigma(G, \nu)$. We shall prove that the C^*-algebras $M^*_{\sigma_i}(G, \nu)$ $i = 1, 2$ associated with different cross sections are $*$-isomorphic

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1 Introduction

The C^*-algebra of a locally compact groupoid was introduced by J. Renault in [8]. The construction extends the case of a group: the space of continuous functions with compact support on groupoid is made into a $*$-algebra and endowed with the smallest C^*-norm making its representations continuous.

In [7] Arlan Ramsay and Martin E. Walter have associated to a locally compact groupoid G a C^*- algebra denoted $M^*(G, \nu)$. They have considered the universal representation ω of $C^*(G, \nu)$ -the usual C^*-algebra associated to a Haar system $\nu = \{\nu^u, u \in G^{(0)}\}$ (constructed as in [8]) . Since every cyclic representation of $C^*(G, \nu)$ is the integrated form of a representation of G, it follows that ω can be also regarded as a representation of $\mathcal{B}_c(G)$, the space of compactly supported Borel bounded function on G. Arlan Ramsay and Martin E. Walter have used the notation $M^*(G, \nu)$ for the operator norm closure of $\omega(\mathcal{B}_c(G))$. Since ω is an $*$-isomorphism on $C^*(G, \nu)$, we can regarded $C^*(G, \nu)$ as a subalgebra of $M^*(G, \nu)$.

We assume that the orbit space of the groupoid G is proper and we choose a Borel subset F of $G^{(0)}$ meeting each orbit exactly once and such that $F \cap [K]$ has a compact closure for each compact subset K of $G^{(0)}$. For each appropriated cross section $\sigma : G^{(0)} \to G^F$ for $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, we construct a C^*-algebra $M^*_\sigma(G,\nu)$ which can be viewed as a subalgebra of $M^*(G,\nu)$. In [2] we have proved that if $\nu_1 = \{\nu_1^u, u \in G^{(0)}\}$ and $\nu_2 = \{\nu_2^u, u \in G^{(0)}\}$ are two Haar systems on G, then the C^*-algebras $M^*_\sigma(G,\nu_1)$ and $M^*_\sigma(G,\nu_2)$ are $*$-isomorphic. We have also proved that for a transitive (or more generally, a locally transitive) groupoid G, the C^*-algebras $C^*(G,\nu)$, $M^*(G,\nu)$ and $M^*_\sigma(G,\nu)$ coincide, and for a principal proper groupoid G, we have proved that

$$C^*(G,\nu) \subset M^*_\sigma(G,\nu) \subset M^*(G,\nu).$$

In this paper we shall prove that the C^*-algebras $M^*_{\sigma_i}(G,\nu)$ $i = 1, 2$ associated with different cross sections are $*$-isomorphic

2 Definitions and Notation

We shall use the definition of a topological groupoid given by J. Renault in [8]. For a groupoid G, $G^{(2)}$ will denote the composable pairs. The inverse map is written $x \to x^{-1}$ $[: G \to G]$ and the product map is written $(x,y) \to xy$ $\left[: G^{(2)} \to G\right]$. The maps r and d on G, defined by the formulae $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$, are called the range and the source maps. It follows easily from the definition that they have a common image called the unit space of G, which is denoted $G^{(0)}$. Its elements are units in the sense that $xd(x) = r(x)x = x$. Units will usually be denoted by letters as u, v, w while arbitrary elements will be denoted by x, y, z. It is useful to note that a pair (x,y) lies in $G^{(2)}$ precisely when $d(x) = r(y)$, and that the cancellation laws hold (e.g. $xy = xz$ iff $y = z$). The fibres of the range and the source maps are denoted $G^u = r^{-1}(\{u\})$ and $G_v = d^{-1}(\{v\})$, respectively. More generally, given the subsets A, $B \subset G^{(0)}$, we define $G^A = r^{-1}(A)$, $G_B = d^{-1}(B)$ and $G^A_B = r^{-1}(A) \cap d^{-1}(B)$. The reduction of G to $A \subset G^{(0)}$ is $G|A = G^A_A$. The relation $u\tilde{}v$ iff $G^u_v \neq \phi$ is an equivalence relation on $G^{(0)}$. Its equivalence classes are called orbits and the orbit of a unit u is denoted $[u]$. A groupoid is called transitive iff it has a single orbit. The quotient space for this equivalence relation is called the orbit space of G and denoted $G^{(0)}/G$. We denote by $\pi : G^{(0)} \to G^{(0)}/G, \pi(u) = \dot{u}$ the quotient map. A subset of $G^{(0)}$ is said saturated if it contains the orbits of its elements. For any subset A of $G^{(0)}$, we denote by $[A]$ the union of the orbits $[u]$ for all $u \in A$.

A topological groupoid consists of a groupoid G and a topology compatible with the groupoid structure. This means that:

(1) $x \to x^{-1}$ $[: G \to G]$ is continuous.

(2) (x,y) $\left[: G^{(2)} \to G\right]$ is continuous where $G^{(2)}$ has the induced topology from $G \times G$.

We are exclusively concerned with topological groupoids which are second countable, locally compact Hausdorff. It was shown in [6] that measured groupoids may be assume to have locally compact topologies, with no loss in generality.

If X is a locally compact space, $C_c(X)$ denotes the space of complex-valuated continuous functions with compact support. The Borel sets of a topological space are taken to be the σ-algebra generated by the open sets. The space of compactly supported bounded Borel function on X is denoted by $\mathcal{B}_c(X)$.

Let G be a locally compact second countable groupoid equipped with a Haar system, i.e. a family of positive Radon measures on G, $\{\nu^u,\, u \in G^{(0)}\}$, such that

1) For all $u \in G^{(0)}$, $supp(\nu^u) = G^u$.

2) For all $f \in C_c(G)$,

$$u \to \int f(x)\, d\nu^u(x) \quad \left[: G^{(0)} \to \mathbf{C}\right]$$

is continuous.

3) For all $f \in C_c(G)$ and all $x \in G$,

$$\int f(y)\, d\nu^{r(x)}(y) = \int f(xy)\, d\nu^{d(x)}(y)$$

As a consequence of the existence of continuous Haar systems, $r, d : G \to G^{(0)}$ are open maps ([10]). Therefore, in this paper we shall always assume that $r : G \to G^{(0)}$ is an open map

If μ is a measure on $G^{(0)}$, then the measure $\nu = \int \nu^u d\mu(u)$, defined by

$$\int f(y)\, d\nu(y) = \int \left(\int f(y)\, d\nu^u(y)\right) d\mu(u), \quad f \geq 0 \text{ Borel}$$

is called the measure on G induced by μ. The image of ν by the inverse map $x \to x^{-1}$ is denoted ν^{-1}. μ is said quasi-invariant if its induced measure ν is equivalent to its inverse ν^{-1} A measure belongings to the class of a quasi-invariant measure is also quasi-invariant. We say that the class is invariant.

If μ is a quasi-invariant measure on $G^{(0)}$ and ν is the measure induced on G, then the Radon-Nikodym derivative $\Delta = \frac{d\nu}{d\nu^{-1}}$ is called the modular function of μ.

In order to define the C^*-algebra of a groupoid the space of continuous functions with compact support on groupoid is made into a $*$-algebra and endowed with the smallest C^*-norm making its representations continuous. For $f, g \in C_c(G)$ the convolution is defined by:

$$f * g(x) = \int f(xy)\, g\left(y^{-1}\right) d\nu^{d(x)}(y)$$

and the involution by

$$f^*(x) = \overline{f\left(x^{-1}\right)}.$$

Under these operations, $C_c(G)$ becomes a topological $*$-algebra.

A representation of $C_c(G)$ is a $*$-homomorphism from $C_c(G)$ into $\mathcal{B}(H)$, for some Hilbert space H, that is continuous with respect to the inductive limit topology on $C_c(G)$ and the weak operator topology on $\mathcal{B}(H)$. The full C^*-algebra $C^*(G)$ is defined as the completion of the involutive algebra $C_c(G)$ with respect to the full C^*-norm

$$\|f\| = \sup \|L(f)\|$$

where L runs over all non-degenerate representation of $C_c(G)$ which are continuous for the inductive limit topology.

Every representation $\left(\mu, G^{(0)} * \mathcal{H}, L\right)$(see Definition 3.20/p.68 [4]) of G can be integrated into a representation, still denoted by L, of $C_c(G)$. The relation between the two representation is:

$$\langle L(f)\xi_1, \xi_2\rangle = \int f(x)\langle L(x)\xi_1(d(x)), \xi_2(r(x))\rangle \Delta^{-\frac{1}{2}}(x)\, d\nu^u(x)\, d\mu(u)$$

where $f \in C_c(G)$, $\xi_1, \xi_2 \in \int_{G^{(0)}}^{\oplus} \mathcal{H}(u)\, d\mu(u)$.

Conversely, every non-degenerate $*$-representation of $C_c(G)$ is obtained in this fashion (see [8] or [4]).

3 The decomposition of a Haar system over the principal groupoid

We shall need some results on the structure of the Haar systems developed by J. Renault in Section 1 of [9] and also by A. Ramsay and M.E. Walter in Section 2 of [7].

In Section 1 of [9] Jean Renault constructs a Borel Haar system for G'. One way to do this is to choose a function F_0 continuous with conditionally support which is nonnegative and equal to 1 at each $u \in G^{(0)}$. Then for each $u \in G^{(0)}$ choose a left Haar measure β_u^u on G_u^u so the integral of F_0 with respect to β_u^u is 1.

Renault defines $\beta_v^u = x\beta_v^v$ if $x \in G_v^u$ (where $x\beta_v^v(f) = \int f(xy)\, d\beta_v^v(y)$ as usual). If z is another element in G_v^u, then $x^{-1}z \in G_v^v$, and since β_v^v is a left Haar measure on G_v^v, it follows that β_v^u is independent of the choice of x. If K is a compact subset of G, then $\sup\limits_{u,v} \beta_v^u(K) < \infty$. Renault also defines a 1-cocycle δ on G such that for every $u \in G^{(0)}$, $\delta|_{G_u^u}$ is the modular function for β_u^u. δ and $\delta^{-1} = 1/\delta$ are bounded on compact sets in G.

Let

$$R = (r, d)(G) = \{(r(x), d(x)),\ x \in G\}$$

be the graph of the equivalence relation induced on $G^{(0)}$. This R is the image of G under the homomorphism (r, d), so it is a σ-compact groupoid. With this apparatus in place, Renault describes a decomposition of the Haar system $\left\{\nu^u, u \in G^{(0)}\right\}$ for G over the equivalence relation R (the principal groupoid associated to G). He proves that there is a unique Borel Haar system α for R with the property that

$$\nu^u = \int \beta_t^s d\alpha^u(s,t) \quad \text{for all } u \in G^{(0)}.$$

In Section 2 [7] A. Ramsay and M.E. Walter prove that

$$\sup_u \alpha^u((r,d)(K)) < \infty, \text{ for all compact } K \subset G$$

For each $u \in G^{(0)}$ the measure α^u is concentrated on $\{u\} \times [u]$. Therefore there is a measure μ^u concentrated on $[u]$ such that $\alpha^u = \varepsilon_u \times \mu^u$, where ε_u is the unit point mass at u. Since $\left\{\alpha^u, u \in G^{(0)}\right\}$ is a Haar system, we have $\mu^u = \mu^v$ for all $(u,v) \in R$, and the function

$$u \to \int f(s)\,\mu^u(s)$$

is Borel for all $f \geq 0$ Borel on $G^{(0)}$. For each u the measure μ^u is quasi-invariant (Section 2 [7]). Therefore μ^u is equivalent to $d_*(v^u)$ (Lemma 4.5/p. 277 [5]).

If η is a quasi-invariant measure for $\left\{\nu^u, u \in G^{(0)}\right\}$, then η is a quasi-invariant measure for $\left\{\alpha^u, u \in G^{(0)}\right\}$. Also if Δ_R is the modular function associated to $\left\{\alpha^u, u \in G^{(0)}\right\}$ and η, then $\Delta = \delta\Delta_R \circ (r,d)$ can serve as the modular function associated to $\left\{\nu^u, u \in G^{(0)}\right\}$ and η.

Since $\mu^u = \mu^v$ for all $(u,v) \in R$, the system of measures $\{\mu^u\}_u$ may be indexed on the elements of the orbit space $G^{(0)}/G$.

Definition 1 *We shall call the pair of the system of measures*

$$\left(\{\beta_v^u\}_{(u,v)\in R}, \{\mu^{\dot{u}}\}_{\dot{u}\in G^{(0)}/G}\right)$$

(described above) the decomposition of the Haar system $\left\{\nu^u, u \in G^{(0)}\right\}$ *over the principal groupoid associated to* G*. Also we shall call* δ *the* 1*-cocycle associated to the decomposition.*

Definition 2 *A locally compact groupoid* G *is proper if the map* $(r,d) : G \to G^{(0)} \times G^{(0)}$ *is proper (i.e. the inverse map of each compact subset of* $G^{(0)} \times G^{(0)}$ *is compact).(Definition 2.1.9/p. 37 [1]).*

Throughout this paper we shall assume that G is a second countable locally compact groupoid for which the orbit space is Hausdorff and the map

$$(r,d) : G \to R,\ (r,d)(x) = (r(x), d(x))$$

is open, where R is endowed with the product topology induced from $G^{(0)} \times G^{(0)}$. Therefore R will be a locally compact groupoid. The fact that R is a closed subset of $G^{(0)} \times G^{(0)}$ and that it is endowed with the product topology is equivalent to the fact R is a proper groupoid.

In the subsequent considerations by a groupoid with proper orbit space we shall mean a groupoid G for which the orbit space is Hausdorff and the map

$$(r,d) : G \to R,\ (r,d)(x) = (r(x), d(x))$$

is open, where R is endowed with the product topology induced from $G^{(0)} \times G^{(0)}$.

4 A C^*-algebra associated to a locally compact groupoid with proper orbit space

Let G be a locally compact second countable groupoid with proper orbit space. Let

$$\pi : G^{(0)} \to G^{(0)}/G$$

be the quotient map. Let F be a Borel subset of $G^{(0)}$ meeting each orbit exactly once. We shall associated a C^*- algebra to each cross section $\sigma : G^{(0)} \to G^F$ for $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, satisfying the following properties:

1. $\sigma(e(v)) = e(v)$ for all $v \in G^{(0)}$, where $e(u)$ is the unique element in the orbit of u contained in F.
2. $\sigma(K)$ is relatively compact in G for all compact sets $K \subset G^{(0)}$.

First let us show that there exist such cross sections. Since the quotient space is proper, $G^{(0)}/G$ is Hausdorff. We have assumed that the range map r is open. As a consequence, the map π is open. Applying Lemma 1.1 [3] to the locally compact second countable spaces $G^{(0)}$ and $G^{(0)}/G$ and to the continuous open surjection $\pi : G^{(0)} \to G^{(0)}/G$, it follows that there is a Borel set F in $G^{(0)}$ such that:

1. F contains exactly one element in each orbit $[u] = \pi^{-1}(\pi(u))$.
2. For each compact subset K of $G^{(0)}$, $F \cap [K] = F \cap \pi^{-1}(\pi(K))$ has a compact closure.

For each unit u let us define $e(u) = F \cap [u]$ ($e(u)$ is the unique element in the orbit of u contained in F). For each Borel subset B of $G^{(0)}$, π is continuous and one-to-one on $B \cap F$ and hence $\pi(B \cap F)$ is Borel in $G^{(0)}/G$. Therefore the map $e : G^{(0)} \to G^{(0)}$ is Borel (for each Borel subset B of $G^{(0)}$, $e^{-1}(B) = [B \cap F] = \pi^{-1}(\pi(B \cap F))$ is Borel in $G^{(0)}$). Also for each compact subset K of $G^{(0)}$, $e(K)$ has a compact closure because $e(K) \subset F \cap [K]$.

Since the orbit space $G^{(0)}/G$ is proper the map

$$(r,d) : G \to R,\ (r,d)(x) = (r(x), d(x))$$

is open and R is closed in $G^{(0)} \times G^{(0)}$. Applying Lemma 1.1 [3] to the locally compact second countable spaces G and R and to the continuous open surjection

$(r,d): G \to R$, it follows that there is a *regular cross section* $\sigma_0 : R \to G$. This means that σ_0 is Borel, $(r,d)(\sigma_0(u,v)) = (u,v)$ for all $(u,v) \in R$, and $\sigma_0(K)$ is relatively compact in G for each compact subset K of R.

Let us define $\sigma : G^{(0)} \to G^F$ by $\sigma(u) = \sigma_0(e(u), u)$ for all u. It is easy to note that σ is a cross section for $d : G^F \to G^{(0)}$ and $\sigma(K)$ is relatively compact in G for all compact $K \subset G^{(0)}$.

Replacing σ by

$$v \to \sigma(e(v))^{-1} \sigma(v)$$

we may assume that $\sigma(e(v)) = e(v)$ for all v. Therefore there exist cross section σ satisfying the conditions from the beginning of this section.

Let us define $q : G \to G_F^F$ by

$$q(x) = \sigma(r(x)) x \sigma(d(x))^{-1}, \, x \in G.$$

Let $\nu = \left\{\nu^u : u \in G^{(0)}\right\}$ be a Haar system on G and let $\left(\{\beta_v^u\}, \{\mu^{\dot{u}}\}\right)$ be its decompositions over the principal groupoid . Let δ be the 1-cocycle associated to the decomposition.

Let us denote by $\mathcal{B}_\sigma(G)$ the linear span of the functions of the form

$$x \to g_1(r(x)) g(q(x)) g_2(d(x))$$

where g_1, g_2 are compactly supported Borel bounded function on $G^{(0)}$ and g is a Borel bounded function on G_F^F such that if S is the support of g, then the closure of $\sigma(K_1)^{-1} S \sigma(K_2)$ is compact in G for all compact subsets K_1, K_2 of $G^{(0)}$. $\mathcal{B}_\sigma(G)$ is a subspace of $\mathcal{B}_c(G)$, the space of compactly supported Borel bounded function on G.

If $f_1, \, f_2 \in \mathcal{B}_\sigma(G)$ are defined by

$$\begin{aligned} f_1(x) &= g_1(r(x)) g(q(x)) g_2(d(x)) \\ f_2(x) &= h_1(r(x)) h(q(x)) h_2(d(x)) \end{aligned}$$

then

$$f_1 * f_2(x) = g * h(q(x)) g_1(r(x)) h_2(d(x)) \int g_2(u) h_1(u) d\mu^{\pi(r(x))}(u)$$

$$f_1^*(x) = \overline{g_2(r(x))} \overline{g\left(q(x)^{-1}\right)} \overline{g_1(d(x))}$$

Thus $\mathcal{B}_\sigma(G)$ is closed under convolution and involution.

Let ω be the universal representation of $C^*(G,\nu)$ the usual C^*-algebra associated to a Haar system $\nu = \left\{\nu^u, \, u \in G^{(0)}\right\}$ (constructed as in [8]) . Since every cyclic representation of $C^*(G,\nu)$ is the integrated form of a representation of G, it follows that ω can be also regarded as a representation of $\mathcal{B}_c(G)$, the space of compactly supported Borel bounded function on G. Arlan Ramsay and Martin E. Walter have used the notation $M^*(G,\nu)$ for the operator norm closure of $\omega(\mathcal{B}_c(G))$. Since ω is an $*$-isomorphism on $C^*(G,\nu)$, we can regarded $C^*(G,\nu)$ as a subalgebra of $M^*(G,\nu)$.

Definition 3 *We denote by* $M_{\sigma}^{*}(G,\nu)$ *the operator norm closure of* $\omega(\mathcal{B}_{\sigma}(G))$.

In [2] we have proved the follow theorem

Theorem 4 *Let* G *be a locally compact second countable groupoid with proper orbit space. Let* $\{\nu_i^u, u \in G^{(0)}\}$, $i = 1,2$ *be two Haar systems on* G. *Let* F *be a Borel subset of* $G^{(0)}$ *containing only one element* $e(u)$ *in each orbit* $[u]$. *Let* $\sigma : G^{(0)} \to G^F$ *be a cross section for* $d : G^F \to G^{(0)}$ *with* $\sigma(e(v)) = e(v)$ *for all* $v \in G^{(0)}$ *and* $\sigma(K)$ *relatively compact in* G *for all compact sets* $K \subset G^{(0)}$. *Then the* C^**-algebras* $M_{\sigma}^{*}(G,\nu_1)$ *and* $M_{\sigma}^{*}(G,\nu_2)$ *are* $*$*-isomorphic.*

5 The independence of the C^*-algebra $M_{\sigma}(G,\nu)$ on the cross section σ

In order to prove that the C^*-algebras $M_{\sigma_i}^{*}(G,\nu)$ $i = 1,2$ associated with different cross sections are $*$-isomorphic, we need to emphasize the structure of the quasi invariant measures in term of decomposition of the Haar system over the associated principal groupoid.

Remark 5 *Let* G *be a locally compact second countable groupoid with proper orbit space. Let* $\{\nu^u, u \in G^{(0)}\}$ *be a Haar system on* G *and* $(\{\beta_v^u\}, \{\mu^{\dot{u}}\})$ *be its decomposition over the associated principal groupoid. If* μ *is a quasi-invariant probability measure for the Haar system, then* $\mu_1 = \int \mu^{\pi(u)} d\mu(u)$ *is a Radon measure which is equivalent to* μ. *Indeed, let* $f \geq 0$ *Borel on* $G^{(0)}$ *such that* $\mu(f) = 0$. *Since* μ *is quasi-invariant, it follows that for* μ *a.a.* u $\nu^u(f \circ d) = 0$, *and since* $\mu^{\pi(u)}$ *is equivalent to* $d_*(\nu^u)$, *it results* $\mu^{\pi(u)}(f) = 0$ *for* μ *a.a.* u. *Conversely, if* $\mu_1(f) = 0$, *then* $\mu^{\pi(u)}(f) = 0$ *for* μ *a.a.* u, *and therefore* $\nu^u(f \circ d) = 0$. *Thus the quasi-invariance of* μ *implies* $\mu(f) = 0$. *Thus each Radon quasi-invariant measure is equivalent to a Radon measure of the form* $\int \mu^{\dot{u}} d\tilde{\mu}(\dot{u})$, *where* $\tilde{\mu}$ *is a probability measure on the orbit space* $G/G^{(0)}$.

Theorem 6 *Let* G *be a locally compact second countable groupoid with proper orbit space. Let* $\{\nu^u, u \in G^{(0)}\}$ *be a Haar systems on* G. *Let* F_i, $i = 1,2$, *be two Borel subsets of* $G^{(0)}$ *containing only one element* $e_i(u)$ *in each orbit* $[u]$. *For each* $i = 1,2$, *let* $\sigma_i : G^{(0)} \to G^{F_i}$ *be a cross section for* $d_{F_i} : G^{F_i} \to G^{(0)}$, $d_{F_i}(x) = d(x)$, *satisfying the conditions*

1. $\sigma_i(e_i(v)) = e_i(v)$ *for all* $v \in G^{(0)}$
2. $\sigma_i(K)$ *is relatively compact in* G *for all compact sets* $K \subset G^{(0)}$.

Then the C^**-algebras* $M_{\sigma_1}^{*}(G,\nu)$ *and* $M_{\sigma_2}^{*}(G,\nu)$ *are* $*$*-isomorphic.*

Proof. Let $\left(\left\{\beta_v^u\right\},\left\{\mu^{\dot{u}}\right\}\right)$ be the decomposition of the Haar system over the principal groupoid . Let δ be the 1-cocycle associated to the decomposition.

For each i, let $q_i : G \to G_{F_i}^{F_i}$ be defined by

$$q_i\left(x\right)=\sigma_i\left(r\left(x\right)\right)x\sigma_i\left(d\left(x\right)\right)^{-1},\, x\in G.$$

We shall define a $*$-homomorphism Φ from $\mathcal{B}_{\sigma_1}\left(G\right)$ to $\mathcal{B}_{\sigma_2}\left(G\right)$. It suffices to define Φ on the set of function on G of the form

$$x\to g_1\left(r\left(x\right)\right)g\left(q_1\left(x\right)\right)g_2\left(d\left(x\right)\right)$$

Let us define Φ by

$$\Phi\left(f\right)=\left(x\to C\left(x\right)g_1\left(r\left(x\right)\right)g_2\left(d\left(x\right)\right)g\circ q_{12}\circ q_2\left(x\right)\right)$$

where f is defined by

$$f\left(x\right)=g_1\left(r\left(x\right)\right)g\left(q_1\left(x\right)\right)g_2\left(d\left(x\right)\right)$$

and

$$C\left(x\right)=\delta\left(\sigma_1\left(e_2\left(r\left(x\right)\right)\right)\right)^{-1}.$$

It is easy to see that $C\left(x\right)=C\left(y\right)$ if x and y belong to the same transitivity component of G.

If f_1 and f_2 are defined by

$$\begin{aligned}f_1\left(x\right) &= g_1\left(r\left(x\right)\right)g\left(q_1\left(x\right)\right)g_2\left(d\left(x\right)\right)\\ f_2\left(x\right) &= h_1\left(r\left(x\right)\right)h\left(q_1\left(x\right)\right)h_2\left(d\left(x\right)\right)\end{aligned}$$

then

$$f_1*f_2\left(x\right)=g*h\left(q_1\left(x\right)\right)g_1\left(r\left(x\right)\right)h_2\left(d\left(x\right)\right)\int g_2\left(u\right)h_1\left(u\right)d\mu^{\pi\left(r\left(x\right)\right)}\left(u\right)$$

and

$$\begin{aligned}&\Phi\left(f_1*f_2\right)\\ =\ &C\left(x\right)\left(g*h\right)\circ q_{12}\circ q_2\left(x\right)g_1\left(r\left(x\right)\right)h_2\left(d\left(x\right)\right)\int g_2\left(u\right)h_1\left(u\right)d\mu^{\pi\left(r\left(x\right)\right)}\left(u\right)\end{aligned}$$

Since

$$\begin{aligned}\Phi\left(f_1\right) &= C\left(x\right)g\circ q_{12}\circ q_2\left(x\right)g_1\left(r\left(x\right)\right)g_2\left(d\left(x\right)\right)\\ \Phi\left(f_2\right) &= C\left(x\right)h\circ q_{12}\circ q_2\left(x\right)h_1\left(r\left(x\right)\right)h_2\left(d\left(x\right)\right)\end{aligned}$$

it follows that

$$\begin{aligned}&\Phi\left(f_1\right)*\Phi\left(f_2\right)\\ =\ &C\left(x\right)^2 g\circ q_{12}*h\circ q_{12}\circ q_2\left(x\right)g_1\left(r\left(x\right)\right)h_2\left(d\left(x\right)\right)\cdot\\ &\cdot\int g_2\left(u\right)h_1\left(u\right)d\mu^{\pi\left(r\left(x\right)\right)}\left(u\right)\end{aligned}$$

On the other hand, for all $z \in G_{F_2}^{F_2}$

$$\begin{aligned} (g*h)\circ q_{12}(z) &= \int g(q_{12}(z)y)\, h\left(y^{-1}\right) d\beta_{e_1(r(z))}^{e_1(r(z))}(y) \\ &= \delta(\sigma_1(e_2(r(z))))^{-1} .A(z) \end{aligned}$$

where

$$\begin{aligned} & A(z) \\ =& \int g\left(q_{12}(z)\,\sigma_1(r(y))\, y\sigma_1(d(y))^{-1}\right) h\left(\sigma_1(d(y))\, y^{-1}\sigma_1(r(y))^{-1}\right) d\beta_{e_2(r(z))}^{e_2(r(z))}(y). \\ =& \int g(q_{12}(z)\, q_{12}(y))\, h\left(q_{12}(y)^{-1}\right) d\beta_{e_2(r(z))}^{e_2(r(z))}(y) \\ =& \int g(q_{12}(zy))\, h\left(q_{12}\left(y^{-1}\right)\right) d\beta_{e_2(r(z))}^{e_2(r(z))}(y) \\ =& (g\circ q_{12}(y) * h\circ q_{12})(z) \end{aligned}$$

Therefore $\Phi(f_1 * f_2) = \Phi(f_1) * \Phi(f_2)$.

Since

$$f_1^*(x) = \overline{g_2(r(x))}\,\overline{g\left(q(x)^{-1}\right)}\,\overline{g_1(d(x))}$$

it follows that

$$\begin{aligned} \Phi(f_1^*) &= C(x)\,\overline{g\left((q_{12}\circ q_2(x))^{-1}\right)}\,\overline{g_2(r(x))\, g_1(d(x))} \\ &= \Phi(f_1)^* \end{aligned}$$

Let $\left(L, \mathcal{H} * G^{(0)}, \mu\right)$ be a representation of G. We have seen in Remark 5 that each Radon quasi-invariant measure μ with respect to the Haar system ν is equivalent to a Radon measure of the form $\int \mu^{\dot{u}} d\tilde{\mu}(\dot{u})$, where $\tilde{\mu}$ is a probability measure on the orbit space $G/G^{(0)}$. We note that $\int \mu^{\dot{u}} d\tilde{\mu}(\dot{u})$ is equivalent to $\eta_1 = \int \rho_1 \cdot \mu^{\dot{u}} d\tilde{\mu}(\dot{u})$ which is equivalent to $\eta_2 = \int \rho_2 \cdot \mu^{\dot{u}} d\tilde{\mu}(\dot{u})$, where $\rho_1(w) = \delta(\sigma_1(u))^{-1}$ and $\rho_2(w) = \delta(\sigma_2(u))^{-1}$ for all $w \in G^{(0)}$.

Then $\left(L, \mathcal{H} * G^{(0)}, \mu\right)$,$\left(L, \mathcal{H} * G^{(0)}, \eta_1\right)$ and $\left(L, \mathcal{H} * G^{(0)}, \eta_2\right)$ are equivalent representations. Let L_1 be the integrated form of a representation $\left(L, \mathcal{H} * G^{(0)}, \eta_1\right)$ and let L_2 be the integrated form of a representation $\left(L, \mathcal{H} * G^{(0)}, \eta_2\right)$. Let B be the Borel function defined by:

$$B(u) = \delta(\sigma_2(u))^{1/2}\, \delta(\sigma_1(u))^{-1/2}\, L_2(\sigma_2(u))^{-1}\, L_1(\sigma_1(e_2(u)))^{-1}\, L_1(\sigma_1(u))$$

and $W : \int_{G^{(0)}}^{\oplus} \mathcal{H}(u)\, d\eta_1(u) \to \int_{G^{(0)}}^{\oplus} \mathcal{H}(u)\, d\eta_2(u)$ be defined by

$$W(\zeta) = (u \to B(u)(\zeta(u)))$$

For $\zeta_1, \zeta_2 \in \int_{G^{(0)}}^{\oplus} \mathcal{H}(u)\, d\eta_1(u)$ and f of the form

$$f(x) = g_1(r(x))\, g(q(x))\, g_2(d(x)),$$

we have

$$\langle L_1(f)\zeta_1, \zeta_2\rangle = \langle W^* L_2(\Phi(f))\, W\zeta_1, \zeta_2\rangle$$

Therefore $\|L_1(f)\| = \|L_2(\Phi(f))\|$. Consequently,$\|f\| = \|\Phi(f)\|$ (full C^*-norms). Thus we can extend Φ to a $*$-homomorphism between the $M^*_{\sigma_1}(G,\nu)$ and $M^*_{\sigma_2}(G,\nu)$. It is not hard to see that Φ is in fact a $*$-isomorphism:

$$\Phi^{-1}(f) = \left(x \to C(x)^{-1} g_1(r(x))\, g_2(d(x))\, g \circ q_{21} \circ q_1(x)\right)$$

for each f of the form

$$f(x) = g_1(r(x))\, g(q(x))\, g_2(d(x)).$$

where $q_{21}(x) = \sigma_1(e_2(r(x)))^{-1} x\sigma_1(e_2(d(x)))$ for all $x \in G_{F_2}^{F_2}$. ■

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References

[1] C. Anantharaman-Delaroche, J. Renault, *Amenable groupoids*, Monographie de L'Enseignement Mathematique No **36**, Geneve, 2000.

[2] M. Buneci, *Isomorphic groupoid C^*-algebras associated with different Haar systems*, submitted to New York J. Math.

[3] G. Mackey, *Induced representations of locally compact groups. I*, Ann. of Math., **55**(1952), 101-139.

[4] P. Muhly, *Coordinates in operator algebra*, (Book in preparation).

[5] A. Ramsay, *Virtual groups and groups actions*, Adv. in Math. **6**(1971), 253-322.

[6] A. Ramsay, *Topologies on measured groupoids*, J. Funct. Anal. **47**(1982), 314-343.

[7] A. Ramsay and M. E. Walter, *Fourier-Stieltjes Algebras of locally compact groupoids*, J. Funct. Anal. **148**(1997), 314-367.

[8] J. Renault, *A groupoid approach to C^*- algebras*, Lecture Notes in Math., Springer-Verlag, **793**, 1980.

[9] J. Renault,*The ideal structure of groupoid crossed product algebras*, J. Operator Theory, **25**(1991), 3-36.

[10] J. Westman, *Nontransitive groupoid algebras*, Univ. of California at Irvine, 1967.

University Constantin Brâncuşi of Târgu-Jiu
Bulevardul Republicii, Nr. 1,
210152 Târgu-Jiu
Romania
e-mail: ada@utgjiu.ro