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AMENABLE EQUIVARIANT MAPS AND SEMI-DIRECT PRODUCTS

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Abstract. We shall consider a measure groupoid G, two Borel left G-spaces, S and T, and a Borel G-equivariant map $\pi : T \to S$. We shall show that if π is amenable, then S is an amenable G-space. Conversely, we shall show that if S is an amenable G-space and if there is a G-equivariant map $\rho : T \to G$ (not necessarily amenable), then π is an amenable map.

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1. Introduction

The notion of amenability for groups, their actions on spaces, and more generally for semigroups and groupoids has been study for more than fifty years. It is now well known that the existence of an invariant mean on a locally compact group is equivalent to a great number of far-reaching properties in the harmonic analysis of the group: Folner property, the fixed-point property or the equality of the reduced and the full C^*-algebras associated to the group. This remarkable notion has found natural generalizations in other branches of mathematics. The notion of amenability for groupoids was introduced in [9] and was extensively studied in [1]. In fact in [1], the authors have defined a more general notion: the amenability of an equivariant map. This notion covers the amenability of a groupoid (using the range map $r : G \to G^{(0)}$) but also the amenability of a G-space S, as defined by Greenleaf [7] and Eymard [6] (using the map from S to a point).

The purpose of this paper is to obtain new characterization of the amenability of an equivariant map. The main results of this paper are Theorem 5.2 and Theorem 5.3.

2. Groupoids and semi-direct products

In order to establish notation and to give the reader a source of some essential information about groupoids, semi-direct products and amenable map, we include here a list of definitions that can be found in [9], [8] and [1]. We shall use the definition of the groupoid given by J. Renault in [9] (Definition I.2.1./p. 16). For a groupoid G, $G^{(0)}$ will denote its unit space and $G^{(2)}$ the set of the composable pairs. Usually, elements of G will be denoted by letters as x, y, or z, and the elements of $G^{(0)}$ by letters as u, v, or w. The inverse map is written $x \to x^{-1}$ $[: G \to G]$ and the product map is written $(x, y) \to xy$ $\left[: G^{(2)} \to G\right]$. The range and the source maps from G to $G^{(0)}$ will be denoted respectively by r and d. The fibers of the range and the source maps are denoted $G^u = r^{-1}(\{u\})$ and $G_v = d^{-1}(\{v\})$, respectively. More generally, given the subsets A, $B \subset G^{(0)}$, we define $G^A = r^{-1}(A)$, $G_B = d^{-1}(B)$ and $G_B^A = r^{-1}(A) \cap d^{-1}(B)$. The reduction of G is $G|A = G_A^A$.

A Borel groupoid is a groupoid G endowed with a Borel structure such that $G^{(2)}$ is a Borel set in the product structure $G \times G$, the product map and the inverse map are Borel functions. We are exclusively concerned with analytic Borel structure.

A Borel Haar system on a Borel groupoid G is a family of positive σ-finite measures on G, $\left\{\lambda^u,\, u \in G^{(0)}\right\}$, having the following properties:

1. For all $u \in G^{(0)}$, λ^u is concentrated on G^u.

2. For all $f \geq 0$ Borel on G

$$u \to \int f(x)\, d\lambda^u(x) \quad \left[: G^{(0)} \to \overline{\mathbf{R}}\right]$$

   is a real-extended Borel map.

3. For all $f \geq 0$ Borel on G and all $x \in G$,

$$\int f(y)\, d\lambda^{r(x)}(y) \; = \; \int f(xy)\, d\lambda^{d(x)}(y)$$

4. There is a positive Borel function f_0 on G such that $\lambda^u(f_0) = 1$ for all u.

The image of λ^u by the inverse map $x \to x^{-1}$ is denoted λ_u.

If μ is a finite measure on $G^{(0)}$, then the measure $\nu = \int \lambda^u d\mu(u)$, defined by

$$\int f(y)\, d\nu(y) = \int \left(\int f(y)\, d\lambda^u(y) \right) d\mu(u)\,, \text{ f} \geq 0 \text{ Borel}$$

is called the measure on G induced by μ. The image of ν by the inverse map $x \to x^{-1}$ is denoted ν^{-1}. The measure μ is said quasi-invariant if its induced measure ν is equivalent to its inverse ν^{-1}. If μ is a quasi-invariant measure on $G^{(0)}$ and λ is the measure induced on G, then the Radon-Nikodym derivative $\Delta = \frac{d\nu}{d\nu^{-1}}$ is called the modular function of μ.

A (left) Borel G-space is a Borel space S endowed with a Borel surjection $r : S \to G^{(0)}$ and a Borel map $(x, s) \to xs$ from the space

$$G * S = \{(x, s) : d(x) = r(s)\}$$

to S, satisfying the following conditions:

1. $r(xs) = r(x)$ for all $(x, s) \in G * S$ and $r(s)\, s = s$ for all $s \in S$.
2. if $(x_1,\, x_2) \in G^{(2)}$ and $(x_2,\, s) \in G * S$, then $(x_1 x_2)\, s = x_1 (x_2 s)$.

The space

$$S * G = \{(s, x) : r(s) = r(x)\}$$

has a groupoid structure (called semi-direct product) with the following operations

$$\begin{aligned} (s, x)\left(x^{-1}s, y\right) &= (s, xy) \\ (s, x)^{-1} &= \left(x^{-1}s, x^{-1}\right) \end{aligned}$$

Let us fix a Borel Haar system $\lambda = \left\{\lambda^u,\ u \in G^{(0)}\right\}$ on G. Let S be a Borel G-space. Then $\left\{\varepsilon_s \times \lambda^{r(s)},\ s \in S\right\}$ is a Borel Haar system for $S * G$, where ε_s is the unit point mass at s. A positive measure μ on S is called quasi-invariant with respect to λ if the measure $\mu \circ \lambda$ on $S * G$ defined by

$$\int f d\mu \circ \lambda = \int f\left(s, x\right) d\lambda^{r(s)}\left(x\right) d\mu\left(s\right)$$

for all Borel nonnegative function on $S * G$, is equivalent to its image under the inverse map $(s, x) \to (x^{-1}s, x^{-1})$.

3. Amenable Borel equivariant maps

Concerning the amenability, we shall use the following definitions and notation given in [1].

Let T, S be two Borel spaces and $\pi : T \to S$ a Borel map. A Borel π-system (or a Borel system of measures for π) is a family $\beta = \{\beta^s : s \in S\}$ of measures on $\pi^{-1}\left(\{s\}\right)$ such that:

1. For every nonnegative Borel function f on T, the function:

$$s \to \beta\left(f\right) = \int f\left(t\right) d\beta^s\left(t\right) \text{ is Borel.}$$

2. There is a positive Borel function g on T such that $\beta\left(g\right) = 1$.

The condition 2 is equivalent to:

($*$) Y is the union of an increasing sequence $\left(A_n\right)_n$ of Borel subsets such that $s \to \beta^s\left(A_n\right)$ is bounded for all n, and $\beta^s \neq 0$, for all $s \in S$. (Lemme 3/ p. 37 [5]).

Let β be such a π-system and c be a class of measures on S. We shall fix a positive measure μ belonging to c, and the space $L^\infty\left(S, c\right)$ will be identified to $L^\infty\left(S, \mu\right)$, which will be denoted by $L^\infty\left(S\right)$. We introduce the measure $\mu \circ \beta$ on T defined by:

$$\int f d\mu \circ \beta = \int \beta\left(f\right) d\mu$$

for every nonnegative Borel function $f : T \to \mathbf{R}$.

A Borel map π from T to S is said to be *equivariant* if $r(\pi(t)) = r(t)$ for all $t \in T$ and $\pi(xt) = x\pi(t)$ for all $(x,t) \in G * T$.

If $\pi : T \to S$ an equivariant Borel map, then a Borel π-system of measures $\beta = \{\beta^s : s \in S\}$ is said *invariant* (or G-invariant) if $x^{-1}\beta^s = \beta^{x^{-1}s}$ for every $(s,x) \in S * G$, where $x^{-1}\beta^s$ is defined by

$$\int f(t)\, d\left(x^{-1}\beta^s\right)(t) = \int f\left(x^{-1}t\right) d\beta^s(t).$$

Let $\beta = \{\beta^s : s \in S\}$ be a Borel G-invariant π-system of measures, $\lambda = \left\{\lambda^u : u \in G^{(0)}\right\}$ be a Borel Haar system on G, and μ be a quasi-invariant measure on S with repect to λ. The *amenability* of the G-equivariant map $\pi : T \to S$ (with respect to(λ, μ, β)) is equivalent to the existence of an invariant measurable π-system of means (Proposition 3.1.27/p.66, Proposition 3.2.5/p.70 [1]). An invariant measurable π-system of means is a family $m = \{m^s, \in S\}$ of states (or means) m^s on $L^\infty(T, \beta^s)$ such that for every $\phi \in L^\infty(T)$

1. $s \to m^s(\phi)$ is μ-measurable
2. $xm^{x^{-1}s}(\phi) = m^s(\phi)$ for $\mu \circ \lambda$-almost every $(s,x) \in S * G$, where $xm^{x^{-1}s}(\phi) = m^{x^{-1}s}(t \to \phi(xt))$.

In [1] the amenability of a measure groupoid (G, λ, μ) (a groupoid G endowed with a Borel Haar system λ and a quasi invariant measure μ) was defined as the amenability of the range map with respect to (λ, μ, λ) (Definition 3.2.8/p. 71 [1]).

4. Groupoid representations

In order to establish necessary and sufficient condition for the amenability of an equivariant map $\pi : T \to S$, we need the characterization of an amenable groupoid in terms of amenable representation. In order to establish notation and for the reader convenience we recall here the notion of groupoid representation.

Let $\mathcal{H} = \{\mathcal{H}(s)\}_{s\in S}$ be a family of Hilbert spaces indexed by a set S. Let us form the disjoint union

$$S * \mathcal{H} = \{(s,\xi) : \xi \in \mathcal{H}(s)\},$$

and let $p : S*\mathcal{H} \to S$ be the natural projection, $p(s,\xi) = s$. A pair $(S*\mathcal{H}, p)$ is called Hilbert bundle over S. For each $s \in S$, the space $\mathcal{H}(s)$, which can be identified with $p^{-1}(\{s\}) = \{s\} \times \mathcal{H}(s)$, is called the fibre over s. A section of the bundle is a function $f : S \to S*\mathcal{H}$ such that $p(f(s)) = s$ for all $s \in S$. Given a section f, we may write $f(s) = \left(s, \hat{f}(s)\right)$, for a uniquely determined element

$$\hat{f} \in \prod_{s\in S} \mathcal{H}(s) = \left\{\phi : S \to \bigcup_{s\in S} \mathcal{H}(s),\ \phi(s) \in \mathcal{H}(s) \text{ for all } s\right\};$$

and given an element $\hat{f} \in \prod_{s\in S} \mathcal{H}(s)$ we may define a section $f(s) = \left(s, \hat{f}(s)\right)$. Because of this link between sections of $S*\mathcal{H}$ and elements of $\prod_{s\in S} \mathcal{H}(s)$ we shall often abuse notation and write $f(s)$ instead of $\hat{f}(s)$. An *analytic Borel Hilbert bundle* is a Hilbert bundle $(S*\mathcal{H}, p)$ where $S*\mathcal{H}$ is endowed with an analytic Borel structure such that the following axioms are satisfied:

1. A subset E is Borel if and only if $p^{-1}(E)$ is Borel.
2. There is a sequence $\{f_n\}_n$ of sections, called a fundamental sequence, such that
   a) each function $\tilde{f}_n : S*\mathcal{H} \to \mathbf{C}$, defined by $\tilde{f}_n(s,\xi) = (f_n(s), \xi)_{\mathcal{H}(s)}$, is Borel.
   b) for each pair of fundamental sections, f_n and f_m, the function $s \to (f_n(s), f_m(s))_{\mathcal{H}(s)}$ is Borel.
   c) the functions $\left\{\tilde{f}_n\right\}_n$ and p separate the points of $S*\mathcal{H}$.

Let G be a groupoid and $\{\lambda^u,\ u \in G^{(0)}\}$ be a Haar system on G. Let $G^{(0)} * \mathcal{H}$ be a Hilbert bundle. We write $Iso\left(G^{(0)} * \mathcal{H}\right)$ for

$$\{(u,L,v) \mid\ L : \mathcal{H}(v) \to \mathcal{H}(u) \text{ is a Hilbert space isomorphism}\}$$

endowed with the weakest Borel structure so that the maps

$$(u, L, v) \to (Lf_n(v), f_m(u))$$

are Borel for every n and m, where $(f_n)_n$ is fundamental sequence for $G^{(0)} * \mathcal{H}$.

$Iso\left(G^{(0)} * \mathcal{H}\right)$ is a groupoid in the operations:

$$\begin{aligned}(u, L_1, v)(v, L_2, w) &= (u, L_1 L_2, w)\\ (u, L, v)^{-1} &= \left(v, L^{-1}, u\right)\end{aligned}$$

A unitary representation of G consists of a quasi-invariant measure μ, a Hilbert bundle $G^{(0)} * \mathcal{H}$, a conull subset U of $G^{(0)}$, and a Borel map

$$L : G|_U \to Iso\left(G^{(0)} * \mathcal{H}|_U\right)$$

where $G^{(0)} * \mathcal{H}|_U$ is the restriction of $G^{(0)} * \mathcal{H}$ to U, such that

1. $L(x) = \left(d(x), \hat{L}(x), r(x)\right)$ and $\hat{L}(x) : \mathcal{H}(d(x)) \to \mathcal{H}(r(x))$ is a Hilbert space isomorphism for $\mu \circ \lambda$-almost all $x \in G|_U$.
2. $\hat{L}(u) = I_u$, the identity operator on $\mathcal{H}(u)$, for μ-almost every $u \in U$.
3. $\hat{L}(x)\hat{L}(y) = \hat{L}(xy)$ for $\int (\lambda^u \times \lambda_u)\, d\mu(u)$-almost every $(x, y) \in G^{(2)}$.
4. $\hat{L}(x^{-1}) = \hat{L}(x)^{-1}$ for $\mu \circ \lambda$-almost all x.

If $(G^{(0)} * \mathcal{H}, L, \mu)$ is a representation of the groupoid G, we abuse notation and write $L(x)$ instead of $\hat{L}(x)$ $\left(L(x) = \left(d(x), \hat{L}(x), r(x)\right)\right)$. For any representation L there is a Borel homomorphism $L_0 : G \to Iso\left(G^{(0)} * \mathcal{H}\right)$ that preserves the unit space $G^{(0)}$ in the sense that

$$L_0(x) = \left(d(x), \hat{L}_0(x), r(x)\right),$$

where $\hat{L}_0(x) : \mathcal{H}(d(x)) \to \mathcal{H}(r(x))$ is a Hilbert space isomorphism, such that L_0 agrees with L almost everywhere with respect to $\mu \circ \lambda$

5. Amenability of an equivariant map and

the amenability of a semi-direct product

Notation 5.1. The space of the bounded operators on the Hilbert space H is denoted $\mathcal{B}(H)$.

Let $S * \mathcal{H}$ be an analytic Borel Hilbert bundle over S, and let $(f_n)_n$ be a fundamental sequence for this bundle.

Let μ be a measure on S. We write $L^\infty(S, \mathcal{H}, \mu)$ for the set of $A : S \to \bigcup_{s \in S} \mathcal{B}(\mathcal{H}(s))$ having the following properties:

1. $A(s) \in \mathcal{B}(\mathcal{H}(s))$ for μ-almost every $s \in S$.
2. $s \to \langle A(s) f_n(s), f_m(s) \rangle$ is μ-measurable for all $m, n \in \mathbf{N}$
3. $\| s \to \|A(s)\| \|_\infty < \infty$.

In order to prove the equivalence between the amenability of an equivariant map defined on a groupoid and the amenability of a semi-direct product we shall use the notion of amenable representation of a groupoid introduced in [3] (Definition 2/p. 130):

Definition 5.1. Let G be a Borel groupoid endowed with a Borel Haar system $\lambda = \left\{ \lambda^u,\, u \in G^{(0)} \right\}$. Let μ be a quasi invariant measure for the Haar system λ. A unitary representation of the groupoid G, $(G^{(0)} * \mathcal{H}, L, \mu)$, is said to be amenable if there exists an invariant family of states $\{M^u, u \in G^{(0)}\}$ on $L^\infty\left(G^{(0)}, \mathcal{H}, \mu\right)$, i.e.

1. $M^u \in \mathcal{B}(\mathcal{H}(u))^*$, $M^u \geq 0$, $M^u(I) = 1$(this means that M^u is a state on $\mathcal{H}(u)$) for μ-almost every $u \in G^{(0)}$.
2. $u \to M^u(A(u))$ is μ-measurable for all $A \in L^\infty\left(G^{(0)}, \mathcal{H}, \mu\right)$.
3. $M^{r(x)}(L(x) A L(x^{-1})) = M^{d(x)}(A)$, for $\mu \circ \lambda$-almost every $x \in G$ and for all $A \in \mathcal{B}(\mathcal{H}(d(x)))$.

This definition extends from groups to groupoids the notion of amenability for an arbitrary unitary representation, introduced by E.B. Bekka in [2].

In [3](Theorem 6/p. 132) we characterized the amenable groupoids by amenability of all their unitary representations:

Theorem 5.1. *Let G be a Borel groupoid endowed with a Borel Haar system $\lambda = \left\{\lambda^u,\ u \in G^{(0)}\right\}$. Let μ be a quasi invariant measure for the Haar system λ. Then the following conditions are equivalent:*

1. *(G, λ, μ) is amenable.*
2. *Every unitary representation $\left(G^{(0)} * \mathcal{H}, L, \mu\right)$ of G is amenable.*
3. *The left regular representation Reg_μ is amenable.*

We shall use this theorem for a semi-direct product $S * G$. If $\lambda = \left\{\lambda^u,\ u \in G^{(0)}\right\}$ is a Borel Haar system on G, then we endow $S * G$ with the Borel Haar system $\left\{\varepsilon_s \times \lambda^{r(s)},\ s \in S\right\}$. A unitary representation of $S * G$, $(S * \mathcal{H}, L, \mu)$, is amenable if there exists an invariant family of states $\{M^s, s \in S\}$ on $L^\infty(S, \mathcal{H}, \mu)$. This means that

1. $M^s \in \mathcal{B}(\mathcal{H}(s))^*$, $M^s \geq 0$, $M^s(I) = 1$ for μ-almost every $s \in S$.
2. $s \to M^s(A(s))$ is μ-measurable for all $A \in L^\infty(S, \mathcal{H}, \mu)$.
3. $M^s\left(L(s,x)\,AL(s,x)^{-1}\right) = M^{x^{-1}s}(A)$, $\mu \circ \lambda$-almost every $(s,x) \in S * G$ and for all $A \in \mathcal{B}(\mathcal{H}(x^{-1}s)))$.

Let us define an analog of Reg_μ for an equivariant map. Let $\pi : T \to S$ be a G-equivariant map (S is (left) Borel G-space and also G is a Borel G-space by left multiplication). Let $\beta = \{\beta^s : s \in S\}$ be a G-invariant π-system of measure and μ be a quasi-invariant measure for the Haar system on $S * G$. Let g_0 be a positive Borel function such that $\beta(g_0) = 1$. In order to obtain a Hilbert bundle $S * \mathcal{H}_T$ on S, let $\mathcal{H}_T(s) = L^2(T, \beta^s)$. The Borel structure on $S * \mathcal{H}_T$ is given by a sequence of sections $(f_n)_n$ defined as follows: let $(h_n)_n$ be a sequence of bounded , non-negative Borel functions on G that separate points and let us define $f_n : S \to S * \mathcal{H}_T$, by $f_n(s) = \left(s, \hat{f}_n(s)\right)$, where $\hat{f}_n(s)(x) = h_n(x)\, g_0^{-1/2}(x)\,,\ x \in \pi^{-1}(\{s\})$.

Let μ be a quasi-invariant measure for the Haar system on $S * G$. Let us define a representation $(\mu, S * \mathcal{H}_T, R_{\pi,\beta,\mu})$ where

$$\hat{R}_{\pi,\beta,\mu}(s,x) : L^2\left(T, \beta^{x^{-1}s}\right) \to L^2(T, \beta^s)$$

is defined by the formula

$$\left(\hat{R}_{\pi,\beta,\mu}\left(s,x\right)f\right)(t)=f\left(x^{-1}t\right)$$

Let us called this representation of $S*G$ *the unitary representation associated to π induced by the G-action* and denote it $R_{\pi,\beta,\mu}$.

Theorem 5.2. *Let G be a Borel groupoid endowed with a Borel Haar system, $\lambda=\left\{\lambda^u,\,u\in G^{(0)}\right\}$. Let T and S be two (left) Borel G-spaces and let $\pi: T\to S$ be a G-equivariant map. Let $\beta=\{\beta^s: s\in S\}$ be a G-invariant π-system of measure and μ be a quasi-invariant measure for the Haar system on $S*G$. If the above defined representation $(\mu, S*\mathcal{H}_T, R_{\pi,\beta,\mu})$ of $S*G$ is amenable, then $\pi: G\to S$ is an amenable map with respect to (λ,μ,β).*

Proof. For each $s\in S$ let τ^s_ϕ be the multiplication operator on $L^2(T,\beta^s)$ by $\phi\in L^\infty(T,\beta^s)$. Then we have $\tau^s_1=1$, $\tau^s_\phi\geq 0$ if $\phi\geq 0$, and

$$R_{\pi,\beta,\mu}\left(s,x\right)\tau_\phi^{x^{-1}s}R_{\pi,\beta,\mu}\left(s,x\right)^{-1}=\tau^s_{{}_x\phi},\text{ for all }\phi\in L^\infty\left(T,\beta^s\right).$$

For each $s\in S$ let us define $m^s(\phi)=M^s\left(\tau^s_\phi\right)$ for every $\phi\in L^\infty(T,\beta^s)$. We have

$$\begin{aligned} m^s\left({}_x\phi\right) &= M^s\left(\tau^s_{{}_x\phi}\right)=M^s\left(R_{\pi,\beta,\pi}\left(s,x\right)\tau_\phi^{x^{-1}s}R_{\pi,\beta,\pi}\left(s,x\right)^{-1}\right)\\ &= M^{x^{-1}s}\left(\tau_\phi^{x^{-1}s}\right)=m^{x^{-1}s}\left(\phi\right)\end{aligned}$$

for $\mu\circ\lambda_S$-almost every $(s,x)\in S*G$ and every $\phi\in L^\infty(T,\beta^{x^{-1}s})$. Thus $\{m^s, s\in S\}$ is an invariant measurable π-system of means with respect to (β,λ,μ).

Corollary 5.1. *Let G be a Borel groupoid endowed with a Borel Haar system $\lambda=\left\{\lambda^u,\,u\in G^{(0)}\right\}$. Let S be a (left) Borel G-space and let $\pi: T\to S$ be a G-equivariant map. Let $\beta=\{\beta^s: s\in S\}$ be a G-invariant π-system of measure and μ be a quasi-invariant measure for the Haar system $\lambda_S=\left\{\varepsilon_s\times\lambda^{r(s)},\ s\in S\right\}$ on $S*G$. If $(S*G,\lambda_S,\mu)$ is amenable, then π is amenable with respect to (λ,μ,β).*

Proof. According to Theorem 5.1, $(S*G,\lambda_S,\mu)$ is amenable if and only if any unitary representation $(S*\mathcal{H},L,\mu)$ of $S*G$ is amenable. Thus, if $S*G$ is amenable, then $(\mu, S*\mathcal{H}_T, R_{\pi,\beta,\mu})$ is an amenable representation of $S*G$.

Applying Theorem 5.2, it follows that $\pi : T \to S$ is an amenable map with respect to (λ, μ, β).

Theorem 5.3. *Let G be a Borel groupoid endowed with a Borel Haar system $\lambda = \left\{\lambda^u,\, u \in G^{(0)}\right\}$. Let T and S be two (left) Borel G-spaces and let $\pi : T \to S$ be a G-equivariant map. Let $\beta = \{\beta^s : s \in S\}$ be a G-invariant π-system of measure and μ be a quasi-invariant measure for the Haar system on $S * G$. If there is a G-equivariant map $\rho : T \to G$ and if π is an amenable map with respect to (λ, μ, β), then $(S * G, \lambda_S, \mu)$ is amenable.*

Proof. We shall prove that every unitary representation $(S * \mathcal{H}, L, \mu)$ of $S * G$ is amenable. Let $\{m^s, s \in S\}$ be an invariant measurable π-system of means with respect to (λ, μ, β) and let $(S * \mathcal{H}, L, \mu)$ be a unitary representation of $S * G$. Since $(S * \mathcal{H}, L, \mu)$ is a representation of $S * G$, there is a Borel homomorphism $L_0 : S * G \to Iso\,(S * \mathcal{H})$ such that L_0 agrees with L almost everywhere with respect to $\mu \circ \lambda_S$, where $\lambda_S = \left\{\varepsilon_s \times \lambda^{r(s)},\ s \in S\right\}$. We replace L by L_0. Let us choose $C \in S * \mathcal{B}(\mathcal{H})$ with $C(s) \geq 0$ and $\|C\,(s)\|_1 = 1$ for all $s \in S$. For each $s \in S$ and each $A \in \mathcal{B}(\mathcal{H}(s))$ let us define $\phi^s_A : T^s \to \mathbf{C}$ by

$$\begin{aligned}\phi^s_A\,(t) &= Tr\left(AL\,(\pi\,(t)\,,\rho\,(t))\,C\left(\rho\,(t)^{-1}\,\pi\,(t)\right)L\,(\pi\,(t)\,,\rho\,(t))^{-1}\right),\ \ t \in T^s\\ &= Tr\left(AL\,(s,\rho\,(t))\,C\left(\rho\,(t)^{-1}\,s\right)L\,(s,\rho\,(t))^{-1}\right),\ \ t \in T^s\end{aligned}$$

Then we have

$$\begin{aligned}\left|\phi^s_A\,(t)\right| &\leq \left\|AL\,(s,\rho\,(t))\,C\left(\rho\,(t)^{-1}\,s\right)L\,(s,\rho\,(t))^{-1}\right\|_1\\ &\leq \left\|L\,(s,\rho\,(t))^{-1}\,AL\,(s,\rho\,(t))\right\|\left\|C\left(\rho\,(t)^{-1}\,s\right)\right\|_1\\ &\leq \|A\|\,.\end{aligned}$$

Thus $\phi^s_A \in L^\infty(T, \beta^s)$. For each $s \in S$ let us define $M^s : \mathcal{B}(\mathcal{H}(s)) \to \mathbf{C}$, by

$$M^s\,(A) = m^s\,(\phi^s_A)\,, \quad A \in \mathcal{B}\,(\mathcal{H}\,(s))\,.$$

Let us verify the invariance of the family of states $\{M^s, s \in S\}$. By definition, we have

$$\begin{aligned}&M^{x^{-1}s}\,(A) =\\ =\ & m^{x^{-1}s}\left(t \to Tr\left(AL\left(x^{-1}s,\rho\,(t)\right)C\left(\rho\,(t)^{-1}\left(x^{-1}s\right)\right)L\left(x^{-1}s,\rho\,(t)\right)^{-1}\right)\right)\\ =\ & m^s\,(f)\end{aligned}$$

where, $f : T \to \mathbf{C}$, is defined by

$$
\begin{aligned}
f(t) = & \\
= \; & Tr\left(AL\left(x^{-1}s, \rho\left(x^{-1}t\right)\right) C\left(\rho\left(x^{-1}t\right)^{-1}\left(x^{-1}s\right)\right) L\left(x^{-1}s, \rho\left(x^{-1}t\right)\right)^{-1}\right) \\
= \; & Tr\left(AL\left(x^{-1}s, x^{-1}\rho(t)\right) C\left(\left(\rho(t)^{-1}x\right)\left(x^{-1}s\right)\right) L\left(x^{-1}s, x^{-1}\rho(t)\right)^{-1}\right) \\
= \; & Tr\left(AL\left(x^{-1}s, x^{-1}\right) L(s,\rho(t)) C\left(\rho(t)^{-1}s\right) L(s,\rho(t))^{-1} L\left(x^{-1}s, x^{-1}\right)^{-1}\right) \\
= \; & Tr\left(L\left(x^{-1}s, x^{-1}\right)^{-1} AL\left(x^{-1}s, x^{-1}\right) L(s,\rho(t)) C\left(\rho(t)^{-1}s\right) L(s,\rho(t))^{-1}\right) \\
= \; & Tr\left(L(s,x) AL(s,x)^{-1} L(s,\rho(t)) C\left(\rho(t)^{-1}s\right) L(s,\rho(t))^{-1}\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
M^{x^{-1}s}(A) &= m^s(f) \\
&= m^s\left(\phi^s_{L(s,x)AL(s,x)^{-1}}\right) \\
&= M^s\left(L(s,x)\, AL(s,x)^{-1}\right)
\end{aligned}
$$

for $\mu \circ \lambda_S$-almost every $(s,x) \in S * G$ and $A \in \mathcal{B}\left(\mathcal{H}\left(x^{-1}s\right)\right)$.

Thus every unitary representation $(S*\mathcal{H}, L, \mu)$ of $(S * G, \lambda_S)$ is amenable, and consequently, $(S * G, \lambda_S, \mu)$ is amenable (according Theorem 5.1).

Remark 5.1. Let us apply the preceding theorem in the particular case when $T = G$. More precisely, let G be a Borel groupoid endowed with a Borel Haar system $\lambda = \left\{\lambda^u,\ u \in G^{(0)}\right\}$. Let S be a (left) Borel G-space and let $\pi : G \to S$ be a G-equivariant map (G is a Borel G-space by left multiplication). Let $\beta = \{\beta^s : s \in S\}$ be a G-invariant π-system of measure and μ be a quasi-invariant measure for the Haar system $\lambda_S = \left\{\varepsilon_s \times \lambda^{r(s)},\ s \in S\right\}$ on $S * G$. Obviously, there is an equivariant map $\rho : G \to G$ (we may take, for example, $\rho(x) = x$ for all x). Then, according to the preceding theorem, π is amenable with respect to (λ, μ, β) if and only if $(S * G, \lambda_S, \mu)$ is amenable. Thus we recover a result in [4].

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