Morphisms of locally compact groupoids endowed with Haar systems

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Abstract

We shall generalize the notion of groupoid morphism given by Zakrzewski ([19], [20]) to the setting of locally compact σ-compact Hausdorff groupoids endowed with Haar systems. To each groupoid Γ endowed with a Haar system λ we shall associate a C^*-algebra $C^*(\Gamma, \lambda)$, and we construct a covariant functor $(\Gamma, \lambda) \to C^*(\Gamma, \lambda)$ from the category of locally compact, σ-compact, Hausdorff groupoids endowed with Haar systems to the category of C^*-algebras (in the sense of [18]). If Γ is second countable and measurewise amenable, then $C^*(\Gamma, \lambda)$ coincides with the full and the reduced C^*-algebras associated to Γ and λ.

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1 Introduction

The purpose of this paper is extend the notion of morphism of groupoids introduced in [19, 20] to locally compact σ-compact groupoids endowed with Haar systems and to use the extension to construct a covariant functor from this category to the category of C^*-algebras.

If Γ and G are two locally compact, σ-compact, Hausdorff groupoids and if $\lambda = \{\lambda^u, u \in \Gamma^{(0)}\}$ (respectively, $\nu = \{\nu^t, t \in G^{(0)}\}$) is a Haar system on Γ (respectively, on G), then by a morphism from (Γ, λ) to (G, ν) we shall mean a left action of Γ on G, which commutes with multiplication on G, and which satisfies a "compatibility" condition with respect to the Haar systems on Γ and G. This notion of morphism reduces to a group homomorphism if Γ and G are groups, and to a map in the reverse direction if Γ and G are sets.

To each groupoid Γ endowed with a Haar system λ we shall associate a C^*-algebra $C^*(\Gamma, \lambda)$ in the following way. We shall consider the space $C_c(\Gamma)$ of complex-valuated continuous functions with compact support on Γ, which is a topological $*$-algebra under the usual convolution and involution. To each morphism h from (Γ, λ) to a groupoid (G, ν) and each unit $t \in G^{(0)}$ we shall associate a $*$-representation $\pi_{h,t}$ of $C_c(\Gamma)$. For any morphism h from (Γ, λ) to a groupoid (G, ν) we shall define

$$\|f\|_h = \sup_t \|\pi_{h,t}(f)\|$$

for all $f \in C_c(\Gamma)$. The C^*-algebra $C^*(\Gamma,\lambda)$ will be defined to be the completion of $C_c(\Gamma,\lambda)$ in the norm $\|\cdot\| = sup_h \|\cdot\|_h$, where h runs over all morphism defined on (Γ,λ). We shall show the following inequalities

$$\|f\|_{red} \leq \|f\| \leq \|f\|_{full}$$

for all $f \in C_c(\Gamma,\lambda)$, where $\|\cdot\|_{red}$ and $\|\cdot\|_{full}$ are the usual reduced and full C^*-norms on $C_c(\Gamma,\lambda)$. Therefore, according to prop. 6.1.8/p.146 [1], if (Γ,λ) is measurewise, amenable then $C^*(\Gamma,\lambda) = C^*_{full}(\Gamma,\lambda) = C^*_{red}(\Gamma,\lambda)$, where $C^*_{full}(\Gamma,\lambda)$(respectively, $C^*_{red}(\Gamma,\lambda)$) is the full (respectively, the reduced) C^*-algebra associated to Γ and λ. If the principal associated groupoid of Γ is a proper groupoid (but Γ is not necessarily measurewise amenable), then for any quasi invariant measure μ on $\Gamma^{(0)}$ and any $f \in C_c(\Gamma,\lambda)$ we shall prove that

$$\|II_\mu(f)\| \leq \|f\|,$$

where II_μ is the trivial representation on μ of $C_c(\Gamma,\lambda)$.

Now we are going to establish notation. Relevant definitions can be found in several places (e.g. [12], [7]). For a groupoid Γ the set of *composable pairs* will be denoted by $\Gamma^{(2)}$ and the *multiplication map* by $\Gamma^{(2)} \ni (x,y) \mapsto xy \in \Gamma$; the *inverse map* by $\Gamma \ni x \mapsto x^{-1} \in \Gamma$. The set of *units* by $\Gamma^{(0)}$ and the *domain* and *range* maps by $d : \Gamma \ni x \mapsto d(x) := x^{-1}x \in \Gamma^{(0)}$, and $r : \Gamma \ni x \mapsto d(x) := xx^{-1} \in \Gamma^{(0)}$ respectively.

The fibres of the range and the source maps are denoted $\Gamma^u = r^{-1}(\{u\})$ and $\Gamma_v = d^{-1}(\{v\})$, respectively. More generally, given the subsets A, $B \subset \Gamma^{(0)}$, we define $\Gamma^A = r^{-1}(A)$, $\Gamma_B = d^{-1}(B)$ and $\Gamma^A_B = r^{-1}(A) \cap d^{-1}(B)$. The *reduction* of Γ to $A \subset \Gamma^{(0)}$ is $\Gamma|_A = \Gamma^A_A$. The relation $u \sim v$ iff $\Gamma^u_v \neq \phi$ is an equivalence relation on $\Gamma^{(0)}$. Its equivalence classes are called *orbits*, the orbit of a unit u is denoted $[u]$ and the quotient space for this equivalence relation is called the *orbit space* of G and denoted $G^{(0)}/G$. A groupoid is called *transitive* iff it has a single orbit, or equivalently, if the map

$$\theta : \Gamma \to \{(r(x), d(x)),\ x \in \Gamma\},\ \theta(x) = (r(x), d(x))$$

is surjective. A groupoid Γ is called *principal*, if the above map θ is injective. A subset of $G^{(0)}$ is said *saturated* if it contains the orbits of its elements. For any subset A of $\Gamma^{(0)}$, we denote by $[A]$ the union of the orbits $[u]$ for all $u \in A$.

A *topological groupoid* consists of a groupoid Γ and a topology compatible with the groupoid structure. This means that:

(1) $x \to x^{-1}$ $[: \Gamma \to \Gamma]$ is continuous.

(2) (x,y) $\left[: \Gamma^{(2)} \to \Gamma\right]$ is continuous where $\Gamma^{(2)}$ has the induced topology from $\Gamma \times \Gamma$.

We are exclusively concerned with topological groupoids which are locally compact Hausdorff.

In the following we will use abbreviation *lcH*-groupoids.

If Γ is Hausdorff, then $\Gamma^{(0)}$ is closed in Γ, and $\Gamma^{(2)}$ closed in $\Gamma \times \Gamma$. It was shown in [10] that measured groupoids (in the sense of def. 2.3./p.6 [4]) may be assume to have locally compact topologies, with no loss of generality.

A subset A of a locally compact groupoid Γ is called *r-(relatively) compact* iff $A \cap r^{-1}(K)$ is (relatively) compact for each compact subset K of $\Gamma^{(0)}$. Similarly, one may define *d-(relatively) compact* subsets of Γ. A subset of Γ which is r-(relatively) compact and d-(relatively) compact is said *conditionally-(relatively) compact*. If the unit space $\Gamma^{(0)}$ is paracompact, then there exists a fundamental system of conditionally-(relatively) compact neighborhoods of $\Gamma^{(0)}$ (see the proof of prop. II.1.9/p.56 [12]).

If X is a locally compact space, $C_c(X)$ denotes the space of complex-valued continuous functions with compact support. The Borel sets of a topological space are taken to be the σ-algebra generated by open sets.

If Γ is a locally compact groupoid, then for each $u \in \Gamma^{(0)}$, $\Gamma_u^u = \Gamma|_{\{u\}}$ is a locally compact group. We denote by

$$\Gamma' = \{x \in \Gamma : r(x) = d(x)\} = \bigcup_{u \in \Gamma^{(0)}} \Gamma_u^u$$

the *isotropy group bundle* of Γ. It is closed in Γ.

Recall that a (continuous) *Haar system* on a lcH-groupoid Γ is a family of positive Radon measures on Γ, $\lambda = \left\{\lambda^u,\, u \in \Gamma^{(0)}\right\}$, such that

1. For all $u \in \Gamma^{(0)}$, $supp(\lambda^u) = \Gamma^u$.
2. For all $f \in C_c(\Gamma)$,

$$u \to \int f(x)\, d\lambda^u(x) \quad \left[: \Gamma^{(0)} \to \mathbf{C}\right]$$

   is continuous
3. For all $f \in C_c(\Gamma)$ and all $x \in \Gamma$,

$$\int f(y)\, d\lambda^{r(x)}(y) \;=\; \int f(xy)\, d\lambda^{d(x)}(y)\,.$$

Unlike the case of locally compact group, Haar system on groupoid need not exists, and if it does, it will not usually be unique. The continuity assumption 2) has topological consequences for Γ. It entails that the range map $r : \Gamma \to \Gamma^{(0)}$, and hence the domain map $d : \Gamma \to \Gamma^{(0)}$ is open (prop. I.4 [17]). *Therefore, in this paper we shall always assume that $r : \Gamma \to \Gamma^{(0)}$ is an open map.* For each λ^u, we denote by λ_u the image of λ^u by the inverse map $x \to x^{-1}$ (i.e. $\int f(y)\, d\lambda_u(y) = \int f\left(y^{-1}\right) d\lambda^u(y), \quad f \in C_c(\Gamma)$) .

If μ is a Radon measure on $\Gamma^{(0)}$, then the measure $\lambda^\mu = \int \lambda^u d\mu(u)$, defined by

$$\int f(y)\, d\lambda^\mu(y) = \int \left(\int f(y)\, d\lambda^u(y)\right) d\mu(u)\,, \quad f \in C_c(\Gamma)$$

is called the *measure on Γ induced by μ*. The image of λ^μ by the inverse map $x \to x^{-1}$ is denoted $(\lambda^\mu)^{-1}$. μ is said *quasi-invariant* if its induced measure λ^μ

is equivalent to its inverse $(\lambda^\mu)^{-1}$. A measure belonging to the class of a quasi-invariant measure is also quasi-invariant. We say that the class is *invariant*.

If μ is a quasi-invariant measure on $\Gamma^{(0)}$ and λ^μ is the measure induced on G, then the Radon-Nikodym derivative $\Delta = \frac{d\lambda^\mu}{d(\lambda^\mu)^{-1}}$ is called the *modular function* of μ. According to cor. 3.14/p.19 [4], there is a μ-conull Borel subset U_0 of $\Gamma^{(0)}$ such that the restriction of Δ to $\Gamma|_{U_0}$ is a homomorphism.

A family of positive Borel measures on Γ, $\lambda = \left\{\lambda^u,\, u \in \Gamma^{(0)}\right\}$ is called a *Borel Haar system* if in the definition of the (continuous) Haar system we replace the condition 2 by

$2'$ For all $f \geq 0$ Borel on Γ, the map

$$u \to \int f(x)\, d\lambda^u(x) \quad \left[: \Gamma^{(0)} \to \mathbf{R}\right]$$

is Borel, and there is a nonnegative Borel function F on Γ such that

$$\int F(x)\, d\lambda^u(x) = 1 \text{ for all } u.$$

For a C^*-algebra A let $\mathcal{B}(A)$ be the algebra of bounded linear map acting on A. We say that $a \in \mathcal{B}(A)$ *is adjointable* iff there is an element $b \in \mathcal{B}(A)$ such that $y^*(a(x)) = (b(y))^* x$ for all $x, y \in A$. The set of adjointable $a \in \mathcal{B}(A)$ is the *multiplier algebra* of A, and will be denoted by $M(A)$. A *morphism* from a C^*-algebra A to a C^*-algebra B, is a $*$-homomorphism $\phi : A \to M(B)$ such that the set $\phi(A)B$ is dense in B. Such morphism extends uniquely to a $*$-homomorphism $\hat{\phi}$ from $M(A)$ to $M(B)$ by $\hat{\phi}(m)(\phi(a)b) = \phi(m(a))b$ for $m \in M(A)$, $a \in A$ and $b \in B$. If ϕ_1 is a morphism from A to B, and ϕ_2 is a morphism from B to C, the composition is defined by $\hat{\phi}_2\phi_1 : A \to M(C)$. C^* algebras with above defined morphisms form a C^*-category (see [18]).

2 The category of locally compact groupoids endowed with Haar systems

2.1 Definition of morphism

Let Γ and G be two σ-compact, *lcH*-groupoids. Let $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ (respectively, $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$) be a Haar system on Γ (respectively, on G).

By a morphism from (Γ, λ) to (G, ν) we mean a left action of Γ on G, which commutes with multiplication on G, and which satisfies a "compatibility" condition with respect to the Haar systems on Γ and G.

Definition 1 *Let Γ be a groupoid and X be a set. We say Γ acts (to the left) on X if there is a map $\rho : X \to \Gamma^{(0)}$ (called a momentum map) and a map $(\gamma, x) \to \gamma \cdot x$ from*

$$\Gamma *_\rho X = \{(\gamma, x) : d(\gamma) = \rho(x)\}$$

to X, called (left) action, such that:

1. *$\rho(\gamma \cdot x) = r(\gamma)$ for all $(\gamma, x) \in \Gamma *_\rho X$.*
2. *$\rho(x) \cdot x = x$ for all $x \in X$.*
3. *If $(\gamma_2, \gamma_1) \in \Gamma^{(2)}$ and $(\gamma_1, x) \in \Gamma *_\rho X$, then $(\gamma_2\gamma_1) \cdot x = \gamma_2 \cdot (\gamma_1 \cdot x)$.*

If Γ is a topological groupoid and X is a topological space, then we say that a left action is continuous if the mappings ρ and $(\gamma, x) \to \gamma \cdot x$ are continuous, where $\Gamma *_\rho X$ is endowed with the relative product topology coming from $\Gamma \times X$.

The difference with the definition of action in [7] (def. 2.12/p. 32 and rem. 2.30/p.45) or in [8] is that we do not assume that the momentum map is surjective and open.

The action is called *free* if $(\gamma, x) \in \Gamma *_\rho X$ and $\gamma \cdot x = x$ implies $\gamma \in \Gamma^{(0)}$.

The continuous action is called *proper* if the map $(\gamma, x) \to (\gamma \cdot x, x)$ from $\Gamma *_\rho X$ to $X \times X$ is proper (i.e. the inverse image of each compact subset of $X \times X$ is a compact subset of $\Gamma *_\rho X$).

In the same manner, we define a *right action* of Γ on X, using a continuous map $\sigma : X \to \Gamma^{(0)}$ and a map $(x, \gamma) \to x \cdot \gamma$ from

$$X *_\sigma \Gamma = \{(x, \gamma) : \sigma(x) = r(\gamma)\}$$

to X.

The simplest example of proper and free action is the case when the groupoid Γ acts upon itself by either right or left translation (multiplication).

Definition 2 *Let Γ_1, Γ_2 be two groupoids and X be set. Let us assume that Γ_1 acts to the left on X with momentum map $\rho : X \to \Gamma_1^{(0)}$, and that Γ_2 acts to the right on X with momentum map $\sigma : X \to \Gamma_2^{(0)}$. We say that the action commute if*

1. *$\rho(x \cdot \gamma_2) = \rho(x)$ for all $(x, \gamma_2) \in X *_\sigma \Gamma_2$ and $\sigma(\gamma_1 \cdot x) = \sigma(x)$ for all $(\gamma_1, x) \in \Gamma_1 *_\rho X$.*
2. *$\gamma_1 \cdot (x \cdot \gamma_2) = (\gamma_1 \cdot x) \cdot \gamma_2$ for all $(\gamma_1, x) \in \Gamma_1 *_\rho X$, $(x, \gamma_2) \in X *_\sigma \Gamma_2$.*

Definition 3 *Let Γ and G be two groupoids. By an algebraic morphism from Γ to G we mean a left action of Γ on G which commutes with the multiplication on G.*

The morphism is said continuous if the action of Γ on G is continuous (assuming that Γ and G are topological spaces).

Let us note that if we have a morphism in the sense of the preceding definition and if $\rho : G \to \Gamma$ is the momentum map of the left action, then $\rho = \rho \circ r$. Indeed, for any $x \in G$, we have $\rho(x) = \rho\left(xx^{-1}\right) = \rho(r(x))$ because of the fact that left action of Γ on G commutes with the multiplication on G.

Therefore an algebraic morphism $h : \Gamma \longrightarrow\!\!\!\!\triangleright\, G$ is given by two maps

1. $\rho_h : G^{(0)} \to \Gamma^{(0)}$

2. $(\gamma, x) \to \gamma \cdot_h x$ from $\Gamma \star_h G$ to G, where

$$\Gamma \star_h G = \{(\gamma, x) \in \Gamma \times G : d(\gamma) = \rho_h(r(x))\}$$

satisfying the following conditions:

(1) $\rho_h(r(\gamma \cdot_h x)) = r(\gamma)$ for all $(\gamma, x) \in \Gamma \star_h G$.

(2) $\rho_h(r(x)) \cdot_h x = x$ for all $x \in G$.

(3) $(\gamma_1\gamma_2) \cdot_h x = \gamma_1 \cdot_h (\gamma_2 \cdot_h x)$ for all $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$ and all $(\gamma_2, x) \in \Gamma \star_h G$.

(4) $d(\gamma \cdot_h x) = d(x)$ for all $(\gamma, x) \in \Gamma \star_h G$.

(5) $(\gamma \cdot_h x_1) x_2 = \gamma \cdot_h (x_1 x_2)$ for all $(\gamma, x_1) \in \Gamma \star_h G$ and $(x_1, x_2) \in G^{(2)}$.

In the case continuous morphism the map ρ_h is a continuous map. The map ρ_h is not necessarily open or surjective. However, the image of ρ_h is always a saturated subset of $\Gamma^{(0)}$. Indeed, let $v \sim u = \rho_h(t)$ and let $\gamma \in \Gamma$ be such that $r(\gamma) = v$ and $d(\gamma) = u$. Then v belongs to the image of ρ_h because $v = r(\gamma) = \rho_h(r(\gamma \cdot_h t))$.

Remark 4 *Let h be an algebraic morphism from Γ to G (in the sense of Def. 3). Then h is determined by ρ_h and the restriction of the action to*

$$\left\{(\gamma, t) \in \Gamma \times G^{(0)} : d(\gamma) = \rho_h(t)\right\} .$$

Indeed, using the condition 5, one obtains

$$\gamma \cdot_h x = (\gamma \cdot_h r(x))\, x$$

Let us also note that

$$\begin{aligned}(\gamma_1\gamma_2) \cdot_h x &= ((\gamma_1\gamma_2) \cdot_h r(x))\, x = \gamma_1 \cdot_h (\gamma_2 \cdot_h r(x))\, x \\ &= (\gamma_1 \cdot_h r(\gamma_2 \cdot_h r(x)))\, (\gamma_2 \cdot_h r(x))\, x.\end{aligned}$$

Consequently, for any $\gamma \in \Gamma$ and any $t \in G^{(0)}$ with $\rho_h(t) = d(\gamma)$, we have

$$\left(\gamma^{-1} \cdot_h r(\gamma \cdot_h t)\right)(\gamma \cdot_h t) = \left(\gamma^{-1}\gamma\right) \cdot_h t = d(\gamma) \cdot_h t = \rho_h(t) \cdot_h t = t.$$

Thus for any $\gamma \in \Gamma$ and any $t \in G^{(0)}$ with $\rho_h(t) = d(\gamma)$,

$$(\gamma \cdot_h t)^{-1} = \gamma^{-1} \cdot_h r(\gamma \cdot_h t) .$$

Therefore, algebraically, the notion of morphism in the sense of def. 3 is the same with that introduced in [19] (p. 351). In order to prove the equivalence of these definitions, we can use Prop. 2.7/p. 5[15], taking $f = \rho_h$ and $g(\gamma, t) = \gamma \cdot_h t$.

Remark 5 *Let* $h : \Gamma \longrightarrow\!\triangleright G$ *be a continuous morphism of* lcH-*groupoids (in the sense of Def. 3). Then* G *is left* Γ*-space under the action* $(\gamma, x) \to \gamma \cdot_h x$*, and a right* G*-space under the multiplication on* G*.* G *is a correspondence in the sense of Def. 2/p. 234[16] if and only if the left action of* Γ *on* G *is proper and* ρ_h *is open and injective.* G *is a regular bibundle in the sense of Def. 6/p.103 [6] if and only if the action of* Γ *is free and transitive along the fibres of* d *(this means that for all* $u \in G^{(0)}$ *and* x *satisfying* $d(x) = x$*, there is* $\gamma \in \Gamma$ *such that* $\gamma \cdot_h u = x$*). Therefore, the notion of morphism introduced in Def. 3 is not cover by the notions used in [16] and [6].*

Lemma 6 *Let* Γ *and* G *be two groupoids and let* h *be an algebraic morphism from* Γ *to* G *(in the sense of Definition 3). Then the function*

$$(\gamma, t) \to \gamma \cdot_{h_0} t := r(\gamma \cdot_h t)$$

from $\left\{(\gamma, t) \in \Gamma \times G^{(0)} : d(\gamma) = \rho_h(t)\right\}$ *to* $G^{(0)}$ *defines an action of* Γ *to* $G^{(0)}$ *with the momentum map* ρ_h*.*

Proof. Let $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$ and $(\gamma_1, x) \in \left\{(\gamma, t) \in \Gamma \times G^{(0)} : d(\gamma) = \rho_h(t)\right\}$. Using the computation in the preceding remark, we obtain

$$(\gamma_1\gamma_2) \cdot_{h_0} x = r((\gamma_1\gamma_2) \cdot_h x) = r(((\gamma_1\gamma_2) \cdot_h r(x))\, x)$$

$$= r(((\gamma_1\gamma_2) \cdot_h r(x))) = r(\gamma_1 \cdot_h (\gamma_2 \cdot_h r(x)))$$
$$= \gamma_1 \cdot_{h_0} (\gamma_2 \cdot_{h_0} r(x)).$$

■

Notation 7 *Let* Γ *and* G *be two groupoids and let* h *be an algebraic morphism from* Γ *to* G *(in the sense of def. 3). Let us denote*

$$\begin{aligned} G \rtimes_h \Gamma &= \{(x, \gamma) \in G \times \Gamma : \rho_h(r(x)) = r(\gamma)\} \\ G^{(0)} \rtimes_{h_0} \Gamma &= \left\{(t, \gamma) \in G^{(0)} \times \Gamma : \rho_h(t) = r(\gamma)\right\} \end{aligned}$$

$G \rtimes_h \Gamma$*, respectively* $G^{(0)} \rtimes_{h_0} \Gamma$*, can be viewed as groupoid under the operations*

$$(x, \gamma)^{-1} = \left(\gamma^{-1} \cdot_h x, \gamma^{-1}\right)$$

$$(x, \gamma_1)\left(\gamma_1^{-1} \cdot_h x, \gamma_2\right) = (x, \gamma_1\gamma_2)$$

respectively,

$$(t, \gamma)^{-1} = \left(\gamma^{-1} \cdot_{h_0} t, \gamma^{-1}\right)$$

$$(t, \gamma_1)\left(\gamma_1^{-1} \cdot_{h_0} t, \gamma_2\right) = (t, \gamma_1\gamma_2).$$

(where $\gamma \cdot_{h_0} t := r(\gamma \cdot_h t)$ *as in preceding lemma).*

If Γ and G are lcH-groupoids, then $G \rtimes_h \Gamma$ and $G^{(0)} \rtimes_{h_0} \Gamma$ are lcH-groupoids. If $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ is a Haar system on Γ and if the morphism h is continuous, then $\left\{\varepsilon_x \times \lambda^{\rho_h(r(x))}, x \in G\right\}$ is a Haar system on $G \rtimes_h \Gamma$ (where ε_x is the unit point mass at x) and $\left\{\varepsilon_t \times \lambda^{\rho_h(t)}, t \in G^{(0)}\right\}$ is a Haar system on $G^{(0)} \rtimes_{h_0} \Gamma$.

Definition 8 *Let* Γ *and* G *be two* σ*-compact,* lcH*-groupoids. Let* $\lambda = \{\lambda^u, u \in \Gamma^{(0)}\}$ *(respectively,* $\nu = \{\nu^t, t \in G^{(0)}\}$*) be a Haar system on* Γ *(respectively, on* G*). By a morphism* $h : (\Gamma, \lambda) \longrightarrow\!\!\triangleright\, (G, \nu)$ *we mean a continuous morphism from* Γ *to* G *(in the sense of def.3) which satisfies in addition the following condition:*

(6) *There exists a continuous positive function* Δ_h *on*

$$G \rtimes_h \Gamma = \{(x, \gamma) \in G \times \Gamma : \rho_h(r(x)) = r(\gamma)\}$$

such that

$$\int\int g(\gamma^{-1} \cdot_h x, \gamma^{-1})\Delta_h(\gamma^{-1} \cdot_h x, \gamma^{-1}) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x)$$

$$= \int\int g(\gamma, x) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x)$$

for all $t \in G^{(0)}$ *and all Borel nonnegative functions* g *on* $G \rtimes_h \Gamma$.

Remark 9 *The condition 8 in the preceding definition means that each measure* ν_t *is quasi-invariant with respect to the Haar system* $\{\varepsilon_x \times \lambda^{\rho_h(r(x))}, x \in G\}$ *on* $G \rtimes_h \Gamma$.

Example 10 *Let* Γ *be a* σ*-compact, lcH-groupoid, endowed with a Haar system* $\lambda = \{\lambda^u, u \in \Gamma^{(0)}\}$*. Let us define a morphism* $l : (\Gamma, \lambda) \longrightarrow\!\!\triangleright\, (\Gamma, \lambda)$ *by* $\rho_l = id_{\Gamma^{(0)}}$ *and* $\gamma \cdot_l x = \gamma x$ *(multiplication on* Γ*). It is easy to check that the conditions in the Definition 8 are satisfied with* $\Delta_l \equiv 1$.

Lemma 11 *Let* $h : (\Gamma, \lambda) \longrightarrow\!\!\triangleright\, (G, \nu)$ *be a morphism of* σ*-compact, lcH-groupoids with Haar systems. Then the function* Δ_h *that appears in the condition 8 of the Definition 8 satisfies* $\Delta_h(x, \gamma) = \Delta_h(r(x), \gamma)$ *for all* $(x, \gamma) \in G \rtimes_h \Gamma$.

Proof. Let $f \geq 0$ be a Borel function on $G \rtimes_h \Gamma$. For each $t \in G^{(0)}$ and each $x_0 \in G^t$, we have

$$\begin{aligned}
&\int f(x, \gamma)\, d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x) \\
=\ &\int f\left(x x_0^{-1}, \gamma\right) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_{d(x_0)}(x) \\
=\ &\int f\left(\gamma^{-1} \cdot_h x x_0^{-1}, \gamma^{-1}\right) \Delta_h\left(\gamma^{-1} \cdot_h x, \gamma^{-1}\right) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_{d(x_0)}(x) \\
=\ &\int f\left(\left(\gamma^{-1} \cdot_h r(x)\right) x x_0^{-1}, \gamma^{-1}\right) \Delta_h\left(\left(\gamma^{-1} \cdot_h r(x)\right) x, \gamma^{-1}\right) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_{d(x_0)}(x) \\
=\ &\int f\left(\left(\gamma^{-1} \cdot_h r(x)\right) x, \gamma^{-1}\right) \Delta_h\left(\left(\gamma^{-1} \cdot_h r(x)\right) x x_0, \gamma^{-1}\right) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x) \\
=\ &\int f\left(\gamma^{-1} \cdot_h x, \gamma^{-1}\right) \Delta_h\left(\gamma^{-1} \cdot_h x x_0, \gamma^{-1}\right) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x) \\
=\ &\int f(x, \gamma)\, \Delta_h(x x_0, \gamma)\, \Delta_h\left(\gamma^{-1} \cdot_h x, \gamma^{-1}\right) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x)
\end{aligned}$$

Thus for all $t \in G^{(0)}$ and $x_0 \in G^t$and almost all $(x,\gamma) \in G \rtimes_h \Gamma$,

$$\begin{aligned} 1 &= \Delta_h(xx_0,\gamma)\,\Delta_h\left(\gamma^{-1}\cdot_h x,\gamma^{-1}\right) \\ &= \Delta_h(xx_0,\gamma)\,\Delta_h(x,\gamma)^{-1}. \end{aligned}$$

Therefore $\Delta_h(xx_0,\gamma) = \Delta_h(x,\gamma)$ for $\int \lambda^{\rho_h(r(x))}(\gamma)\,d\nu_t(x)$ -a.a.$(x,\gamma) \in G \rtimes_h \Gamma$. Since Δ_h is a continuous function and $\int \lambda^{\rho_h(r(x))}(\gamma)\,d\nu_t(x)$ is a measure of full support on

$$\{(x,\gamma) \in G \rtimes_h \Gamma :\, d(x) = t\},$$

it follows that $\Delta_h(xx_0,\gamma) = \Delta_h(x,\gamma)$ for all $(x,\gamma) \in G \rtimes_h \Gamma$ with $d(x) = r(x_0)$. Particularly, for $x_0 = x^{-1}$, it follows that $\Delta_h(r(x),\gamma) = \Delta_h(x,\gamma)$. ∎

We shall prove that for particular classes of groupoids we can choose a Haar system on G such that the condition 8 is satisfied. In order to do this we need some results on the structure of the Haar systems, as developed by J. Renault in Section 1 of [14] and also by A. Ramsay and M.E. Walter in Section 2 of [11]. In [14] Jean Renault constructs a Borel Haar system for G'(the isotropy group bundle of a locally compact groupoid G which has a fundamental system of conditionally-compact neighborhoods of $G^{(0)}$). One way to do this is to choose a function F_0 continuous with conditionally support which is nonnegative and equal to 1 at each $t \in G^{(0)}$. Then for each $t \in G^{(0)}$ choose a left Haar measure β_t^t on G_t^t so the integral of F_0 with respect to β_t^t is 1. If the restriction of r to $G'$is open, then $\left\{\beta_t^t,\, t \in G^{(0)}\right\}$ is a Haar system for G'(Lemma 1.3/p. 6 [14]).

Renault defines $\beta_s^t = x\beta_s^s$ if $x \in G_s^t$ (where $x\beta_s^s(f) = \int f(xy)\,d\beta_s^s(y)$). If z is another element in G_s^t, then $x^{-1}z \in G_s^s$, and since β_s^s is a left Haar measure on G_s^s, it follows that β_s^t is independent of the choice of x. If K is a compact subset of G, then $\sup_{t,s} \beta_s^t(K) < \infty$. Renault also defines a 1-cocycle δ_G on G such that for every $t \in G^{(0)}$, $\delta|_{G_t^t}$ is the modular function for β_t^t. With this apparatus in place, Renault describes a decomposition of the Haar system $\left\{\nu^t,\, t \in G^{(0)}\right\}$ for G over the equivalence relation R (the principal groupoid associated to G). He proves that there is a unique Borel Haar system $\tilde{\nu}$ for R with the property that

$$\nu^t = \int \beta_s^q d\tilde{\nu}^t(q,s) \quad \text{for all } t \in G^{(0)}.$$

In Section 2 [11] A. Ramsay and M.E. Walter prove that

$$\sup_t \tilde{\nu}^t((r,d)(K)) < \infty, \text{ for all compact } K \subset G$$

For each $t \in G^{(0)}$ the measure $\tilde{\nu}^t$ is concentrated on $\{t\} \times [t]$. Therefore there is a measure $\tilde{\beta}^t$ concentrated on $[t]$ such that $\tilde{\nu}^t = \varepsilon_t \times \tilde{\beta}^t$, where ε_t is the unit point mass at t. Since $\left\{\tilde{\nu}^t, t \in G^{(0)}\right\}$ is a Haar system, we have $\tilde{\beta}^t = \tilde{\beta}^s$ for all $(t,s) \in R$, and the function

$$t \rightarrow \int f(s)\,\tilde{\beta}^t(s)$$

is Borel for all $f \geq 0$ Borel on $G^{(0)}$. If μ is a quasi-invariant measure for $\left\{\nu^t, t \in G^{(0)}\right\}$, then μ is a quasi-invariant measure for $\left\{\tilde{\nu}^t, t \in G^{(0)}\right\}$. Also if $\Delta_{\mu,R}$ is the modular function associated to $\left\{\tilde{\nu}^t, t \in G^{(0)}\right\}$ and μ, then $\Delta_\mu = \delta_G \Delta_{\mu,R} \circ (r, d)$ can serve as the modular function associated to $\left\{\nu^t, t \in G^{(0)}\right\}$ and μ. For each t the measure $\tilde{\beta}^t$ is quasi-invariant (Section 2 [11]). It is easy to see that $\Delta_{\tilde{\beta}^t,R} = 1$, and consequently $\Delta_{\tilde{\beta}^t} = \delta_G$. Thus if G is a transitive groupoid $\tilde{\beta}^t$ is a quasi-invariant measure of full support having a continuous modular function ($\Delta_{\tilde{\beta}^t} = \delta_G$). More generally, let us assume that the associated principal groupoid R associated to G is proper. This means that R is a closed subset of $G^{(0)} \times G^{(0)}$ endowed with product topology (or equivalently, $G^{(0)}/G$ is a Hausdorff space) and the map

$$(r, d) : G \to R, \; (r, d)(x) = (r(x), d(x))$$

is an open map when R is endowed with the relative product topology coming from $G^{(0)} \times G^{(0)}$. If μ is a quasi-invariant Radon measure for the Haar system $\left\{\nu^t, \, t \in G^{(0)}\right\}$, then $\mu_1 = \int \tilde{\beta}^t d\mu(t)$ is a Radon measure which is equivalent to μ (see Remark 6/p. 232 [2]). It is easy to prove that μ_1 has a continuous modular function $\Delta_{\mu_1} = \delta_G$.

We shall call the pair of the system of measures

$$\left(\left\{\beta_s^t\right\}_{(t,s)\in R}, \left\{\tilde{\beta}^{\dot{t}}\right\}_{\dot{t}\in G^{(0)}/G} \right)$$

(described above) *the decomposition of the Haar system* $\left\{\nu^t, t \in G^{(0)}\right\}$ *over the principal groupoid associated to* G. Also we shall call δ_G the 1-*cocycle associated to the decomposition.*

Proposition 12 *Let* Γ *and* G *be two* σ*-compact, lcH-groupoids. Let* $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ *(respectively,* $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$*) be a Haar system on* Γ *(respectively, on* G*). Let* $\left(\left\{\beta_s^t\right\}_{(t,s)\in R}, \left\{\tilde{\beta}^{\dot{t}}\right\}_{\dot{t}\in G^{(0)}/G} \right)$ *be the decomposition of the Haar system* $\left\{\nu^t, t \in G^{(0)}\right\}$ *over the principal groupoid associated to* G*. Let* h *be continuous morphism from* Γ *to* G *(in the sense of Definition 3). If there is a continuous positive function* $\Delta : G^{(0)} \rtimes_{h_0} \Gamma \to \mathbf{R}$*, such that* Δ *is the modular function of* $\tilde{\beta}^{\dot{t}}$ *with respect to the Haar system* $\left\{\varepsilon_t \times \lambda^{\rho_h(t)}, t \in G^{(0)}\right\}$ *on* $G^{(0)} \rtimes_{h_0} \Gamma$ *for each* $\dot{t} \in G^{(0)}/G$*, then* $h : (\Gamma, \lambda) \longrightarrow\!\!\!\triangleright\; (G, \nu)$ *is a morphism in the sense of Definition 8).*

Proof. Let δ_G be the 1-cocycle associated to the decomposition $\left(\left\{\beta_s^t\right\}_{(t,s)\in R}, \left\{\tilde{\beta}^{\dot{t}}\right\}_{\dot{t}\in G^{(0)}/G} \right)$.

Let $g{:}G \rtimes_h \Gamma \to \mathbf{R}$ be Borel nonnegative function. Then we have

$$\begin{aligned}
&\int f\left(\gamma^{-1} \cdot_h x, \gamma^{-1}\right) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x) \\
&= \int f\left(\gamma^{-1} \cdot_h x^{-1}, \gamma^{-1}\right) d\lambda^{\rho_h(d(x))}(\gamma)\, d\nu^t(x) \\
&= \int f\left(\gamma^{-1} \cdot_h x^{-1}, \gamma^{-1}\right) d\lambda^{\rho_h(d(x))}(\gamma)\, d\beta_s^t(x)\, d\tilde{\beta}^t(s) \\
&= \int f\left(\gamma^{-1} \cdot_h x^{-1}, \gamma^{-1}\right) d\beta_s^t(x)\, d\lambda^{\rho_h(s)}(\gamma)\, d\tilde{\beta}^t(s) \\
&= \int f\left(\gamma^{-1} \cdot_h x, \gamma^{-1}\right) \delta_G(x)^{-1}\, d\beta_t^s(x)\, d\lambda^{\rho_h(s)}(\gamma)\, d\tilde{\beta}^t(s) \\
&= \int f\left(\left(\gamma^{-1} \cdot_h r(x)\right) x, \gamma^{-1}\right) \delta_G(x)^{-1}\, d\beta_t^s(x)\, d\lambda^{\rho_h(s)}(\gamma)\, d\tilde{\beta}^t(s) \\
&= \int f\left(x, \gamma^{-1}\right) \delta_G\left(\left(\gamma^{-1} \cdot_h s\right)^{-1} x\right)^{-1} d\beta_t^{r\left(\gamma^{-1} \cdot_h s\right)}(x)\, d\lambda^{\rho_h(s)}(\gamma)\, d\tilde{\beta}^t(s) \\
&= \int f\left(x, \gamma^{-1}\right) \delta_G\left(\left(\gamma \cdot_h r\left(\gamma^{-1} \cdot_h s\right)\right) x\right)^{-1} d\beta_t^{r\left(\gamma^{-1} \cdot_h s\right)}(x)\, d\lambda^{\rho_h(s)}(\gamma)\, d\tilde{\beta}^t(s) \\
&= \int f(x, \gamma)\, \delta_G\left(\left(\gamma^{-1} \cdot_h s\right) x\right)^{-1} d\beta_t^s(x)\, \Delta\left(r\left(\gamma^{-1} \cdot_h s\right), \gamma^{-1}\right) d\lambda^{\rho_h(s)}(\gamma)\, d\tilde{\beta}^t(s) \\
&= \int f\left(x^{-1}, \gamma\right) \delta_G\left(\left(\gamma^{-1} \cdot_h s\right)\right)^{-1} d\beta_s^t(x)\, \Delta\left(r\left(\gamma^{-1} \cdot_h s\right), \gamma^{-1}\right) d\lambda^{\rho_h(s)}(\gamma)\, d\tilde{\beta}^t(s) \\
&= \int f\left(x^{-1}, \gamma\right) \delta_G\left(\left(\gamma^{-1} \cdot_h d(x)\right)\right)^{-1} \Delta\left(r\left(\gamma^{-1} \cdot_h d(x)\right), \gamma^{-1}\right) d\lambda^{\rho_h(s)}(\gamma)\, d\nu^t(x) \\
&= \int f(x, \gamma)\, \delta_G\left(\left(\gamma^{-1} \cdot_h r(x)\right)\right)^{-1} \Delta\left(r\left(\gamma^{-1} \cdot_h r(x)\right), \gamma^{-1}\right) d\lambda^{\rho_h(s)}(\gamma)\, d\nu_t(x).
\end{aligned}$$

The condition 8 in the Definition 8 is satisfied taking

$$\Delta_h(x, \gamma) = \delta_G\left(\gamma^{-1} \cdot_h r(x)\right) \Delta(r(x), \gamma).$$

■

Corollary 13 *Let* Γ *and* G *be two* σ*-compact,* lcH*-groupoids. Let* $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ *be a Haar system on* Γ*. Let* h *be continuous morphism from* Γ *to* G *(in the sense of Definition 3). If* G *is transitive and there is a quasi-invariant measure with respect to the Haar system* $\left\{\varepsilon_t \times \lambda^{\rho_h(t)}, t \in G^{(0)}\right\}$ *on* $G^{(0)} \rtimes_{h_0} \Gamma$ *having the support* $G^{(0)}$ *and continuous modular function, then we can choose a Haar system* ν *on* G *such that* $h : (\Gamma, \lambda) \longrightarrow\!\!\triangleright\, (G, \nu)$ *is a morphism in the sense of Definition 8).*

Proof. Let $\tilde{\beta}$ be a quasi-invariant measure with respect to the Haar system $\left\{\varepsilon_t \times \lambda^{\rho_h(t)}, t \in G^{(0)}\right\}$ on $G^{(0)} \rtimes_{h_0} \Gamma$ having *supp*$(\tilde{\beta}) = G^{(0)}$ and continuous modular function. Then

$$\left\{\int \beta_s^t d\tilde{\beta}(s),\, t \in G^{(0)}\right\}$$

is Haar system on G satisfying the hypothesis of Proposition 12. ■

Remark 14 *If the associated principal groupoid of* $G^{(0)} \rtimes_{h_0} \Gamma$ *is proper, then there is a quasi-invariant measure with respect to the Haar system* $\left\{\varepsilon_t \times \lambda^{\rho_h(t)}, t \in G^{(0)}\right\}$*, having the support* $G^{(0)}$ *and continuous modular function. The associated principal groupoid of* $G^{(0)} \rtimes_{h_0} \Gamma$ *is proper if and only if the set*

$$R_{\Gamma, G^{(0)}} = \left\{\left(t, r\left(\gamma^{-1} t\right)\right) : t \in G^{(0)}, \gamma \in \Gamma,\, r(\gamma) = \rho_h(t)\right\}$$

is a closed in $G^{(0)} \times G^{(0)}$ *(endowed with the product topology) and the map*

$$\left(t, \gamma\right) \to \left(t, r\left(\gamma^{-1}t\right)\right)$$

is an open map from $G^{(0)} \rtimes_{h_0} \Gamma$ *to* $R_{\Gamma, G^{(0)}}$*, when* $R_{\Gamma, G^{(0)}}$ *is endowed with the relative product topology coming from* $G^{(0)} \times G^{(0)}$*.*

2.2 Composition of morphisms

Definition 15 *Let* $h : (\Gamma, \lambda) \longrightarrow\!\!\triangleright (G_1, \nu)$ *and* $k : (G_1, \lambda) \longrightarrow\!\!\triangleright (G_2, \eta)$ *be two morphism of locally compact groupoids endowed with Haar systems. Let* $kh : (\Gamma, \lambda) \longrightarrow\!\!\triangleright (G_2, \eta)$ *be defined by*

1. $\rho_{kh} : G_2^{(0)} \to \Gamma^{(0)}$

$$\rho_{kh}\left(x_2\right) = \rho_h\left(\rho_k\left(x_2\right)\right) \text{ for all } x_2 \in G_2.$$

2. $(\gamma, x_2) \to \gamma \cdot_{kh} x_2 := \left(\gamma \cdot_h \rho_k\left(r\left(x_2\right)\right)\right) \cdot_k x_2$ *from* $\Gamma \star_{kh} G_2$ *to* G_2*, where*

$$\Gamma \star_{kh} G_2 = \left\{(\gamma, x_2) \in \Gamma \times G_2 : d\left(\gamma\right) = \rho_{kh}\left(r\left(x_2\right)\right)\right\}$$

Remark 16 *Let* $h : (\Gamma, \lambda) \longrightarrow\!\!\triangleright (G_1, \nu)$ *and* $k : (G_1, \lambda) \longrightarrow\!\!\triangleright (G_2, \eta)$ *be two morphism of locally compact groupoids endowed with Haar systems. Let* $kh : (\Gamma, \lambda) \longrightarrow\!\!\triangleright (G_2, \eta)$ *be as in Definition 15. Then for all* $\gamma \in \Gamma$*, all* $x_1 \in G_1$ *with* $\rho_h\left(r\left(x_1\right)\right) = d\left(\gamma\right)$ *and all* $x_2 \in G_2$ *with* $\rho_k\left(r\left(x_2\right)\right) = d\left(x_1\right)$*, we have*

$$\gamma \cdot_{kh} \left(x_1 \cdot_k x_2\right) = \left(\gamma \cdot_h \rho_k\left(r\left(x_1 \cdot_k x_2\right)\right)\right) \cdot_k \left(x_1 \cdot_k x_2\right) = \left(\gamma \cdot_h r\left(x_1\right) x_1\right) \cdot_k x_2$$
$$= \left(\gamma \cdot_h x_1\right) \cdot_k x_2$$

Lemma 17 *Let* (Γ, λ)*,* (G_1, ν) *and* (G_2, η) *be* σ*-compact, lcH-groupoids. If* $h : (\Gamma, \lambda) \longrightarrow\!\!\triangleright (G_1, \nu)$ *and* $k : (G_1, \lambda) \longrightarrow\!\!\triangleright (G_2, \eta)$ *are morphisms, then* $kh : (\Gamma, \lambda) \longrightarrow\!\!\triangleright (G_2, \eta)$ *is a morphism.*

Proof. Let us check the conditions 3, 5 and 6 in the def. 8. For all $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$ and all $x_2 \in G_2$ with $(\gamma_2, x) \in \Gamma \star_{kh} G_2$, we have

$$\begin{aligned}
\left(\gamma_1\gamma_2\right) \cdot_{kh} x_2 &= \left(\left(\gamma_1\gamma_2\right) \cdot_h \rho_k\left(r\left(x_2\right)\right)\right) \cdot_k x_2 \\
&= \left(\gamma_1 \cdot_h r\left(\gamma_2 \cdot_h \rho_k\left(r\left(x_2\right)\right)\right)\right)\left(\gamma_2 \cdot_h \rho_k\left(r\left(x_2\right)\right)\right) \cdot_k x_2 \\
&= \left(\left(\gamma_1 \cdot_h r\left(\gamma_2 \cdot_h \rho_k\left(r\left(x_2\right)\right)\right)\right)\right) \cdot_k \left(\left(\gamma_2 \cdot_h \rho_k\left(r\left(x_2\right)\right)\right) \cdot_k x_2\right) \\
&= \left(\left(\gamma_1 \cdot_h r\left(\gamma_2 \cdot_h \rho_k\left(r\left(x_2\right)\right)\right)\right)\right) \cdot_k \left(\gamma_2 \cdot_{kh} x_2\right) \\
&= \left(\left(\gamma_1 \cdot_h \rho_k\left(r\left(\gamma_2 \cdot_h \rho_k\left(r\left(x_2\right)\right)\right)\right)\right) \cdot_k x\right) \cdot_k \left(\gamma_2 \cdot_{kh} x_2\right) \\
&= \gamma_1 \cdot_{kh} \left(\gamma_2 \cdot_{kh} x_2\right).
\end{aligned}$$

For all $(\gamma, x_2) \in \Gamma \star_{kh} G_2$ and $(x_2, y_2) \in G_2^{(2)}$, we have

$$(\gamma \cdot_{kh} x_2)\, y_2 = ((\gamma \cdot_h \rho_k (r (x_2))) \cdot_k x_2) \cdot_k y_2$$

$$= (\gamma \cdot_h \rho_k (r (x_2))) \cdot_k (x_2 y_2) = (\gamma \cdot_h \rho_k (r (x_2 y_2))) \cdot_k (x_2 y_2)$$

$$= \gamma \cdot_{kh} (x_2 y_2)$$

Let $P : G_1 \to \mathbf{R}$ be a continuous function with conditionally compact support such that

$$\int P (x_1)\, d\nu^t (x_1) = 1, \text{ for all } t \in G_1^{(0)}$$

If $f : \Gamma \star_{kh} G_2 \to \mathbf{R}$ is a Borel nonnegative function, then

$$\int \int f \left(\gamma^{-1} \cdot_{kh} x_2, \gamma^{-1}\right) d\lambda^{\rho_{kh}(r(x_2))} (\gamma)\, d\eta_s (x_2)$$

$$= \int \int \int f \left(\gamma^{-1} \cdot_{kh} x_2, \gamma^{-1}\right) d\lambda^{\rho_{kh}(r(x_2))} (\gamma)\, P (x_1)\, d\nu^{\rho_k(r(x_2))} (x_1)\, d\eta_s (x_2) .$$

The following sequence of changes of variables

1. $(x_2, x_1) \to \left(x_1^{-1} \cdot_k x_2, x_1\right)$ (using the quasi-invariance of η_s)2. $x_1 \to x_1^{-1}$
3. $(x_1, \gamma) \to \left(\gamma^{-1} \cdot_h x_1, x_1\right)$ (using the quasi-invariance of $\nu_{\rho_k(r(x_2))}$)
4. $x_1 \to x_1^{-1}$
5. $(x_2, x_1) \to \left(x_1^{-1} \cdot_k x_2, x_1\right)$ (using the quasi-invariance of η_s)

transforms the preceding integral into

$$\int \int \int f (x_2, \gamma)\, g (x_2, x_1, \gamma)\, P \left(\gamma^{-1} \cdot_h x_1\right) d\nu^{\rho_k(r(x_2))} (x_1)\, d\lambda^{\rho_{kh}(r(x_2))} (\gamma)\, d\eta_s (x_2)$$

where

$$\begin{aligned} g (x_2, x_1, \gamma) &= \Delta_k \left(x_1^{-1} \cdot_k x_2, \left(\gamma^{-1} \cdot_h x_1\right)^{-1}\right)^{-1} \Delta_h (x_1, \gamma)^{-1} \Delta_k (x_2, x_1)^{-1} \\ &= \left(\Delta_k (x_2, x_1)\, \Delta_k \left(x_1^{-1} \cdot_k x_2, \left(\gamma^{-1} \cdot_h x_1\right)^{-1}\right)\right)^{-1} \Delta_h (x_1, \gamma)^{-1} \\ &= \Delta_k \left(x_2, \left(\gamma^{-1} \cdot_h r (x_1)\right)^{-1}\right)^{-1} \Delta_h (x_1, \gamma)^{-1} . \end{aligned}$$

Let us note that for all $(x_1, \gamma) \in G_1 \rtimes_h \Gamma$, and all $x_2 \in G_2$ with $\rho_k (r (x_2)) = r (x_1)$, $\Delta_k \left(x_2, \left(\gamma^{-1} \cdot_h r (x_1)\right)\right) \Delta_h (x_1, \gamma)$ does not depend on x_1 but only on $r (x_1) = \rho_k (r (x_2))$, and also it does not depend on x_2 but only on $r (x_2)$. For each $(x_2, \gamma) \in G_2 \rtimes_k \Gamma$, let us denote

$$\Delta_{kh} (x_2, \gamma) = \Delta_k \left(x_2, \left(\gamma^{-1} \cdot_h r (x_1)\right)^{-1}\right) \Delta_h (x_1, \gamma) .$$

Consequently,

$$\int \int f \left(\gamma^{-1} \cdot_{kh} x_2, \gamma^{-1}\right) d\lambda^{\rho_{kh}(r(x_2))} (\gamma)\, d\eta_s (x_2)$$

$$= \int \int \int f (x_2, \gamma)\, \Delta_{kh} (x_2, \gamma)^{-1} P \left(\gamma^{-1} \cdot_h x_1\right) d\nu^{\rho_k(r(x_2))} (x_1)\, d\lambda^{\rho_{kh}(r(x_2))} (\gamma)\, d\eta_s (x_2)$$

$$= \int \int \int f (x_2, \gamma)\, \Delta_{kh} (x_2, \gamma)^{-1} P (x_1)\, d\nu^{r\left(\gamma^{-1} \cdot_h \rho_k(r(x_2))\right)} (x_1)\, d\lambda^{\rho_{kh}(r(x_2))} (\gamma)\, d\eta_s (x_2)$$

$$= \int \int \int f (x_2, \gamma)\, \Delta_{kh} (x_2, \gamma)^{-1} d\lambda^{\rho_{kh}(r(x_2))} (\gamma)\, d\eta_s (x_2)$$

Therefore the condition 8 in def. 8 is satisfied if we take

$$\Delta_{kh}(x_2,\gamma) = \Delta_k\left(x_2, \left(\gamma^{-1} \cdot_h r(x_1)\right)\right) \Delta_h(x_1,\gamma), (x_2,\gamma) \in G_2 \rtimes_{kh} \Gamma.$$

■

Remark 18 *If $h : (\Gamma,\lambda) \longrightarrow\!\!\triangleright (G_1,\nu)$ and $k : (G_1,\lambda) \longrightarrow\!\!\triangleright (G_2,\eta)$ are morphisms, then it is easy to see that*

$$\Delta_{kh}(x_2,\gamma) = \Delta_k\left(\gamma^{-1} \cdot_{kh} x_2, \gamma^{-1} \cdot_h x_1\right)^{-1} \Delta_h(x_1,\gamma)\,\Delta_k(x_2,x_1)$$

for any $(x_2,\gamma) \in G_2 \rtimes_{kh} \Gamma$ and any $x_1 \in G_1^{\rho_k(r(x_2))}$.

Proposition 19 *The class of σ-compact, lcH-groupoids with the morphisms in the sense of def. 8 form a category.*

Proof. A straightforward computation shows that the composition of morphisms (in the sense of def. 15) is associative. For each groupoid Γ let l_Γ be the morphism defined in Example 10. If $h : \Gamma \longrightarrow\!\!\triangleright G$ and $k : G \longrightarrow\!\!\triangleright \Gamma$ are morphisms in the sense of Definition 8, then $hl_\Gamma = h$ and $l_\Gamma k = k$. ■

2.2.1 Examples of morphisms

In this subsection we study what becomes a morphism $h : (\Gamma,\lambda) \longrightarrow\!\!\triangleright (G,\nu)$ for a particular groupoid G. We shall consider the following cases:

1. *Groups.* A group G is a groupoid with $G^{(2)} = G \times G$ and $G^{(0)} = \{e\}$ (the unit element).

2. *Sets.* A set X is a groupoid letting

   $$X^{(2)} = diag(X) = \{(x,x), x \in G\}$$

   and defining the operations by $xx = x$, and $x^{-1} = x$.

   Sets and groups are particular cases of group bundles (this means groupoids for which $r(x) = d(x)$ for all x).

3. *Equivalence relations.* Let $\mathcal{E} \subset X \times X$ be (the graph of) an equivalence relation on the set X. Let $\mathcal{E}^{(2)} = \{((x_1,y_1),(x_2,y_2)) \in \mathcal{E} \times \mathcal{E} : y_1 = x_2\}$. With product $(x,y)(y,z) = (x,z)$ and $(x,y)^{-1} = (y,x)$, $\mathcal{E}$ is a principal groupoid. $\mathcal{E}^{(0)}$ may be identified with X. Two extreme cases deserve to be single out. If $\mathcal{E} = X \times X$, then $\mathcal{E}$ is called the trivial groupoid on X, while if $\mathcal{E} = diag(X)$, then $\mathcal{E}$ is called the co-trivial groupoid on X (and may be identified with the groupoid in example 2).

   If G is any groupoid, then

   $$R = \{(r(x), d(x)),\ x \in G\}$$

is an equivalence relation on $G^{(0)}$. The groupoid defined by this equivalence relation is called the *principal groupoid associated with* G.

Any locally compact principal groupoid can be viewed as an equivalence relation on a locally compact space X having its graph $\mathcal{E} \subset X \times X$ endowed with a locally compact topology compatible with the groupoid structure. This topology can be finer than the product topology induced from $X \times X$. We shall endow the principal groupoid associated with a groupoid G with the quotient topology induced from G by the map

$$(r,d) : G \to R,\ (r,d)(x) = (r(x), d(x))$$

This topology consists of the sets whose inverse images by (r,d) in G are open.

Let Γ and G be two σ-compact, lcH-groupoids, endowed with the Haar systems $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$, respectively, $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$. Let $\left(\{\beta_s^t\}_{(t,s)\in R}, \left\{\tilde{\beta}^t\right\}_{\dot{t}\in G^{(0)}/G}\right)$ be the decomposition of the Haar system $\left\{\nu^t, t \in G^{(0)}\right\}$ over the principal groupoid associated to G and let δ_G be its associated 1-cocycle.

Let $h : (\Gamma, \lambda) \longrightarrow (G, \nu)$ be a morphism in the sense of Def. 8. Let us show that if G is a group bundle , then $\Gamma|_{\rho_h\left(G^{(0)}\right)}$ is also a group bundle and the condition 8 in the Def. 8 is automatically satisfied. Indeed, let $\gamma \in \Gamma|_{\rho_h\left(G^{(0)}\right)}$. Then there is $t \in G^{(0)}$ such that $d(\gamma) = \rho_h(t)$. We

$$r(\gamma) = \rho_h(r(\gamma \cdot_h t)) = \rho_h(d(\gamma \cdot_h t)) = \rho_h(t) = d(\gamma).$$

Therefore $\Gamma|_{\rho_h\left(G^{(0)}\right)}$ is a group bundle. Let us prove that the condition 8 in the def. 8 is automatically satisfied if G is a group bundle. If the restriction of r to $G'$is open, then $\left\{\beta_t^t,\, t \in G^{(0)}\right\}$ is a Haar system for G'(Lemma 1.3/p. 6 [14]). In our case G is a group bundle, consequently, $G = G'$. If $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$ is a Haar system on G, then for each $t \in G^{(0)}$, ν^t is a (left) Haar measure on the locally compact group G_t^t. By the uniqueness of the Haar measure on G_t^t, it follows that there is $P(t) \in \mathbf{R}_+^*$, such that $\nu^t = P(t)\,\beta_t^t$. Thus the restriction of δ_G to G_t^t is the modular function for ν^t. Reasoning in the same way, for each $u \in \rho_h\left(G^{(0)}\right)$, λ^u is a (left) Haar measure on the locally compact group Γ_u^u, and the restriction of δ_Γ to Γ_u^u is the modular function for λ^u. For each

$f \in C_c(G \rtimes_h \Gamma)$ and $t \in G$, we have

$$\int\int f\left(\gamma^{-1} \cdot_h x, \gamma^{-1}\right) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x)$$
$$= \int\int f\left(\gamma^{-1} \cdot_h x, \gamma^{-1}\right) d\lambda^{\rho_h(t)}(\gamma)\, d\nu_t(x)$$
$$= \int\int f\left(\gamma^{-1} \cdot_h x, \gamma^{-1}\right) \delta_G(x)^{-1} d\nu^t(x)\, d\lambda^{\rho_h(t)}(\gamma)$$
$$= \int\int f\left(x, \gamma^{-1}\right) \delta_G\left(\left(\gamma^{-1} \cdot_h t\right)^{-1} x\right)^{-1} d\nu^t(x)\, d\lambda^{\rho_h(t)}(\gamma)$$
$$= \int\int f\left(x, \gamma^{-1}\right) \delta_G\left(\gamma^{-1} \cdot_h t\right) \delta_G(x)^{-1} d\nu^t(x)\, d\lambda^{\rho_h(t)}(\gamma)$$
$$= \int\int f(x, \gamma)\, \delta_G(\gamma \cdot_h t)\, \delta_\Gamma\left(\gamma^{-1}\right) \delta_G(x)^{-1} d\nu^t(x)\, d\lambda^{\rho_h(t)}(\gamma)$$
$$= \int\int f(x, \gamma)\, \delta_G(\gamma \cdot_h t)\, \delta_\Gamma\left(\gamma^{-1}\right) d\lambda^{\rho_h(t)}(\gamma)\, d\nu_t(x).$$

Hence taking $\Delta_h(x, \gamma) = \delta_\Gamma(\gamma)\, \delta_G(\gamma \cdot_h r(x))^{-1}$the condition 8 in the Definition 8 is satisfied. Let us note that if G is a group bundle, then each morphism $h : (\Gamma, \lambda) \longrightarrow\!\triangleright (G, \nu)$, for which $\rho_h : G^{(0)} \to \Gamma^{(0)}$ is a homeomorphism, can be viewed as a continuous homomorphism φ from Γ to G for which the restriction $\varphi|_{\Gamma^{(0)}} = \varphi^{(0)} : \Gamma^{(0)} \to G^{(0)}$ is a homeomorphism. Indeed if $\varphi : \Gamma \to G$ is a groupoid homomorphism (this means that if $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$, then $(\varphi(\gamma_1), \varphi(\gamma_2)) \in G^{(2)}$ and $\varphi(\gamma_1\gamma_2) = \varphi(\gamma_1)\varphi(\gamma_2)$) and if $\varphi^{(0)} : \Gamma^{(0)} \to G^{(0)}$ is a homeomorphism, then taking $\rho_h = \left(\varphi^{(0)}\right)^{-1} : G^{(0)} \to \Gamma^{(0)}$, and defining

$$\gamma \cdot_h x = \varphi(\gamma)\, x$$

we obtain a morphism in the sense of Definition 8. Conversely, if $h : (\Gamma, \lambda) \longrightarrow\!\triangleright (G, \nu)$ a morphism in the sense of Definition 8, for which $\rho_h : G^{(0)} \to \Gamma^{(0)}$ is a homeomorphism, then let us define $\varphi : \Gamma \to G$ by

$$\varphi(\gamma) = \gamma \cdot_h \rho_h^{-1}(d(\gamma)),\ \gamma \in \Gamma.$$

If $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$, then

$$\begin{aligned} d(\varphi(\gamma_1)) &= d\left(\gamma_1 \cdot_h \rho_h^{-1}(d(\gamma_1))\right) = \rho_h^{-1}(d(\gamma_1)) = \rho_h^{-1}(r(\gamma_2)) = \rho_h^{-1}(d(\gamma_2)) \\ &= d\left(\gamma_2 \cdot_h \rho_h^{-1}(d(\gamma_2))\right) = d(\varphi(\gamma_2)) = r(\varphi(\gamma_2)). \end{aligned}$$

Consequently, $(\varphi(\gamma_1), \varphi(\gamma_2)) \in G^{(2)}$ and

$$\begin{aligned}
\varphi(\gamma_1\gamma_2) &= (\gamma_1\gamma_2) \cdot_h \rho_h^{-1}(d(\gamma_2)) \\
&= \gamma_1 \cdot_h \left(\gamma_2 \cdot_h \rho_h^{-1}(d(\gamma_2))\right) \\
&= \gamma_1 \cdot_h r\left(\gamma_2 \cdot_h \rho_h^{-1}(d(\gamma_2))\right)\left(\gamma_2 \cdot_h \rho_h^{-1}(d(\gamma_2))\right) \\
&= \gamma_1 \cdot_h d\left(\gamma_2 \cdot_h \rho_h^{-1}(d(\gamma_2))\right)\left(\gamma_2 \cdot_h \rho_h^{-1}(d(\gamma_2))\right) \\
&= \left(\gamma_1 \cdot_h \rho_h^{-1}(d(\gamma_2))\right)\left(\gamma_2 \cdot_h \rho_h^{-1}(d(\gamma_2))\right) \\
&= \left(\gamma_1 \cdot_h \rho_h^{-1}(d(\gamma_1))\right)\left(\gamma_2 \cdot_h \rho_h^{-1}(d(\gamma_2))\right) \\
&= \varphi(\gamma_1)\varphi(\gamma_2).
\end{aligned}$$

The restriction of φ to $\Gamma^{(0)}$ is a homeomorphism, because it coincides with ρ_h^{-1}.

Therefore if Γ and G are locally compact groups, then the notion of morphism (cf. Definition 8) reduces to the usual notion of group homomorphism.

If G is a set (see example 2 at beginning of the subsection) and if $h : (\Gamma, \lambda) \longrightarrow\!\triangleright (G, \nu)$ is a morphism, then

$$\gamma \cdot_h x = d(\gamma \cdot_h x) = d(x) = x$$

for each (γ, x) with $d(\gamma) = \rho_h(r(x)) = \rho_h(x)$. In this case a morphism is uniquely determined by the map $\rho_h : G \to \Gamma^{(0)}$.

Let us now assume that $G \subset X \times X$ is an equivalence relation, where X is locally compact, σ-compact, Hausdorff space. Let us endow G with the relative product topology from $X \times X$. Let us also assume that there is a Haar system on G, $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$, and let Γ be another groupoid endowed with the Haar systems $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$. Any morphism in the sense of Definition 8, $h : (\Gamma, \lambda) \longrightarrow\!\triangleright (G, \nu)$, defines a continuous action of Γ on X, by

$$\gamma \cdot x = r(\gamma \cdot_h (x, x)).$$

Conversely, let us consider an action of Γ on X with the momentum map $\rho : X \to \Gamma^{(0)}$, satisfying $\gamma \cdot x\tilde{\ }x$. Then taking $\rho_h = \rho$, and

$$\gamma \cdot_h (x, y) = (\gamma \cdot x, y)$$

we obtain an continuous morphism (in the sense of Definition 3). The condition 8 is not necessarily satisfied. In order to see that it is enough to consider $\Gamma = G = X \times X$ (the trivial groupoid on X endowed with the product topology). Any Haar system on $X \times X$ is of the form $\{\varepsilon_x \times \mu, x \in X\}$, where μ is a measure of full support on X, and ε_x is the unit point mass at x. If $\{\varepsilon_x \times \mu_1, x \in X\}$ is a Haar system on $\Gamma = X \times X$, and $\{\varepsilon_x \times \mu_2, x \in X\}$ is a Haar system on $G = X \times X$, then the condition 8 in the Definition 8 is satisfied if and only if μ_1 and μ_2 are equivalent measure (have the same null sets) and the Radon Nikodym derivative is a continuous function.

Let $G = X \times X$ (the trivial groupoid on X endowed with the product topology), let $\nu = \{\varepsilon_x \times \mu, x \in X\}$ be a Haar system on G. Then any continuous morphism h gives rise to a continuous action of Γ on X. Conversely, any continuous action of Γ on X gives rise to a continuous morphism h from Γ to G. In the hypothesis of Corollary 13, we can choose the measure μ such that $h : (\Gamma, \lambda) \longrightarrow\!\!\triangleright\ (G, \nu)$ becomes a morphism in the sense of Definition 8.

Let us assume that the associated principal groupoid of Γ is proper. This means that it is a closed subset of $\Gamma^{(0)} \times \Gamma^{(0)}$ endowed with product topology (or equivalently, $\Gamma^{(0)}/\Gamma$ is a Hausdorff space) and the map

$$(r, d) : \Gamma \to R,\ (r, d)(x) = (r(x), d(x))$$

is an open map when R is endowed with the relative product topology coming from $\Gamma^{(0)} \times \Gamma^{(0)}$. Let μ be a quasi-invariant measure for the Haar system $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ on Γ. It can be shown that there is a quasi-invariant measure μ_0 equivalent to μ such that the modular function of μ_0 is a continuous function δ_Γ. Let S_0 be the support of μ_0. Let us take $X = S_0$ and let us consider the action Γ on S_0 defined by $\rho : S_0 \to \Gamma^{(0)}, \rho(u) = u$ for all $u \in S_0$, and $\gamma \cdot d(\gamma) = r(\gamma)$ for all $\gamma \in \Gamma|_{S_0}$. It is easy to see that μ_0 is a quasi-invariant measure for the Haar system $\{\varepsilon_u \times \lambda^u, u \in S_0\}$ on $S_0 \rtimes \Gamma$, and its modular function is δ_Γ. Thus we can define a morphism $h : (\Gamma, \lambda) \longrightarrow\!\!\triangleright\ (S_0 \times S_0, \nu)$ in the sense of Definition 8 (where $\nu = \{\varepsilon_u \times \mu_0, u \in S_0\}$) by

1. $\rho_h : S_0 \to \Gamma^{(0)}, \rho(u) = u$ for all $u \in S_0$.
2. $\gamma \cdot_h (u, v) = (r(\gamma), v)$

3 Morphisms on a groupoid Γ and the convolution algebra $C_c(\Gamma)$

Let (Γ, λ) and (G, ν) be two σ-compact, *lcH*-groupoids with Haar systems.

We associate to each morphism $h : (\Gamma, \lambda) \longrightarrow\!\!\triangleright\ (G, \nu)$ an application $\hat{h}$ defined on $C_c(\Gamma)$ in the following way. For any $f \in C_c(\Gamma)$,

$$\hat{h}(f) : C_c(G) \to C_c(G).$$

is defined by

$$\hat{h}(f)(\xi)(x) = \int f(\gamma)\, \xi\left(\gamma^{-1} \cdot_h x\right) \Delta_h(x, \gamma)^{-1/2}\, d\lambda^{\rho_h(r(x))}(\gamma)$$

Using a standard argument ([3] 2.2, [12] II.1) we can prove that $\hat{h}(f)(\xi) \in C_c(G)$ for any $\xi \in C_c(G)$. That is, since $G \rtimes_h \Gamma$ is a closed subset of the normal space $G \times \Gamma$, the function

$$(x, \gamma) \to \xi\left(\gamma^{-1} \cdot_h x\right) \Delta_h(x, \gamma)^{-1/2},$$

may be extended to a bounded continuous function F on $G \times \Gamma$. A compactness argument shows that for each $\varepsilon > 0$ and each $x_0 \in G$

$$\{x \in G : |F(x,\gamma) - F(x_0,\gamma)| < \varepsilon \text{ for all } \gamma \in supp(f)\}$$

is an open subset of G which contains x_0. Therefore the function

$$x \to F_x \; [: G \to C_c(G)]$$

where $F_x(y) = f(\gamma) F(x,\gamma)$, is continuous. Consequently,

$$(x,u) \to \int f(\gamma) F(x,\gamma) \, d\lambda^u(\gamma) \;\; \left[: G \times \Gamma^{(0)} \to \mathbf{C}\right]$$

is a continuous function, and so is the function

$$x \to \int f(\gamma) F(x,\gamma) \, d\lambda^{\rho_k(r(x))}(\gamma) \quad [: G \to \mathbf{C}]$$

(being its composition with $x \to (x, \rho_h(r(x)))$.

Lemma 20 *Let $h : (\Gamma, \lambda) \longrightarrow\!\!\triangleright (G, \nu)$ be a morphism of σ-compact, lcH-groupoids with Haar systems and let $\hat{h}$ be the application defined above . Then*

$$\left\{\hat{h}(f)\xi : f \in C_c(\Gamma), \, \xi \in C_c(G)\right\}$$

is dense in $C_c(G)$ with the inductive limit topology.

Proof. We shall use a similar argument as Jean Renault used in proof of prop. II.1.9/p. 56 [12]. Since $\Gamma^{(0)}$ is a paracompact space, it follows that there is a fundamental system of d-relatively compact neighborhood $\{U_\alpha\}_\alpha$ of $\Gamma^{(0)}$. Let U_0 be a d-relatively compact neighborhood of $\Gamma^{(0)}$ such that $U_\alpha \subset U_0$ for all α. Let $\{K_\alpha\}_\alpha$ be a net of compact subsets of $\Gamma^{(0)}$ increasing to $\Gamma^{(0)}$. Let $e_\alpha \in C_c(\Gamma)$ be a nonnegative function such that

$$\begin{aligned} supp(e_\alpha) &\subset U_\alpha \\ \int e_\alpha(\gamma) \, d\lambda^u(\gamma) &= 1 \text{ for all } u \in K_\alpha. \end{aligned}$$

We claim that for any $\xi \in C_c(G)$, $\left\{\hat{h}(e_\alpha)\xi\right\}_\alpha$ converges to ξ in the inductive limit topology. Let $\xi \in C_c(G)$ and $\varepsilon > 0$. Let K be the support of ξ. Then

$$\begin{aligned} U_0 \cdot_h K &= \{\gamma \cdot_h x : \gamma \in U_0, \, x \in K, \; r(\gamma) = \rho_h(r(x))\} \\ &= \left(U_0 \cap d^{-1}\{\rho_h(r(K))\}\right) \cdot_h K \end{aligned}$$

is a compact subset of G. A compactness argument shows that

$$W_\varepsilon = \left\{\gamma \in \Gamma : \left|\xi\left(\gamma^{-1} \cdot_h x\right) - \xi(x)\right| < \varepsilon \text{ for all } x \in U_0 \cdot_h K, \rho_h(r(x)) = r(\gamma)\right\}$$

is an open subset of Γ which contains $\Gamma^{(0)}$. If $\gamma \in W_\varepsilon \cap U_0$, then

$$\left|\xi\left(\gamma^{-1} \cdot_h x\right) - \xi(x)\right| < \varepsilon \text{ for all } x \text{ satisfying } \rho_h(r(x)) = r(\gamma)$$

(because if $x \notin U_0 \cdot_h K$, then x and $\gamma^{-1} \cdot_h x \notin K = supp(\xi)$, and hence $\xi\left(\gamma^{-1} \cdot_h x\right) = \xi(x) = 0$). Since Δ_h is a continuous function and a homomorphism from $G \times_h \Gamma$ to $\mathbf{R}_+^*$, it follows that there exist an open neighborhood L_ε of $\Gamma^{(0)}$ such that

$$\left|\Delta_h^{-1/2}(x,\gamma) - 1\right| < \varepsilon$$

for all $(x,\gamma) \in (K \times L_\varepsilon) \cap G \times_h \Gamma$. Then for any α such that $U_\alpha \subset W_\varepsilon \cap L_\varepsilon$ and $\rho_h(r(U_0 \cdot_h K)) \subset K_\alpha$, $supp\left(\hat{h}(e_\alpha)\xi\right)$ is contained in $U_0 \cdot_h K$. For all $x \in U_0 \cdot_h K$ we have

$$\begin{aligned}
&\left|\hat{h}(e_\alpha)\xi(x) - \xi(x)\right| \\
= &\left|\int e_\alpha(\gamma)\xi\left(\gamma^{-1} \cdot_h x\right)\Delta_h(x,\gamma)^{-1/2} - e_\alpha(\gamma)\xi(x)\, d\lambda^{\rho_h(r(x))}(\gamma)\right| \\
\leq &\int e_\alpha(\gamma)\left|\xi\left(\gamma^{-1} \cdot_h x\right) - \xi(x)\right|\Delta_h(x,\gamma)^{-1/2}\, d\lambda^{\rho_h(r(x))}(\gamma) + \\
&\qquad + |\xi(x)| \int e_\alpha(\gamma)\left|\Delta_h^{-1/2}(x,\gamma) - 1\right| d\lambda^{\rho_h(r(x))}(\gamma) \\
\leq &\ 2\varepsilon + \sup_x |\xi(x)|\,\varepsilon.
\end{aligned}$$

Thus $\left|\hat{h}(e_\alpha)\xi - \xi\right|$ converges to 0 in the inductive limit topology. ∎

Proposition 21 *Let $h : (\Gamma,\lambda) \longrightarrow\!\!\triangleright (G_1,\nu)$ and $k : (G_1,\lambda) \longrightarrow\!\!\triangleright (G_2,\eta)$ be morphisms of σ-compact lcH-groupoids with Haar systems. Then*

$$\hat{k}\left(\hat{h}(f)\xi_1\right)\xi_2 = \hat{kh}(f)\left(\hat{k}(\xi_1)\xi_2\right)$$

for all $f \in C_c(\Gamma)$, $\xi_1 \in C_c(G_1)$ and $\xi_2 \in C_c(G_2)$.

Proof. Let $f \in C_c(\Gamma)$, $\xi_1 \in C_c(G_1)$ and $\xi_2 \in C_c(G_2)$. For all $(x_2,x_1,\gamma) \in G_2 \times G_1 \times \Gamma$, such that $(x_1,\gamma) \in G_1 \rtimes_h \Gamma$ and $(x_2,x_1) \in G_2 \rtimes_h G_1$ let us denote

$$\begin{aligned}
F(x_2,x_1,\gamma) &= \xi_1\left(\gamma^{-1} \cdot_h x_1\right)\xi_2\left(x_1^{-1} \cdot_k x_2\right) \\
g(x_2,x_1,\gamma) &= \Delta_h(x_1,\gamma)^{-1/2}\Delta_k(x_2,x_1)^{-1/2}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\hat{k}\left(\hat{h}(f)\xi_1\right)\xi_2(x_2) \\
&= \int\int f(\gamma)F(x_2,x_1,\gamma)g(x_2,x_1,\gamma)\lambda^{\rho_h(r(x_1))}(\gamma)\, d\nu^{\rho_k(r(x_2))}(x_1) \\
&= \int\int f(\gamma)F(x_2,x_1,\gamma)g(x_2,x_1,\gamma)\, d\nu^{\rho(r(x_2))}(x_1)\, d\lambda^{\rho_{kh}(r(x_2))}(\gamma) \\
&= \int\int f(\gamma)F_1(x_2,x_1,\gamma)g_1(x_2,x_1,\gamma)\, d\nu^{r\left(\gamma^{-1} \cdot_h \rho_k(r(x_2))\right)}(x_1)\, d\lambda^{\rho_{kh}(r(x_2))}(\gamma),
\end{aligned}$$

where

$$
\begin{aligned}
F_1(x_2, x_1, \gamma) &= F\left(x_2, \left(\gamma^{-1} \cdot_h \rho_k\left(r\left(x_2\right)\right)\right)^{-1} x_1, \gamma\right) \\
&= \xi_1(x_1)\, \xi_2\left(\left(\left(\gamma^{-1} \cdot_h \rho_k\left(r\left(x_2\right)\right)\right)^{-1} x_1\right)^{-1} \cdot_k x_2\right) \\
&= \xi_1(x_1)\, \xi_2\left(x_1^{-1}\left(\gamma^{-1} \cdot_h \rho_k\left(r\left(x_2\right)\right)\right) \cdot_k x_2\right) \\
&= \xi_1(x_1)\, \xi_2\left(x_1^{-1} \cdot_k \left(\gamma^{-1} \cdot_{kh} x_2\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& g_1(x_2, x_1, \gamma) = g\left(x_2, \left(\gamma^{-1} \cdot_h \rho_k\left(r\left(x_2\right)\right)\right)^{-1} x_1, \gamma\right) \\
& = \Delta_h\left(\left(\gamma^{-1} \cdot_h \rho_k\left(r\left(x_2\right)\right)\right)^{-1} x_1, \gamma\right)^{-1/2} \Delta_k\left(x_2, \left(\gamma^{-1} \cdot_h \rho_k\left(r\left(x_2\right)\right)\right)^{-1} x_1\right)^{-1/2} \\
& = \Delta_h\left(\rho_k\left(r\left(x_2\right)\right), \gamma\right)^{-1/2} \Delta_k\left(x_2, \left(\gamma^{-1} \cdot_h \rho_k\left(r\left(x_2\right)\right)\right)^{-1} x_1\right)^{-1/2} \\
& = \Delta_h\left(\rho_k\left(r\left(x_2\right)\right), \gamma\right)^{-1/2} \Delta_k\left(x_2, \left(\gamma^{-1} \cdot_h \rho_k\left(r\left(x_2\right)\right)\right)^{-1}\right)^{-1/2} \Delta_k\left(\gamma^{-1} \cdot_{kh} x_2, x_1\right)^{-1/2} \\
& = \Delta_{kh}\left(x_2, \gamma\right)^{-1/2} \Delta_k\left(x_2, \left(\gamma^{-1} \cdot_h \rho_k\left(r\left(x_2\right)\right)\right)^{-1}\right)^{-1/2} \Delta_k\left(\gamma^{-1} \cdot_{kh} x_2, x_1\right)^{-1/2}
\end{aligned}
$$

Consequently,

$$
\hat{k}\left(\hat{h}(f)\, \xi_1\right) \xi_2(x_2) = \hat{kh}(f)\left(\hat{k}(\xi_1)\, \xi_2\right)(x_2), \text{ for all } x_2 \in G_2.
$$

■

For any locally compact, second countable, Hausdorff groupoid G endowed with a Haar system $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$, $C_c(G)$ is an algebra under convolution of function. For f, $g \in C_c(G)$ the convolution is defined by:

$$
f * g(x) = \int f(y)\, g\left(y^{-1}x\right) d\nu^{r(x)}(y)
$$

and the involution by

$$
f^*(x) = \overline{f\left(x^{-1}\right)}.
$$

Moreover, under these operations, $C_c(G)$ becomes a topological $*$-algebra. Let us note that the involutive algebraic structure on $C_c(G)$ defined above depends on the Haar system $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$. When it will be necessary to emphasis the role of ν in this structure, we shall write $C_c(G, \nu)$.

It is easy to see that for any f, $g \in C_c(G, \nu)$

$$
f * g = \hat{l}(f)\, g,
$$

where $l : (G, \nu) \longrightarrow (G, \nu)$ is the morphism defined in Example 10 : $\rho_l = id_{G^{(0)}}$ and $x \cdot_l y = xy$ (multiplication on G).

For each $f \in C_c(G)$, let us denote by $\|f\|_I$ the maximum of $\sup_t \int |f(x)| \, d\nu^t(x)$ and $\sup_t \int |f(x)| \, d\nu_t(x)$. A straightforward computation shows that $\|\cdot\|_I$ is a norm on $C_c(G)$ and

$$\begin{aligned} \|f\|_I &= \|f^*\|_I \\ \|f * g\|_I &\leq \|f\|_I \|g\|_I \end{aligned}$$

for all $f,\, g \in C_c(G)$.

Proposition 22 *Let $h : (\Gamma, \lambda) \longrightarrow (G, \nu)$ be a morphism of σ-compact lcH-groupoids with Haar systems. Then*

$$\xi_2^* * \left(\hat{h}(f)\, \xi_1\right) = \left(\hat{h}(f^*)\, \xi_2\right)^* * \xi_1$$

for all $f \in C_c(\Gamma)$ and $\xi_1,\, \xi_2 \in C_c(G, \nu)$.

Proof. If $f \in C_c(\Gamma)$ and $\xi_1,\, \xi_2 \in C_c(G)$, then

$$\xi_2^* * \left(\hat{h}(f)\, \xi_1\right)(x) =$$

$$= \int \xi_2^*(y) \int f(\gamma)\, \xi_1\left(\gamma^{-1} \cdot_h \left(y^{-1}x\right)\right) \Delta_h\left(y^{-1}x, \gamma\right)^{-1/2} d\lambda^{\rho_h\left(r\left(y^{-1}x\right)\right)}(\gamma)\, d\nu^{r(x)}(y)$$

$$= \int \overline{\xi_2\left(y^{-1}\right)} \int f(\gamma)\, \xi_1\left(\gamma^{-1} \cdot_h \left(y^{-1}x\right)\right) \Delta_h\left(y^{-1}, \gamma\right)^{-1/2} d\lambda^{\rho_h\left(r\left(y^{-1}x\right)\right)}(\gamma)\, d\nu^{r(x)}(y)$$

$$= \int \int \overline{\xi_2(y)} f(\gamma)\, \xi_1\left(\gamma^{-1} \cdot_h (yx)\right) \Delta_h(y, \gamma)^{-1/2} d\lambda^{\rho_h(r(yx))}(\gamma)\, d\nu_{r(x)}(y)$$

The change of variable $(y,\, \gamma) \to \left(\gamma^{-1} \cdot_h y,\, \gamma^{-1}\right)$ transforms the preceding integral into

$$= \int \int \overline{\xi_2\left(\gamma^{-1} \cdot_h y\right)} f\left(\gamma^{-1}\right) \Delta_h(y, \gamma)^{-1/2} d\lambda^{\rho_h(r(y))}(\gamma)\, \xi_1(yx)\, d\nu_{r(x)}(y)$$

$$= \int \int \overline{\xi_2\left(\gamma^{-1} \cdot_h y^{-1}\right)} f\left(\gamma^{-1}\right) \Delta_h\left(y^{-1}, \gamma\right)^{-1/2} d\lambda^{\rho_h(d(y))}(\gamma)\, \xi_1\left(y^{-1}x\right) d\nu^{r(x)}(y)$$

$$= \int \overline{\int f^*(\gamma)\, \xi_2\left(\gamma^{-1} \cdot_h y^{-1}\right) \Delta_h\left(y^{-1},\, \gamma\right)^{-1/2} d\lambda^{\rho_h(d(y))}(\gamma)} \xi_1\left(y^{-1}x\right) d\nu^{r(x)}(y)$$

$$= \left(\hat{h}(f^*)\, \xi_2\right)^* * \xi_1$$

■

4 Representations associated to morphisms

Proposition 23 *Let $h : (\Gamma, \lambda) \longrightarrow (G, \nu)$ be a morphism of σ-compact, lcH-groupoids with Haar systems. For $t \in G^{(0)}$ and $f \in C_c(\Gamma)$, let us define the operator $\pi_{h,t}(f) : L^2(G,\, \nu_t) \longrightarrow L^2(G,\, \nu_t)$ by*

$$\pi_{h,t}(f)\, \xi(x) = \int f(\gamma)\, \xi\left(\gamma^{-1} \cdot_h x\right) \Delta_h(x, \gamma)^{-1/2} d\lambda^{\rho_h(r(x))}(\gamma)$$

for all $\xi \in L^2(G, \nu_t)$ *and* $x \in G$*. Then for any* $f \in C_c(\Gamma)$

$$\|\pi_{h,t}(f)\| \le \|f\|_I ,$$

and $\pi_{h,t}$ *is a representation of* $C_c(\Gamma, \lambda)$ *(a* $*$*-homomorphism from* $C_c(\Gamma, \lambda)$ *into* $\mathrm{B}\left(L^2(G, \nu_t)\right)$*, that is continuous with respect to the inductive limit topology on* $C_c(\Gamma)$ *and the weak operator topology on* $\mathrm{B}\left(L^2(G, \nu_t)\right)$*).*

Proof. If $f \in C_c(\Gamma)$, $\xi, \zeta \in L^2(G, \nu_t)$, then

$$\begin{aligned}
&|\langle \pi_{h,t}(f)\xi, \zeta\rangle| = \\
&= \left|\int\int f(\gamma)\,\xi\left(\gamma^{-1}\cdot_h x\right)\Delta_h(x,\gamma)^{-1/2}\,d\lambda^{\rho_h(r(x))}(\gamma)\,\overline{\zeta(x)}d\nu_t(x)\right| \\
&\le \int\int |f(\gamma)|\left|\xi\left(\gamma^{-1}\cdot_h x\right)\right||\zeta(x)|\,\Delta_h(x,\gamma)^{-1/2}\,d\lambda^{\rho_h(r(x))}(\gamma)\,d\nu_t(x) \\
&\le \left(\int\int |f(\gamma)|\left|\xi\left(\gamma^{-1}\cdot_h x\right)\right|^2\Delta_h(x,\gamma)^{-1}\,d\lambda^{\rho_h(r(x))}(\gamma)\,d\nu_t(x)\right)^{1/2}\cdot \\
&\qquad\qquad \cdot\left(\int\int |f(\gamma)|\,|\zeta(x)|^2\,d\lambda^{\rho_h(r(x))}(\gamma)\,d\nu_t(x)\right)^{1/2} \\
&= \left(\int\int \left|f\left(\gamma^{-1}\right)\right||\xi(x)|^2\,d\lambda^{\rho_h(r(x))}(\gamma)\,d\nu_t(x)\right)^{1/2}\cdot \\
&\qquad\qquad \cdot\left(\int\int |f(\gamma)|\,d\lambda^{\rho_h(r(x))}(\gamma)\,|\zeta(x)|^2\,d\nu_t(x)\right)^{1/2} \\
&\le \|f\|_I\,\|\xi\|_2\,\|\zeta\|_2 .
\end{aligned}$$

Thus $\|\pi_{h,t}(f)\| \le \|f\|_I$ for any $f \in C_c(\Gamma)$. Let us prove that $\pi_{h,t} : C_c(\Gamma) \longrightarrow B\left(L^2(G, \nu_t)\right)$ is a $*$-homomorphism. Let $f \in C_c(\Gamma)$, $\xi, \zeta \in L^2(G, \nu_t)$. We have

$$\begin{aligned}
&\langle \pi_{h,t}(f)\xi, \zeta\rangle = \\
&= \int\int f(\gamma)\,\xi\left(\gamma^{-1}\cdot_h x\right)\Delta_h(x,\gamma)^{-1/2}\,d\lambda^{\rho_h(r(x))}(\gamma)\,\overline{\zeta(x)}d\nu_t(x) \\
&= \int\int f\left(\gamma^{-1}\right)\xi(x)\,\overline{\zeta\left(\gamma^{-1}\cdot_h x\right)}\Delta_h(x,\gamma)^{-1/2}\,d\lambda^{\rho_h(r(x))}(\gamma)\,d\nu_t(x) \\
&= \overline{\int\int f^*(\gamma)\,\zeta\left(\gamma^{-1}\cdot_h x\right)\Delta_h(x,\gamma)^{-1/2}\,\overline{\xi(x)}d\lambda^{\rho_h(r(x))}(\gamma)\,d\nu_t(x)} \\
&= \overline{\langle \pi_{h,t}(f^*)\zeta, \xi\rangle}.
\end{aligned}$$

Hence $\pi_{h,t}(f)^* = \pi_{h,t}(f^*)$ for all $f \in C_c(\Gamma)$. If $f, g \in C_c(\Gamma)$ and $\xi \in L^2(G, \nu_t)$, we have

$$\pi_{h,t}(f*g)\xi(x) =$$

$$= \int f*g(\gamma)\xi\left(\gamma^{-1}\cdot_h x\right)\Delta_h(x,\gamma)^{-1/2}\,d\lambda^{\rho_h(r(x))}(\gamma)$$

$$= \int\int f(\gamma_1)\,g\left(\gamma_1^{-1}\gamma\right)d\lambda^{r(\gamma)}(\gamma_1)\,\xi\left(\gamma^{-1}\cdot_h x\right)\Delta_h(x,\gamma)^{-1/2}\,d\lambda^{\rho_h(r(x))}(\gamma)$$

$$= \int\int f(\gamma_1)\,g\left(\gamma_1^{-1}\gamma\right)\xi\left(\gamma^{-1}\cdot_h x\right)\Delta_h(x,\gamma)^{-1/2}\,d\lambda^{\rho_h(r(x))}(\gamma_1)\,d\lambda^{\rho_h(r(x))}(\gamma)$$

$$= \int\int f(\gamma_1)\,g(\gamma)\,\xi\left((\gamma_1\gamma)^{-1}\cdot_h x\right)\Delta_h(x,\gamma_1\gamma)^{-1/2}\,d\lambda^{d(\gamma_1)}(\gamma)\,d\lambda^{\rho_h(r(x))}(\gamma_1)$$

$$= \int f(\gamma_1)\int g(\gamma)\,\xi\left(\gamma^{-1}\cdot_h\left(\gamma_1^{-1}\cdot_h x\right)\right)\Delta_h\left(\gamma_1^{-1}\cdot_h x,\gamma\right)^{-1/2}d\lambda^{d(\gamma_1)}(\gamma)\cdot$$
$$\cdot\Delta_h(x,\gamma_1)^{-1/2}\,d\lambda^{\rho_h(r(x))}(\gamma_1)$$

$$= \pi_{h,t}(f)\left(\pi_{h,t}(g)\xi\right)(x),$$

for all $x \in G$. Thus $\pi_{h,t}(f*g) = \pi_{h,t}(f)\,\pi_{h,t}(g)$ for all $f, g \in C_c(\Gamma)$. ■

Remark 24 *The representation* $\pi_{h,t} : C_c(\Gamma) \to \mathrm{B}\left(L^2(G,\nu_t)\right)$ *defined in the preceding proposition is non-degenerate in the sense that*

$$\left\{\pi_{h,t}(f)\xi :\ f \in C_c(\Gamma),\xi \in L^2(G,\nu_t)\right\}$$

is dense in $L^2(G,\nu_t)$. *For any* $f \in C_c(\Gamma)$ *and* $\xi \in C_c(G)$, $\pi_{h,t}(f)\xi = \hat{h}(f)\xi$, *where* $\hat{h}$ *is the application introduced in Section 3. Indeed, by Lemma 20*

$$\left\{\pi_{h,t}(f)\xi :\ f \in C_c(\Gamma),\xi \in C_c(G)\right\}$$

is dense in $C_c(G)$ *with the inductive limit topology and a fortiori with the* $L^2(G,\nu_t)$ *topology.*

Proposition 25 *Let* $h : (\Gamma,\lambda) \longrightarrow\!\!\triangleright\ (G_1,\nu)$ *and* $k : (G_1,\lambda) \longrightarrow\!\!\triangleright\ (G_2,\eta)$ *be morphisms of* σ*-compact lcH groupoids with Haar systems. Let* $\hat{h}$ *the application associated to* h *introduced in the Section 3, and* $\pi_{k,s}$ *(respectively,* $\pi_{kh,s}$*) be the representation associated to* k *(respectively,* kh*) defined in Proposition 23. Then*

$$\pi_{k,s}\left(\hat{h}(f)\,\xi_1\right)\xi_2 = \pi_{kh,s}(f)\,\pi_{k,s}(\xi_1)\,\xi_2$$

for all $s \in G_2^{(0)}$, $f \in C_c(\Gamma)$, $\xi_1 \in C_c(G_1)$ *and* $\xi_2 \in L^2(G_2,\eta_s)$.

Proof. It follows using the same computation as in the proof of Proposition 21. ■

5 A C^*-algebra associated to a locally compact σ-compact groupoid

In this section we define a C^*-algebra associated to a locally compact σ-compact groupoid. The construction is similar to that made in Section 5 [15]. Like in [15] we shall show that $(\Gamma,\lambda) \longrightarrow\!\!\triangleright\, C^*(\Gamma,\lambda)$ is a covariant functor from the category of locally compact, σ-compact, Hausdorff groupoids endowed with Haar systems to the category of C^*-algebras.

Definition 26 *Let (Γ,λ) be a σ-compact, lcH-groupoid endowed with a Haar and let $f \in C_c(\Gamma)$. For any morphism $h : (\Gamma,\lambda) \longrightarrow\!\!\triangleright\, (G,\nu)$ (where G is a σ-compact, lcH-groupoid endowed with a Haar system) let us define*

$$\|f\|_h = \sup_t \|\pi_{h,t}(f)\|$$

where $\pi_{h,t}$ is the representation associated to h and t defined in prop. 23. Let us also define

$$\|f\| = \sup_t \|f\|_h$$

where h runs over all morphism defined on (Γ,λ).

Remark 27 *Clearly $\|\cdot\| = \sup_h \|\cdot\|_h$ defined above is a C^*-semi-norm. Let $l : (\Gamma,\lambda) \longrightarrow\!\!\triangleright\, (\Gamma,\lambda)$ be the morphism defined in Example 10:$\rho_l = id_{\Gamma^0}$ and $\gamma \cdot_l x = \gamma x$ (multiplication on Γ). Then the representation associated to l and $u \in \Gamma^{(0)}$, $\pi_{h,u} : C_c(\Gamma) \to \mathrm{B}\left(L^2(G,\ \lambda_u)\right)$, is defined by*

$$\pi_{l,u}(f)\,\xi(x) = \int f(\gamma)\,\xi\left(\gamma^{-1}x\right) d\lambda^{r(x)}(\gamma) = f * \xi(x)$$

by for all $f \in C_c(\Gamma)$ and $\xi \in L^2(G,\ \lambda_u)$. Therefore for all $f \in C_c(\Gamma)$

$$\|f\|_l = \|f\|_{red}\ ,$$

the reduced norm of f (def. 2.36/p. 50 [7], or def. II.2.8./p. 82 [12]). According to prop. II.1.11/p. 58 [12], $\{\pi_{l,u}\}_u$ is a faithful family of representations of $C_c(\Gamma,\lambda)$, so $\|\cdot\|_{red}$ is a norm. Hence $\|\cdot\| = sup_h \|\cdot\|_h$ (where h runs over all morphism defined on (Γ,λ)) is a norm on $C_c(\Gamma,\lambda)$.

Definition 28 *Let (Γ,λ) be a σ-compact, lcH-groupoid endowed with Haar system. The C^*-algebra $C^*(\Gamma,\lambda)$ is defined to be the completion of $C_c(\Gamma,\lambda)$ in the norm $\|\cdot\| = sup_h \|\cdot\|_h$, where h runs over all morphism defined on (Γ,λ).*

Remark 29 *Let Γ be a locally compact, second countable, Hausdorff groupoid endowed with Haar system $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$. For $f \in C_c(\Gamma,\lambda)$, the full C^*-norm is defined by*

$$\|f\|_{full} = \sup_L \|L(f)\|$$

where L is a non-degenerate representation of $C_c(\Gamma,\lambda)$, i.e. a $*$-homomorphism from $C_c(\Gamma,\ \lambda)$ into $\mathrm{B}(H)$, for some Hilbert space H, that is continuous with respect to the inductive limit topology on $C_c(\Gamma)$ and the weak operator topology on $\mathrm{B}(H)$, and is such that the linear span of

$$\{L(g)\,\xi :\ g \in C_c(\Gamma),\ \xi \in H\}$$

is dense in H. We have the following inequalities

$$\|f\|_{red} \leq \|f\| \leq \|f\|_{full}$$

for all $f \in C_c(\Gamma, \lambda)$, where $\|\cdot\| = \sup_h \|\cdot\|_h$ is the norm introduced in Definition 26.The full C^* algebra $C^*_{full}(\Gamma, \lambda)$ and the reduced C^* algebra $C^*_{red}(\Gamma, \lambda)$ are defined respectively as the completion of the algebra $C_c(\Gamma, \lambda)$ for the full norm $\|\cdot\|_{full}$, and the reduced norm $\|\cdot\|_{red}$. According to Proposition 6.1.8/p.146 [1], if (Γ, λ) is measurewise amenable (Definition 3.3.1/p. 82 [1]), then $C^*_{full}(\Gamma, \lambda) = C^*_{red}(\Gamma, \lambda)$. Thus if (Γ, λ) is measurewise amenable, then $C^*(\Gamma, \lambda)$ (the C^*-algebra introduced in Definition 28), $C^*_{full}(\Gamma, \lambda)$ and $C^*_{red}(\Gamma, \lambda)$ coincide.

Notation 30 *Let Γ be a locally compact, second countable, Hausdorff groupoid endowed with Haar system $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$. Let μ be a quasi-invariant measure . Let $\Delta\mu$ be the modular function associated to $\left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ and μ. Let λ_1 be the measure induced by μ on Γ, and $\lambda_0 = \Delta_\mu^{-\frac{1}{2}}\lambda_1$.*

For $f \in L^1(G, \lambda_0)$ let us define

$$\|f\|_{II,\mu} = \sup\left\{\int |f(\gamma)\, j(d(\gamma)\, k(r(\gamma)))\,| d\lambda_0(\gamma)\right\}$$

the supremum being taken over all $j, k \in L^2\left(\Gamma^{(0)}, \mu\right)$ with $\int |j|^2 d\mu = \int |k|^2 d\mu = 1$.

Let

$$II(G, \lambda, \mu) = \left\{f \in L^1(G, \lambda_0), \|f\|_{II,\mu} < \infty\right\}.$$

$II(G, \lambda, \mu)$ is a Banach $*$-algebra under the , convolution

$$f * g(\gamma_1) = \int f(\gamma)\, g\left(\gamma^{-1}\gamma_1\right) d\lambda^{r(\gamma_1)}(\gamma)$$

and the involution

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

$C_c(\Gamma, \lambda)$ is a $*$-subalgebra of $II(G, \lambda, \mu)$ for any quasi-invariant measure μ. If μ_1 and μ_2 are two equivalent quasi-invariant measures, then $\|f\|_{II,\mu_1} = \|f\|_{II,\mu_2}$ for all f. Let us denote

$$\|f\|_{II} = \sup\left\{\|f\|_{II,\mu}\right\}$$

the supremum being taken over all quasi-invariant measure μ on $\Gamma^{(0)}$, and let us note that it is enough to consider one quasi-invariant measure in each class.

For any $f \in II(G, \lambda, \mu)$,

$$k \rightarrow \left(u \rightarrow \int f(\gamma)\, k(d(\gamma))\, d\lambda^u(\gamma)\right).$$

is a bounded operator $II_\mu(f)$ *on* $L^2\left(G^{(0)},\mu\right)$ *and* $\|II_\mu(|f|)\| = \|f\|_{II,\mu}$. *Moreover,* $f \to II_\mu(f)$ *is a norm-decreasing* $*$*-representation* II_μ *of* $II(G,\lambda,\mu)$. *The restriction of* II_μ *to* $C_c(\Gamma,\lambda)$ *is a representation of* $C_c(\Gamma,\lambda)$ *called the trivial representation on* μ.

Every representation $\left(\mu,\Gamma^{(0)} * \mathcal{H}, L\right)$ *(see Definition 3.20/p.68 [7]) of* Γ *can be integrated into a representation, still denoted by* L*, of* $II(G,\lambda,\mu)$. *The relation between the two representation is:*

$$\langle L(f)\xi_1,\xi_2\rangle = \int f(\gamma)\langle L(\gamma)\xi_1(d(\gamma)),\xi_2(r(\gamma))\rangle\lambda_0(\gamma)$$

where $f \in II(G,\lambda,\mu)$, $\xi_1,\xi_2 \in \int_{G^{(0)}}^{\oplus}\mathcal{H}(u)\,d\mu(u)$. *Conversely, every non-degenerate* $*$*-representation of any suitably large* $*$*-algebra of* $II(G,\lambda,\mu)$ *(in particular,* $C_c(\Gamma,\lambda)$*) is equivalent to a representation obtained this fashion (see Section 3 [5], Proposition II.1.17/p. 52[12], Proposition 4.2 [13] or Proposition 3.23/p. 70, Theorem 3.29/p. 74 [7]). If* L *is the integrated form of a representation,* $\left(\mu,G^{(0)} * \mathcal{H}, L\right)$*, of the groupoid* G*, then*

$$|\langle L(f)\xi,\eta\rangle| \leq \left\langle II_\mu(|f|)\widetilde{\xi},\widetilde{\eta}\right\rangle$$

where $\widetilde{\xi}(u) = \|\xi(u)\|$. *Therefore* $\|L(f)\| \leq \|II\mu(|f|)\| = \|f\|_{II,\mu} \leq \|f\|_{II}$.

If $f \in C_c(\Gamma,\lambda)$*, then*

$$\||f|\|_{II,\mu} = \|II_\mu(|f|)\| \leq \|(|f|)\|_{full} \leq \||f|\|_{II}.$$

Thus if $f \in C_c(\Gamma,\lambda)$ *and* $f \geq 0$*, then* $\|f\|_{II} = \|f\|_{full}$.

Proposition 31 *Let* Γ *be a locally compact, second countable, Hausdorff groupoid endowed with Haar system* $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$. *Let* $\|\cdot\|$ *be the norm on the* C^**-algebra introduced in Definition 28, and* $\|\cdot\|_{full}$ *the norm on the full* C^* *algebra associated with* Γ *and* λ *introduced by Renault in [12]. Let us assume the principal associated groupoid of* Γ *is a proper groupoid. Then for any quasi invariant measure* μ *on* $\Gamma^{(0)}$ *and any* $f \in C_c(\Gamma,\lambda)$

$$\|II_\mu(f)\| \leq \|f\|,$$

and if $f \geq 0$*, then* $\|f\| = \|f\|_{full} = \|II(f)\|$.

Proof. For each $f \in C_c(\Gamma,\lambda)$ we have $\|f\| \leq \|f\|_{full}$. If $f \in C_c(\Gamma,\lambda)$ and $f \geq 0$, then $\|f\|_{full} = \|f\|_{II} = \sup\left\{\|f\|_{II,\mu}\right\}$, the supremum being taken over all quasi-invariant measure μ on $\Gamma^{(0)}$. Let μ be a quasi-invariant measure. We have shown at the end of Subsection 2.2.1 that if the principal associated groupoid of Γ is a proper groupoid, then there is a quasi-invariant measure μ_0 equivalent to μ such that the modular function of μ_0 is a continuous function δ_Γ. Let S_0 be the support of μ_0. Let us consider the action Γ on S_0 defined by $\rho : S_0 \to \Gamma^{(0)}$, $\rho(u) = u$ for all $u \in S_0$, and $\gamma \cdot d(\gamma) = r(\gamma)$ for all $\gamma \in$

$\Gamma|_{S_0}$. It is easy to see that μ_0 is a quasi-invariant measure for the Haar system $\nu = \{\varepsilon_u \times \lambda^u, u \in S_0\}$ on $S_0 \rtimes \Gamma$, and its modular function is δ_Γ. Thus we can define a morphism $h_\mu : (\Gamma, \lambda) \to (S_0 \times S_0, \nu)$ in the sense of Definition 8 (where $\nu = \{\varepsilon_u \times \mu_0, u \in S_0\}$) by

1. $\rho_h : S_0 \to \Gamma^{(0)}, \rho(u) = u$ for all $u \in S_0$.
2. $\gamma \cdot_{h_\mu} (u, v) = (r(\gamma), v)$

The representations $\pi_{h_\mu, u}$ associated to the morphism $h_\mu : (\Gamma, \lambda) \longrightarrow\!\!\triangleright (S_0 \times S_0, \nu)$ as in Proposition 23, can be identified with the trivial representation II_μ of $C_c(\Gamma, \lambda)$. Hence for any $f \in C_c(\Gamma, \lambda)$

$$\|f\| \geq \|f\|_{h_\mu} = \|II_\mu(f)\|.$$

Therefore $f \in C_c(\Gamma, \lambda)$ and $f \geq 0$, then

$$\|f\|_{full} = \|f\|_{II} = \sup_\mu \|f\|_{II,\mu} = \sup_\mu \|II_\mu(f)\| = \sup_\mu \|f\|_{h_\mu} \geq \|f\|.$$

■

The following propositions are slightly modified version of props.5.2/p. 27 and 5.3/p. 27 [15].

Proposition 32 *Let $h : (\Gamma, \lambda) \longrightarrow\!\!\triangleright (G, \nu)$ be a morphism of σ-compact, lcH-groupoids with Haar systems and let $\hat{h}$ be the mapping defined in Section 3.Then $\hat{h}$ extends to $*$-homomorphism*

$$C^*(h) : C^*(\Gamma, \lambda) \to M(C^*(G, \nu)),$$

where $M(C^*(G, \nu))$ is multiplier algebra of $C^*(G, \nu)$, with the property that $C^*(h)(C^*(\Gamma, \lambda))\, C^*(G, \nu)$ is dense in $C^*(G, \nu)$.

Proof. Let G_1 be a locally compact, σ-compact, Hausdorff groupoid endowed with a Haar system $\eta = \left\{\eta^s,\ s \in G_1^{(0)}\right\}$ and let $k : (G, \nu) \longrightarrow\!\!\triangleright (G_1, \eta)$ be a morphism. Let $\{\pi_{k,s}\}_s$ be the family of representations defined in Proposition 23. According to Proposition 25, for all $s \in G_1^{(0)}$, $f \in C_c(\Gamma)$, $\xi_1 \in C_c(G)$ and $\xi_2 \in L^2(G_1, \eta_s)$,

$$\pi_{k,s}\left(\hat{h}(f)\,\xi_1\right)\xi_2 = \pi_{kh,s}(f)\,\pi_{k,s}(\xi_1)\,\xi_2.$$

Hence for all $s \in G_1^{(0)}$, $f \in C_c(\Gamma)$ and $\xi \in C_c(G)$,

$$\left\|\pi_{k,s}\left(\hat{h}(f)\,\xi\right)\right\| = \|\pi_{kh,s}(f)\|\,\|\pi_{k,s}(\xi)\| \leq \|f\|_{C^*(\Gamma,\lambda)}\,\|\xi\|_{C^*(G,\nu)}$$

Let $\|\cdot\|_{C^*(\Gamma,\lambda)}$ (and respectively, $\|\cdot\|_{C^*(G,\nu)}$) be the norm on the C^*-algebra introduced in Definition 28. Thus

$$\left\|\hat{h}(f)\,\xi\right\|_{C^*(G,\nu)} \leq \|f\|_{C^*(\Gamma,\lambda)}\,\|\xi\|_{C^*(G,\nu)}$$

Since $C_c(G)$ is dense in $C^*(G,\nu)$, it follows that $\hat{h}(f)$ extends to bounded linear map $C^*(h)(f)$ on $C^*(G,\nu)$. By Proposition 22, for all $f \in C_c(\Gamma)$ and $\xi_1,\, \xi_2 \in C_c(G,\nu)$,

$$\xi_2^* * \left(\hat{h}(f)\,\xi_1\right) = \left(\hat{h}(f^*)\,\xi_2\right)^* * \xi_1.$$

Using the density of $C_c(G)$ in $C^*(G,\nu)$ and the continuity of $C^*(h)(f)$ we have

$$\xi_2^* * \left(C^*(h)(f)\,\xi_1\right) = \left(C^*(h)(f^*)\,\xi_2\right)^* * \xi_1,$$

for all $\xi_1,\, \xi_2 \in C^*(G,\nu)$. Hence $C^*(h)(f)$ admits a Hermitian adjoint $C^*(h)(f^*)$, and therefore $C^*(h)(f) \in M(C^*(G,\nu))$. Since $C_c(\Gamma)$ is dense in $C^*(\Gamma,\lambda)$, it follows that $C^*(h)$ extends to $C^*(\Gamma,\lambda)$.

By Lemma 20

$$\left\{\hat{h}(f)\,\xi : f \in C_c(\Gamma)\,,\, \xi \in C_c(G)\right\}$$

is dense in $C_c(G)$ with the inductive limit topology and a fortiori in $C^*(G,\nu)$. Therefore $C^*(h)(C^*(\Gamma,\lambda))\,C^*(G,\nu)$ is dense in $C^*(G,\nu)$. ■

Proposition 33 *Let Γ, G_1 and G_2 be locally compact, σ-compact, Hausdorff groupoids. Let $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ (respectively, $\nu = \left\{\nu^t, t \in G_1^{(0)}\right\}$, $\eta = \left\{\eta^s, s \in G_2^{(0)}\right\}$) be a Haar system on Γ (respectively, on G_1, G_2). Let $h : (\Gamma,\lambda) \longrightarrow\!\!\triangleright\, (G_1,\nu)$ and $k : (G_1,\lambda) \longrightarrow\!\!\triangleright\, (G_2,\eta)$ be morphisms. Then*

$$C^*(kh) = C^*(k)\,C^*(h)\,.$$

Proof. Let us denote by $\hat{C}^*(k) : M(C^*(G_1,\nu)) \longrightarrow\!\!\triangleright\, M(C^*(G_2,\eta))$ the unique extension of $C^*(k)$. In order to show that $C^*(kh) = \hat{C}^*(k)\,C^*(h)$, it is enough to prove that $C^*(kh)(f) = \hat{C}^*(k)(C^*(h))(f)$ for $f \in C_c(\Gamma)$. As a consequence of Lemma 20

$$\left\{\hat{k}(\xi_1)\,\xi_2 : \xi_1 \in C_c(G_1)\,,\, \xi_2 \in C_c(G_2)\right\}$$

is dense in $C^*(G_2,\eta)$. Thus for proving $C^*(kh)(f) = \hat{C}^*(k)\,C^*(h)(f)$ it is enough to prove

$$\begin{aligned} \hat{kh}(f)\left(\hat{k}(\xi_1)\,\xi_2\right) &= \hat{C}^*(k)\left(\hat{h}(f)\right)\hat{k}(\xi_1)\,\xi_2 \\ &= \hat{k}\left(\hat{h}(f)\,\xi_1\right)\xi_2 \end{aligned}$$

but this is true (see Proposition 21). ■

Remark 34 *We have constructed a covariant functor $(\Gamma,\lambda) \to C^*(\Gamma,\lambda)$, $h \to C^*(h)$ from the category of locally compact, σ-compact, Hausdorff groupoids endowed with Haar systems to the category of C^*-algebras (in the sense of [18]). The hypothesis of σ-compactness is not really necessary. It is enough to work with groupoids which are locally compact and normal, and for which the unit spaces have conditionally-compact neighborhoods (for instance, paracompact unit spaces).*

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