BORELL MORPHISMS AND C^*-ALGEBRAS

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Abstract. In this paper we shall introduced a notion of Borel morphism of locally compact Hausdorff second countable groupoids endowed with Haar systems (extending the notion of continuous morphism from [2]) and we shall construct a C^*-algebra in terms of these morphisms. The same as in [2] we shall need a compatibility condition between the morphisms and the Haar systems. This time the compatibility condition will be completely expressed in terms of quasi-invariant measures.

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Contents

Introduction

Zakrzewski introduced in [13] (in the algebraic setting and in [14] in the differential setting) a notion of morphism between groupoids which reduces to a group homomorphism if groupoids are groups and to an ordinary map in the reverse direction if groupoids are sets. A morphism in the sense of Zakrzewski is a relation satisfying some additional properties. In [2] we have introduced a notion of continuous morphism in the settings of locally compact σ-compact groupoids endowed with Haar systems starting from the notion introduced in [13, 14], and we have constructed a covariant functor from the category of locally compact σ-compact groupoids (with morphisms in the sense of [2]) to a category of C^*-algebras in

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the sense of [12]. We have proved that a morphism from Γ to G in the sense of Zakrzewski [13] can be viewed as a left action of Γ on G, which commutes with multiplication on G.

In [1] we have proved that a morphism from Γ to G in the sense of Zakrzewski [13] is determined and determines a right action of Γ on $G^{(0)}$ and a groupoid homomorphism (in the usual sense) from the action groupoid $G^{(0)} \times \Gamma$ to G. Groupoids together with the morphisms defined in terms of groupoid actions form a category in algebraic setting as well as in topological setting. We have proved that the isomorphisms of the resulted categories can be identified with the groupoid isomorphisms in the usual sense.

In order to define a C^*-algebra associated to a locally compact σ-compact groupoid and to obtain a covariant functor from the category of locally compact groupoids endowed with Haar systems to a category of C^*-algebras in the sense of [12], in [2] we have needed a compatibility condition between the Haar systems and the morphisms.

In this paper we shall introduced a notion of Borel morphism of locally compact Hausdorff second countable groupoids endowed with Haar systems (extending the notion of continuous morphism from [2]) and we shall construct a C^*-algebra in terms of these morphisms. The same as in [2] we shall need a compatibility condition between the morphisms and the Haar systems. We shall prove that working with Borel morphism we can weaken the compatibility condition. Let us recall that in [2] the compatibility undertook the existence of a continuous positive function on a groupoid such that its restrictions to certain subgroupoids were modular functions for different measures. The existence of such function was difficult to verify with the exception of particular classes of groupoids. In the case of Borel morphisms the compatibility with the Haar systems will be expressed in terms of quasi-invariant measures.

1. Groupoids and actions

The purpose of this section is to give the reader a source of some basic information about analysis on groupoids needed on this paper and to introduce the notation.

There are many equivalent definitions of the (algebraic) groupoid. We shall use the same definition as in [7] . A *groupoid* is a set Γ endowed with a *product map (multiplication)*

$$(x, y) \to xy \ \left[: \Gamma^{(2)} \to \Gamma\right]$$

where $G^{(2)}$ is a subset of $G \times G$ called the *set of composable pairs*, and an *inverse map*

$$x \to x^{-1} \ [: \Gamma \to \Gamma]$$

such that the following conditions hold:

(1) If $(x, y) \in \Gamma^{(2)}$ and $(y, z) \in \Gamma^{(2)}$, then $(xy, z) \in \Gamma^{(2)}$, $(x, yz) \in \Gamma^{(2)}$ and $(xy)\, z = x\,(yz)$.
(2) $\left(x^{-1}\right)^{-1} = x$ for all $x \in \Gamma$.

(3) For all $x \in \Gamma$, $\left(x, x^{-1}\right) \in \Gamma^{(2)}$, and if $(z, x) \in \Gamma^{(2)}$, then $(zx)\, x^{-1} = z$.
(4) For all $x \in \Gamma$, $\left(x^{-1}, x\right) \in \Gamma^{(2)}$, and if $(x, y) \in \Gamma^{(2)}$, then $x^{-1}(xy) = y$.

The maps r and d on Γ, defined by the formulae $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$, are called the *range* and the (*domain*) *source* maps. It follows easily from the definition that they have a common image called the unit space of G, which is denoted $\Gamma^{(0)}$. Its elements are units in the sense that $xd(x) = r(x)\, x = x$. It is useful to note that a pair (x, y) lies in $\Gamma^{(2)}$ precisely when $d(x) = r(y)$, and that the cancellation laws hold (e.g. $xy = xz$ iff $y = z$). The fibers of the range and the source maps are denoted $\Gamma^u = r^{-1}(\{u\})$ and $\Gamma_v = d^{-1}(\{v\})$, respectively.

Let Γ and G be groupoids. A function $\varphi : \Gamma \to G$ is a groupoid *homomorphism* if $(\varphi(\gamma_1), \varphi(\gamma_2)) \in G^{(2)}$ and $\varphi(\gamma_1)\varphi(\gamma_2) = \varphi(\gamma_1\gamma_2)$ whenever $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$. Let us note that since $\varphi\left(\gamma_1^{-1}\right)\varphi(\gamma_1)\varphi(\gamma_2) = \varphi\left(\gamma_1^{-1}\gamma_1\gamma_2\right) = \varphi(\gamma_2)$, $\varphi\left(\gamma_1^{-1}\right) = \varphi(\gamma_1)^{-1}$. Hence $\varphi\left(\gamma_1\gamma_1^{-1}\right) = \varphi(\gamma_1)\varphi(\gamma_1)^{-1} \in G^{(0)}$. In the sequel we shall denote by $\varphi^{(0)} : \Gamma^{(0)} \to G^{(0)}$ the restriction of φ to $\Gamma^{(0)}$. A homomorphism $\varphi : \Gamma \to G$ is a groupoid *isomorphism* if φ is bijective.

A *topological groupoid* consists of a groupoid Γ and a topology compatible with the groupoid structure. This means that:

(1) $x \to x^{-1}$ $[: \Gamma \to \Gamma]$ is continuous.
(2) (x, y) $\left[: \Gamma^{(2)} \to \Gamma\right]$ is continuous where $\Gamma^{(2)}$ has the induced topology from $\Gamma \times \Gamma$.

Let Γ and G be topological groupoids. By a *topological homomorphism* from Γ to G we shall mean a homomorphism $\varphi : \Gamma \to G$ which is continuous. A topological homomorphism $\varphi : \Gamma \to G$ is a groupoid *isomorphism in the topological setting* if φ is a homeomorphism.

By a locally compact Hausdorff second countable groupoid we shall mean a topological groupoid whose topology is locally compact Hausdorff and second countable.

If X is a locally compact space, $C_c(X)$ denotes the space of complex-valuated continuous functions with compact support.

By a Borel space $(X, \mathcal{B}(X))$ we mean a space X, together with a σ-algebra $\mathcal{B}(X)$ of subsets of X, called Borel sets. A subspace of a Borel space $(X, \mathcal{B}(X))$ is a subset $S \subset X$ endowed with the relative Borel structure, namely the σ-algebra of all subsets of S of the form $S \cap E$, where E is a Borel subset of X. $(X, \mathcal{B}(X))$ is called countably separated if there is a sequence $(E_n)_n$ of sets in $\mathcal{B}(X)$ separating the points of X: i.e., for every pair of distinct points of X there is $n \in \mathbf{N}$ such that E_n contains one point but not both.

A function from one Borel space into another is called Borel if the inverse image of every Borel set is Borel. A one-one onto function Borel in both directions is called Borel isomorphism.

The Borel sets of a topological space are taken to be the σ-algebra generated by the open sets. $(X, \mathcal{B}(X))$ is called standard if it is Borel isomorphic to a Borel subset of a complete separable metric space. $(X, \mathcal{B}(X))$ is called analytic if it is countably separated and if it is the image of a Borel function from a standard space.

Any locally compact Hausdorff second countable space is a an analytic space.

A *Haar system* on locally compact Hausdorff-groupoid Γ is a family of positive Radon measures on Γ, $\lambda = \left\{\lambda^u,\, u \in \Gamma^{(0)}\right\}$, such that

(1) For all $u \in \Gamma^{(0)}$, $supp(\lambda^u) = \Gamma^u$.
(2) For all $f \in C_c(\Gamma)$,
$$u \to \int f(x)\, d\lambda^u(x) \quad \left[: \Gamma^{(0)} \to \mathbf{C}\right]$$
is continuous
(3) For all $f \in C_c(\Gamma)$ and all $x \in \Gamma$,
$$\int f(y)\, d\lambda^{r(x)}(y) = \int f(xy)\, d\lambda^{d(x)}(y).$$

Unlike the case of locally compact group, Haar system on groupoid need not exists, and if it does, it will not usually be unique. The continuity assumption 2) has topological consequences for Γ. It entails that the range map $r : \Gamma \to \Gamma^{(0)}$, and hence the domain map $d : \Gamma \to \Gamma^{(0)}$ is open (prop. I.4 [11]). *Therefore, in this paper we shall always assume that $r : \Gamma \to \Gamma^{(0)}$ is an open map.* For each λ^u, we denote by λ_u the image of λ^u by the inverse map $x \to x^{-1}$ (i.e. $\int f(y)\, d\lambda_u(y) = \int f\left(y^{-1}\right) d\lambda^u(y), \quad f \in C_c(\Gamma)$) .

If μ is a Radon measure on $\Gamma^{(0)}$, then the measure $\lambda^\mu = \int \lambda^u d\mu(u)$, defined by
$$\int f(y)\, d\lambda^\mu(y) = \int \left(\int f(y)\, d\lambda^u(y)\right) d\mu(u), \quad f \in C_c(\Gamma)$$
is called the *measure on* Γ *induced by* μ. The image of λ^μ by the inverse map $x \to x^{-1}$ is denoted $(\lambda^\mu)^{-1}$. μ is said *quasi-invariant* if its induced measure λ^μ is equivalent to its inverse $(\lambda^\mu)^{-1}$. A measure belonging to the class of a quasi-invariant measure is also quasi-invariant. We say that the class is *invariant*.

If μ is a quasi-invariant measure on $\Gamma^{(0)}$ and λ^μ is the measure induced on G, then the Radon-Nikodym derivative $\Delta = \frac{d\lambda^\mu}{d(\lambda^\mu)^{-1}}$ is called the *modular function of* μ. According to Corollary 3.14/p.19 [3], there is a μ-conull Borel subset U_0 of $\Gamma^{(0)}$ such that the restriction of Δ to $\Gamma|_{U_0}$ is a homomorphism.

We are exclusively concerned with topological groupoids which are second countable, locally compact Hausdorff. It was shown in [10] that measured groupoids may be assume to have locally compact topologies, with no loss in generality. Any second countable, locally compact Hausdorff groupoid will be seen as a Borel (analytic) groupoid (Borel sets are taken to be the σ-algebra generated by the open sets.

The notion of action that we shall use in this paper differ from the standard one).

Definition 1. *Let Γ be a groupoid and X be a set. We say Γ acts (to the left) on X if there is a map $\rho : X \to \Gamma^{(0)}$ (called a momentum map) and a map $(\gamma, x) \to \gamma \cdot x$ from*
$$\Gamma *_\rho X = \{(\gamma, x) : d(\gamma) = \rho(x)\}$$
to X, called (left) action, such that:

(1) *$\rho(\gamma \cdot x) = r(\gamma)$ for all $(\gamma, x) \in \Gamma *_\rho X$.*
(2) *$\rho(x) \cdot x = x$ for all $x \in X$.*
(3) *If $(\gamma_2, \gamma_1) \in \Gamma^{(2)}$ and $(\gamma_1, x) \in \Gamma *_\rho X$, then $(\gamma_2\gamma_1) \cdot x = \gamma_2 \cdot (\gamma_1 \cdot x)$.*

If Γ is a topological groupoid and X is a topological space, then we say that a left action is continuous if the maps ρ and $(\gamma, x) \to \gamma \cdot x$ are continuous, where $\Gamma *_\rho X$ is endowed with the relative product topology coming from $\Gamma \times X$.

If Γ and X are Borel spaces, then we say that a left action is Borel if the maps ρ and $(\gamma, x) \to \gamma \cdot x$ are Borel.

The difference with the definition of action in [5] is that we do not assume that the momentum map is surjective and open. However, the image of ρ is always a saturated subset of $\Gamma^{(0)}$. Indeed, let $v \sim u = \rho(x)$ and let $\gamma \in \Gamma$ be such that $r(\gamma) = v$ and $d(\gamma) = u$. Then v belongs to the image of ρ because $v = r(\gamma) = \rho_h(\gamma \cdot x)$.

The action is called *free* if $(\gamma, x) \in \Gamma *_\rho X$ and $\gamma \cdot x = x$ implies $\gamma \in \Gamma^{(0)}$.

The continuous action is called *proper* if the map $(\gamma, x) \to (\gamma \cdot x, x)$ from $\Gamma *_\rho X$ to $X \times X$ is proper (i.e. the inverse image of each compact subset of $X \times X$ is a compact subset of $\Gamma *_\rho X$).

In the same manner, we define a *right action* of Γ on X, using a map $\sigma : X \to \Gamma^{(0)}$ and a map $(x, \gamma) \to x \cdot \gamma$ from

$$X *_\sigma \Gamma = \{(x, \gamma) : \sigma(x) = r(\gamma)\}$$

to X.

The simplest example of proper and free action is the case when the groupoid Γ acts upon itself by either right or left translation (multiplication).

If $(x, \gamma) \to x \cdot \gamma$ is a right action of Γ on X, then $X \times \Gamma$ can be viewed as groupoid under the operations

$$(x, \gamma)^{-1} = \left(x \cdot \gamma, \gamma^{-1}\right)$$

$$(x, \gamma_1)(x \cdot \gamma_1, \gamma_2) = (x, \gamma_1\gamma_2)$$

and it will be called *action groupoid*.

Definition 2. *Let Γ_1, Γ_2 be two groupoids and X be set. Let us assume that Γ_1 acts to the left on X with momentum map $\rho : X \to \Gamma_1^{(0)}$, and that Γ_2 acts to the right on X with momentum map $\sigma : X \to \Gamma_2^{(0)}$. We say that the action commute if*

(1) *$\rho(x \cdot \gamma_2) = \rho(x)$ for all $(x, \gamma_2) \in X *_\sigma \Gamma_2$ and $\sigma(\gamma_1 \cdot x) = \sigma(x)$ for all $(\gamma_1, x) \in \Gamma_1 *_\rho X$.*
(2) *$\gamma_1 \cdot (x \cdot \gamma_2) = (\gamma_1 \cdot x) \cdot \gamma_2$ for all $(\gamma_1, x) \in \Gamma_1 *_\rho X$, $(x, \gamma_2) \in X *_\sigma \Gamma_2$.*

2. Borel morphisms

In this section we recall the reformulation of the notion of Zakrzewski's morphism in terms of groupoid actions as in [2] and we state convention and notation.

Definition 3. *Let Γ and G be two groupoids. By an algebraic morphism from Γ to G we mean a left action of Γ on G which commutes with the multiplication on G.*

The morphism is said Borel if the action of Γ on G is Borel (assuming that Γ and G are Borel spaces).

If we have a morphism in the sense of the preceding definition and if $\rho : G \to \Gamma$ is the momentum map of the left action, then $\rho = \rho \circ r$. Indeed, for any $x \in G$, we have $\rho(x) = \rho\left(xx^{-1}\right) = \rho(r(x))$ because of the fact that left action of Γ on G commutes with the multiplication on G.

Therefore an algebraic morphism h from Γ to G is given by two maps

(1) $\rho_h : G^{(0)} \to \Gamma^{(0)}$

(2) $(\gamma, x) \to \gamma \cdot_h x$ from $\Gamma \star_h G$ to G, where

$$\Gamma \star_h G = \{(\gamma, x) \in \Gamma \times G : d(\gamma) = \rho_h(r(x))\}$$

satisfying the following conditions:

(1): $\rho_h(r(\gamma \cdot_h x)) = r(\gamma)$ for all $(\gamma, x) \in \Gamma \star_h G$.

(2): $\rho_h(r(x)) \cdot_h x = x$ for all $x \in G$.

(3): $(\gamma_1\gamma_2) \cdot_h x = \gamma_1 \cdot_h (\gamma_2 \cdot_h x)$ for all $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$ and all $(\gamma_2, x) \in \Gamma \star_h G$.

(4): $d(\gamma \cdot_h x) = d(x)$ for all $(\gamma, x) \in \Gamma \star_h G$.

(5): $(\gamma \cdot_h x_1) x_2 = \gamma \cdot_h (x_1 x_2)$ for all $(\gamma, x_1) \in \Gamma \star_h G$ and $(x_1, x_2) \in G^{(2)}$.

In the case of a continuous morphism the map ρ_h is a continuous map but not necessarily open or surjective. However, the image of ρ_h is always a saturated subset of $\Gamma^{(0)}$.

Remark 4. *Let h be an algebraic generalized morphism from Γ to G (in the sense of Definition 3). Then h is determined by ρ_h and the restriction of the action to*

$$\left\{(\gamma, t) \in \Gamma \times G^{(0)} : d(\gamma) = \rho_h(t)\right\} .$$

Indeed, using the condition 5, one obtains

$$\gamma \cdot_h x = (\gamma \cdot_h r(x))\, x$$

Let us also note that

$$(\gamma_1\gamma_2) \cdot_h x = ((\gamma_1\gamma_2) \cdot_h r(x))\, x = \gamma_1 \cdot_h (\gamma_2 \cdot_h r(x))\, x$$

$$= (\gamma_1 \cdot_h r(\gamma_2 \cdot_h r(x)))\,(\gamma_2 \cdot_h r(x))\, x.$$

Consequently, for any $\gamma \in \Gamma$ and any $t \in G^{(0)}$ with $\rho_h(t) = d(\gamma)$, we have

$$\left(\gamma^{-1} \cdot_h r(\gamma \cdot_h t)\right)(\gamma \cdot_h t) = \left(\gamma^{-1}\gamma\right) \cdot_h t = d(\gamma) \cdot_h t = \rho_h(t) \cdot_h t = t.$$

Thus for any $\gamma \in \Gamma$ and any $t \in G^{(0)}$ with $\rho_h(t) = d(\gamma)$,

$$(\gamma \cdot_h t)^{-1} = \gamma^{-1} \cdot_h r(\gamma \cdot_h t)\,.$$

Therefore, algebraically, the notion of morphism in the sense of Definition 3 is the same with that introduced in [13, p. 351]. *In order to prove the equivalence of these definitions, we can use* [8, Proposition 2.7/p. 5], *taking $f = \rho_h$ and $g(\gamma, t) = \gamma \cdot_h t$.*

Let us compare this notion of morphism with other notions.

Remark 5. *Let h from Γ to G be a Borel morphism of locally compact Hausdorff groupoids (in the sense of Definition 3). Then G is left Γ-space under the action $(\gamma, x) \to \gamma \cdot_h x$, and a right G-space under the multiplication on G.*

- (i)**:** *G is a (particular) correspondence from Γ to G in the sense of* [9] *if and only if the left action of Γ on G is continuous and proper and ρ_h is open and bijective (hence a homeomorphism).*
- (ii)**:** *G is a (particular) correspondence from G (with the left action $(x, y) \to yx^{-1}$) to Γ (with the right action $(x, \gamma) \to \gamma^{-1}x$) in the sense of* [9] *if and only if the left action of Γ on G is continuous and proper and the map $(\gamma, t) \to \gamma \cdot_h t$ is surjective from $\left\{(\gamma, t) \in \Gamma \times G^{(0)} : d(\gamma) = \rho_h(t)\right\}$ to G.*
- (iii)**:** *G is a (particular) regular bibundle $\Gamma - G$ in the sense of* [4, Definition 6/p.103] *if and only if the action of Γ is continuous, free and transitive along the fibres of d (this means that for all $t \in G^{(0)}$ and x satisfying $d(x) = t$, there is $\gamma \in \Gamma$ such that $\gamma \cdot_h t = x$).*
- (iv)**:** *G is a (particular) regular bibundle $G - \Gamma$ (where, G acts on G to the left by $(x, y) \to yx^{-1}$ and Γ acts on G to the right by the action $(x, \gamma) \to \gamma^{-1}x$) in the sense of* [4, Definition 6/p.103] *if and only if the action of Γ is continuos and proper and the map ρ_h is injective.*

Therefore, the notion of Borel morphism introduced in Definition 3 is not covered by the notions used in [9] *and* [4].

Remark 6. *Let $\varphi : \Gamma \to G$ be a groupoid homomorphism (in the usual sense). Let us assume that $\varphi^{(0)} : \Gamma^{(0)} \to G^{(0)}$ is a bijective map. Then φ can be viewed as a morphism in the sense of Definition 3. Indeed, let $\rho_h : G^{(0)} \to \Gamma^{(0)}$ be an inverse for $\varphi^{(0)}$, and let us define*

$$\gamma \cdot_h x = \varphi(\gamma) x$$

for any $(\gamma, x) \in \{(\gamma, x) \in \Gamma \times G : d(\gamma) = \rho_h(r(x))\}$. Thus we obtain a morphism in the sense of Definition 3. Similarly, any Borel homomorphism $\varphi : \Gamma \to G$ for which $\varphi^{(0)} : \Gamma^{(0)} \to G^{(0)}$ is a Borel isomorphism can be viewed as a Borel morphism from Γ to G.

Lemma 7. *Let Γ and G be two groupoids and let h be an algebraic morphism from Γ to G (in the sense of Definition 3). Then the function*

$$(\gamma, t) \to \gamma \cdot_{h_0} t := r(\gamma \cdot_h t)$$

from $\left\{(\gamma, t) \in \Gamma \times G^{(0)} : d(\gamma) = \rho_h(t)\right\}$ to $G^{(0)}$ defines an action of Γ to $G^{(0)}$ with the momentum map ρ_h.

Proof. See Lemma 5 [2]. □

Notation 8. *Let Γ and G be two groupoids and let h be an algebraic morphism from Γ to G (in the sense of Definition 3). Let us denote*

$$\begin{aligned} G \rtimes_h \Gamma &= \{(x, \gamma) \in G \times \Gamma : \rho_h(r(x)) = r(\gamma)\} \\ G^{(0)} \rtimes_{h_0} \Gamma &= \left\{(t, \gamma) \in G^{(0)} \times \Gamma : \rho_h(t) = r(\gamma)\right\} \end{aligned}$$

$G \rtimes_h \Gamma$, respectively $G^{(0)} \rtimes_{h_0} \Gamma$, can be viewed as groupoid under the operations

$$(x,\gamma)^{-1} = \left(\gamma^{-1} \cdot_h x, \gamma^{-1}\right)$$
$$(x,\gamma_1)\left(\gamma_1^{-1} \cdot_h x, \gamma_2\right) = (x, \gamma_1\gamma_2)$$

respectively,

$$(t,\gamma)^{-1} = \left(\gamma^{-1} \cdot_{h_0} t, \gamma^{-1}\right)$$
$$(t,\gamma_1)\left(\gamma_1^{-1} \cdot_{h_0} t, \gamma_2\right) = (t, \gamma_1\gamma_2).$$

(where $\gamma \cdot_{h_0} t := r(\gamma \cdot_h t)$ as in preceding lemma).

3. Groupoids endowed with Haar systems

If Γ and G are *locally compact Hausdorff second countable*-groupoids, then $G \rtimes_h \Gamma$ and $G^{(0)} \rtimes_{h_0} \Gamma$ are *locally compact Hausdorff second countable*-groupoids. If $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ is a Haar system on Γ and if the morphism h is continuous, then $\left\{\varepsilon_x \times \lambda^{\rho_h(r(x))}, x \in G\right\}$ is a Haar system on $G \rtimes_h \Gamma$ (where ε_x is the unit point mass at x) and $\left\{\varepsilon_t \times \lambda^{\rho_h(t)}, t \in G^{(0)}\right\}$ is a Haar system on $G^{(0)} \rtimes_{h_0} \Gamma$.

Definition 9. *Let Γ and G be two* locally compact Hausdorff second countable-*groupoids. Let $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ (respectively, $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$) be a Haar system on Γ (respectively, on G). Let h be a Borel morphism (in the sense of Definition 3) from Γ to G. We say that h is compatible with (λ,ν) if for each $t \in G^{(0)}$ the measure ν_t is quasi-invariant with respect to the Haar system $\left\{\varepsilon_x \times \lambda^{\rho_h(r(x))},\ x \in G\right\}$ on $G \rtimes_h \Gamma$.*

Remark 10. *Any continuous morphism h from Γ to G which is compatible with (λ,ν) in the sense of* [2, Definition 8/ p. 8] *is also compatible with (λ,ν) in the sense of Definition 9.*

Proposition 11. *Let Γ and G be two* locally compact Hausdorff second countable-*groupoids. Let $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ (respectively, $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$) be a Haar system on Γ (respectively, on G). If h is Borel morphism compatible with (λ,ν) in the sens of Definition 9, then there exists a Borel positive function Δ_h on*

$$G \rtimes_h \Gamma = \{(x,\gamma) \in G \times \Gamma : \rho_h(r(x)) = r(\gamma)\}$$

such that

$$\int\int g(\gamma^{-1} \cdot_h x, \gamma^{-1}) \Delta_h(\gamma^{-1} \cdot_h x, \gamma^{-1}) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x) =$$

$$= \int\int g(\gamma, x) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x)$$

for all $t \in G^{(0)}$ and all Borel nonnegative functions g on $G \rtimes_h \Gamma$.

Proof. The fact that for each $t \in G^{(0)}$ the measure ν_t is quasi-invariant with respect to the Haar system $\left\{\varepsilon_x \times \lambda^{\rho_h(r(x))},\ x \in G\right\}$ on $G \rtimes_h \Gamma$ implies that for each t there is a positive Borel function Δ_t on $G \rtimes_h \Gamma$ (the modular function of ν_t) such that

$$\int\int g(\gamma^{-1} \cdot_h x, \gamma^{-1})\Delta_t(\gamma^{-1} \cdot_h x, \gamma^{-1})d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x) =$$

$$= \int\int g(\gamma, x)d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x)$$

for all Borel nonnegative functions g on $G \rtimes_h \Gamma$.

It is enough to prove that there exists a Borel positive function Δ_h on $G \rtimes_h \Gamma$ such that for all $t \in G^{(0)}$ the restrictions of Δ_h and Δ_t at $G_t \rtimes_h \Gamma$ coincide $\varepsilon_x \times \lambda^{\rho_h(t)}$-a.e. for ν_t-a.a. $x \in G$.

Firstly let us show that there is continuous system of finite Borel positive measures $\left\{\mu^u, u \in \Gamma^{(0)}\right\}$ such that for each $u \in \Gamma^{(0)}$, λ^u and μ^u are equivalent measures (have the same null sets). Let $(K_n)_n$ be an increasing sequence of compact sets with $\bigcup_n K_n = \Gamma$. For each n, let $f_n : \Gamma \to [0,1]$ be a continuous compactly supported function such that $f_n(\gamma) = 1$ for all $\gamma \in K_n$. Let $a_n(u) = \frac{1}{2^n \lambda^u(f_n)}$ if $\lambda^u(f_n) > 1$, and $a_n(u) = \frac{1}{2^n}$ otherwise. It is not hard to see that $u \to a_n(u)$ is continuous. Let

$$P_1(u, \gamma) = \sum_n a_n(u) f_n(\gamma) \text{ for all } \gamma \in \Gamma$$

Since $|a_n(u) f_n(\gamma)| \le \frac{1}{2^n}$, it follows that $(u, x) \to \sum_n a_n(u) f_n(\gamma)$ is uniformly convergent and therefore $(u, \gamma) \to P_1(u, \gamma)$ is continuous. Thus, for all f in $C_c(G)$,

$$u \to \int f(x) P_1(u, \gamma)\, d\lambda^u(\gamma) \quad \left[: \Gamma^{(0)} \to \mathbf{R}\right]$$

is a compactly supported function. Let us define

$$\mu^u(A) = \int 1_A(\gamma) P_1(u, \gamma)\, d\lambda^u(\gamma)$$

for all relatively compact Borel subsets A of Γ and each $u \in \Gamma^{(0)}$. Then μ^u is a finite measure. Since $P_1(u, \gamma) > 0$ for all (u, γ), it follows that λ^u and μ^u are equivalent measures.

Secondly let us show that there is Borel system of finite positive measures $\{\tilde{\mu}^x, x \in G\}$ such that for each $x \in G$, $\varepsilon_x \times \lambda^{\rho_h(r(x))}$ and $\varepsilon_x \times \tilde{\mu}^x$ are equivalent measures and $\Delta_t \in L^1(\varepsilon_x \times \tilde{\mu}^x)$ or ν_t-a.a. $x \in G$. For each natural number m let us denote

$$E(t, m) = \left\{(x, \gamma) \in G_t \rtimes_h \Gamma \ :\ \Delta_t(x, \gamma) < m\right\}.$$

Then $(E(t, m))_m$ be an increasing sequence of compact sets with $\bigcup_m E(t, m) = G_t \rtimes_h \Gamma$. Let $b_m(x) = 1/\left(m 2^m \lambda^{\rho_h(r(x))}(E(d(x), m))\right)$ if $\lambda^{\rho_h(r(x))}(E(d(x), m)) > 1$, and $b_m(x) = \frac{1}{m 2^m}$ otherwise. Let

$$P_2(x, \gamma) = \sum_m b_m(x) 1_{E(d(x), m)}(x, \gamma) \text{ for all } \gamma \in \Gamma$$

Since $\left|b_m(x)\,1_{E(d(x),m)}(x,\gamma)\right| \leq \frac{1}{2^m}$, it follows that $(u,x) \to \sum_m b_m(x)\,1_{E(d(x),m)}(x,\gamma)$ is uniformly convergent and therefore $P_2(u,\gamma) < \infty$. Moreover P_2 is a Borel function. Let us define

$$\tilde{\mu}^x(g) = \int g(\gamma)\,P_2(x,\gamma)\,d\mu^{\rho_h(r(x))}(\gamma)$$

for all Borel nonnegative functions g on Γ and all $x \in G$. Then $\tilde{\mu}^x$ is a finite positive measure. Since $P_2(x,\gamma) > 0$ for all (x,γ), it follows that $\varepsilon_x \times \tilde{\mu}^{\rho_h(r(x))}$ and $\varepsilon_x \times \tilde{\mu}^x$ are equivalent measures. Also it is easy to see that

$$\int \Delta_t(x,\gamma)\,d\mu^x(\gamma) \leq 1 \text{ for all } t \in G^{(0)} \text{ and all } x \in G_t.$$

Let $\mathcal{K}$ be a countable base of compact sets for the topology of Γ. Let $lin(\mathcal{K})$ be the set of linear combinations of characteristic functions of sets in $\mathcal{K}$ with coefficients from $\mathbf{Q}[i]$ ($\mathbf{Q}$ with $\sqrt{-1}$ adjoined). Since $lin(\mathcal{K})$ is a countable set, we can list it as a sequence k_1, k_2, ... For each $\varepsilon > 0$ and each $x \in G$ let us define $j(\varepsilon,x)$ be the least element of

$$\left\{n : \int \left|\Delta_{d(x)}(x,\gamma) - k_n(\gamma)\right|\,d\tilde{\mu}^x(\gamma) < \varepsilon\right\}$$

Now let us prove that $x \to j(\varepsilon,x)$ is a Borel function. It is enough to prove that for each bounded Borel function h on Γ

$$\left\{x \in G \;:\; \int \left|\Delta_{d(x)}(x,\gamma) - h(\gamma)\right|\,d\tilde{\mu}^x(\gamma) < \varepsilon\right\}.$$

But this is a consequence of the fact that

$$\int \left|\Delta_{d(x)}(x,\gamma) - h(\gamma)\right|\,d\tilde{\mu}^x(\gamma) = \sup_{k \in lin(\mathcal{K})} \int \left(\Delta_{d(x)}(x,\gamma) - h(\gamma)\right) k(\gamma)\,d\tilde{\mu}^x(\gamma),$$

and the fact that $lin(\mathcal{K})$ is a countable.

Let us define $g_\varepsilon(x) = k_{j(\varepsilon,x)}$ and $d_\varepsilon(x,\gamma) = g_\varepsilon(x)(\gamma) = k_{j(\varepsilon,x)}(\gamma)$. Let us apply this process to $x \to \Delta_{d(x)}$ for $\varepsilon = \frac{1}{2}$ and obtain g_1 and d_1. Then let us replace $x \to \Delta_{d(x)}$ with $x \to \Delta_{d(x)} - g_1(x)$ and apply the process with $\epsilon = \frac{1}{2^2}$ to obtain g_2 and d_2, etc. For each n we have

$$\left\|\Delta_{d(x)} - (g_1(x) + g_2(x) + ... + g_n(x)\right\|_1 \leq \frac{1}{2^n}. \text{ (norm in } L^1(\tilde{\mu}^x)\text{)}$$

Therefor for each $n \geq 2$, $\left\|g_n(x)\right\|_1 \leq \frac{3}{2^n}$. It follows that for each x the sum $\sum_n d_n(x,\gamma)$ is finite for μ^x-a.a. γ. The set

$$N = \left\{(x,\gamma) \in G \times \Gamma \;:\; \sum_n d_n(x,\gamma) = \infty\right\}$$

is a Borel set. Let us change d_1 to be 1 on N and d_n to be 0 on N for $n \geq 2$. The function

$$\Delta_h(x,\gamma) = \sum_n d_n(x,\gamma)$$

satisfies the required conditions. □

Definition 12. *Let* Γ *and* G *be two* locally compact Hausdorff second countable-*groupoids and let* $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ *(respectively,* $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$*) be a Haar system on* Γ *(respectively, on* G*). If* h *is Borel morphism compatible with* (λ, ν) *in the sens of Definition 9, a Borel positive function* Δ_h *on* $G \rtimes_h \Gamma$*such that*

$$\int\int g(\gamma^{-1} \cdot_h x, \gamma^{-1})\Delta_h(\gamma^{-1} \cdot_h x, \gamma^{-1}) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x) =$$

$$= \int\int g(\gamma, x) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x)$$

for all $t \in G^{(0)}$ *and all Borel nonnegative functions* g *on* $G \rtimes_h \Gamma$ *will be* called modular function associated to (h, λ, ν).

Proposition 13. *Let* Γ *and* G *be two* locally compact Hausdorff second countable-*groupoids and let* $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ *(respectively,* $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$*) be a Haar system on* Γ *(respectively, on* G*). Let* h *is Borel morphism compatible with* (λ, ν) *in the sens of Definition 9. If* Δ_h *is a modular function associated to* (h, λ, ν) *then it satisfies* $\Delta_h(x, \gamma) = \Delta_h(r(x), \gamma)$ *for all* $t \in G^{(0)}$*,* ν^t*-a.a.* x *and* $\lambda^{\rho_h(r(x))}$*-a.a.* γ*.*

Proof. Let $f \geq 0$ be a Borel function on $G \rtimes_h \Gamma$ and $g \geq 0$ be a Borel function on G . For each $t \in G^{(0)}$, we have

$$\begin{aligned}
& \int g(z) f(x, \gamma)\, d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x)\, d\nu^t(z) \\
= & \int g(z) f\left(xz^{-1}, \gamma\right) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_{d(z)}(x)\, d\nu^t(z) \\
= & \int g(z) f\left(\gamma^{-1} \cdot_h xz^{-1}, \gamma^{-1}\right) \Delta_h\left(\gamma^{-1} \cdot_h x, \gamma^{-1}\right) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_{d(z)}(x)\, d\nu^t(z) \\
= & \int g(z) f\left(\left(\gamma^{-1} \cdot_h r(x)\right) xz^{-1}, \gamma^{-1}\right) \Delta_h\left(\left(\gamma^{-1} \cdot_h r(x)\right) x, \gamma^{-1}\right) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_{d(z)}(x)\, d\nu^t(z) \\
= & \int g(z) f\left(\left(\gamma^{-1} \cdot_h r(x)\right) x, \gamma^{-1}\right) \Delta_h\left(\left(\gamma^{-1} \cdot_h r(x)\right) xz, \gamma^{-1}\right) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x)\, d\nu^t(z) \\
= & \int g(z) f\left(\gamma^{-1} \cdot_h x, \gamma^{-1}\right) \Delta_h\left(\gamma^{-1} \cdot_h xz, \gamma^{-1}\right) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x)\, d\nu^t(z) \\
= & \int g(z) f(x, \gamma)\, \Delta_h(xz, \gamma)\, \Delta_h\left(\gamma^{-1} \cdot_h x, \gamma^{-1}\right) d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x)\, d\nu^t(z)
\end{aligned}$$

Thus for all $t \in G^{(0)}$ and and almost all $(x, \gamma, z) \in G_t \rtimes_h \Gamma \times G^t$,

$$\begin{aligned}
1 &= \Delta_h(xz, \gamma)\, \Delta_h\left(\gamma^{-1} \cdot_h x, \gamma^{-1}\right) \\
&= \Delta_h(xz, \gamma)\, \Delta_h(x, \gamma)^{-1}.
\end{aligned}$$

Therefore $\Delta_h(xz,\gamma) = \Delta_h(x,\gamma)$ for $\int \lambda^{\rho_h(r(y))} d\nu_t(y) \times \nu^t$-a.a.$(x,\gamma,z) \in G_t \rtimes_h \Gamma \times G^t$. Thus

$$\begin{aligned} 0 &= \int \left|\Delta_h(xz,\gamma) - \Delta_h(x,\gamma)\right| d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x)\, d\nu^t(z) \\ &= \int \left|\Delta_h(xz,\gamma) - \Delta_h(x,\gamma)\right| d\nu^t(z)\, d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x) \\ &= \int \left|\Delta_h(xz,\gamma) - \Delta_h(x,\gamma)\right| d\nu^{d(x)}(z)\, d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x) \\ &= \int \left|\Delta_h(z,\gamma) - \Delta_h(x,\gamma)\right| d\nu^{r(x)}(z)\, d\lambda^{\rho_h(r(x))}(\gamma)\, d\nu_t(x) \end{aligned}$$

Hence for $\int \lambda^{\rho_h(r(y))} d\nu_t(y)$-a.a.$(x,\gamma) \in G_t \rtimes_h \Gamma$,

$$\int \left|\Delta_h(z,\gamma) - \Delta_h(x,\gamma)\right| d\nu^{r(x)}(z) = 0.$$

Let P be a continuos positive function on G (constructed as in the proof of Proposition 11) such that

$$\int P(y)\, d\nu^t(y) = 1 \text{ for all } t \in G^{(0)}.$$

Since $P > 0$, for $\int \lambda^{\rho_h(r(y))} d\nu_t(y)$-a.a.$(x,\gamma) \in G_t \rtimes_h \Gamma$,

$$\int \left|\Delta_h(z,\gamma) - \Delta_h(x,\gamma)\right| P(z)\, d\nu^{r(x)}(z) = 0$$

and consequently,

$$\left|\int \left(\Delta_h(z,\gamma) - \Delta_h(x,\gamma)\right) P(z)\, d\nu^{r(x)}(z)\right| = 0$$

or equivalently

$$\int \Delta_h(z,\gamma)\, P(z)\, d\nu^{r(x)}(z) = \int \Delta_h(x,\gamma)\, P(z)\, d\nu^{r(x)}(z).$$

If we denote

$$R(t,\gamma) = \int \Delta_h(z,\gamma)\, P(z)\, d\nu^t(z),$$

then for $\int \lambda^{\rho_h(r(y))} d\nu_t(y)$-a.a.$(x,\gamma) \in G_t \rtimes_h \Gamma$we have

$$\begin{aligned} \Delta_h(x,\gamma) &= \int \Delta_h(x,\gamma)\, P(z)\, d\nu^{r(x)}(z) \\ &= \int \Delta_h(z,\gamma)\, P(z)\, d\nu^{r(x)}(z) \\ &= R(r(x),\gamma) \end{aligned}$$

Thus for ν^t-a.a.x and $\lambda^{\rho_h(r(x))}$-a.a. γ

$$\Delta_h(x,\gamma) = \Delta_h(r(x),\gamma).$$

□

Proposition 14. *Let Γ and G be two* locally compact Hausdorff second countable-*groupoids. Let $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ be a Haar system on Γ. Let h be Borel morphism from Γ to G (in the sense of Definition 3). If G is transitive, then we can choose a Haar system ν on G such that h is compatible with (λ, ν).*

Proof. We can use a similar reasoning as in Corollary 19 [2]. □

Example 15. *Let Γ (respectively, G) be a* locally compact Hausdorff second countable-*groupoid endowed with a Haar system $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ (respectively, $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$). Let h be a Borel morphism from Γ to G. Using similar reasoning as in 2.3* [2] *and the fact that in Borel settings we weakened the compatibility condition we can see that:*

1. *If G is a group bundle , then $\Gamma|_{\rho_h\left(G^{(0)}\right)}$ is also a group bundle and the h is always compatible with (λ, ν).*
2. *If Γ and G are locally compact groups, then the notion of Borel morphism reduces to the usual notion of Borel group homomorphism. The compatibility condition is automatically satisfied.*
3. *If G is a set, then h is uniquely determined by the map $\rho_h : G \to \Gamma^{(0)}$.*
4. *If $G = X \times X$ (the trivial groupoid on X endowed with the product topology), then any h gives rise to a Borel action of Γ on X. Conversely, any Borel action of Γ on X gives rise to a Borel morphism h from Γ to G. The compatibility condition is not automatically satisfied but for each Haar system λ on Γ we can choose a Haar system ν on G such that h is compatible with (λ, ν).*

4. Representations associated to morphisms

For any locally compact, second countable, Hausdorff groupoid G endowed with a Haar system $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$, $C_c(G)$ is an algebra under convolution of functions. For f, $g \in C_c(G)$ the convolution is defined by:

$$f * g(x) = \int f(y)\, g\left(y^{-1}x\right) d\nu^{r(x)}(y)$$

and the involution by

$$f^*(x) = \overline{f\left(x^{-1}\right)}.$$

Moreover, under these operations, $C_c(G)$ becomes a topological $*$-algebra. Let us note that the involutive algebraic structure on $C_c(G)$ defined above depends on the Haar system $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$. When it will be necessary to emphasis the role of ν in this structure, we shall write $C_c(G, \nu)$.

For each $f \in C_c(G)$, let us denote by $\|f\|_I$ the maximum of $\sup_t \int |f(x)|\, d\nu^t(x)$ and $\sup_t \int |f(x)|\, d\nu_t(x)$. A straightforward computation shows that $\|\cdot\|_I$ is a norm on $C_c(G)$ and

$$\begin{aligned} \|f\|_I &= \|f^*\|_I \\ \|f * g\|_I &\leq \|f\|_I \|g\|_I \end{aligned}$$

for all $f, g \in C_c(G)$.

Proposition 16. *Let* Γ *(respectively,* G*) be a* locally compact Hausdorff second countable-*groupoid endowed with a Haar system* $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ *(respectively,* $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$*). Let* h *be a Borel morphism from* Γ *to* G *compatible with* (λ, ν) *in the sens of Definition 9. For* $t \in G^{(0)}$ *and* $f \in C_c(\Gamma)$*, let us define the operator* $\pi_{h,t}(f) : L^2(G, \nu_t) \to L^2(G, \nu_t)$ *by*

$$\pi_{h,t}(f)\xi(x) = \int f(\gamma)\xi\left(\gamma^{-1} \cdot_h x\right)\Delta_h(x,\gamma)^{-1/2} d\lambda^{\rho_h(r(x))}(\gamma)$$

for all $\xi \in L^2(G, \nu_t)$ *and* $x \in G$*. Then for any* $f \in C_c(\Gamma)$

$$\|\pi_{h,t}(f)\| \leq \|f\|_I,$$

and $\pi_{h,t}$ *is a representation of* $C_c(\Gamma, \lambda)$ *(a* $*$*-homomorphism from* $C_c(\Gamma, \lambda)$ *into* $\mathrm{B}\left(L^2(G, \nu_t)\right)$*, that is continuous with respect to the inductive limit topology on* $C_c(\Gamma)$ *and the weak operator topology on* $\mathrm{B}\left(L^2(G, \nu_t)\right)$*).*

Proof. If $f \in C_c(\Gamma)$, $\xi, \zeta \in L^2(G, \nu_t)$, then

$$\begin{aligned}
&\left|\langle \pi_{h,t}(f)\xi, \zeta\rangle\right| = \\
&= \left|\int\int f(\gamma)\xi\left(\gamma^{-1} \cdot_h x\right)\Delta_h(x,\gamma)^{-1/2} d\lambda^{\rho_h(r(x))}(\gamma)\overline{\zeta(x)}d\nu_t(x)\right| \\
&\leq \int\int |f(\gamma)|\left|\xi\left(\gamma^{-1} \cdot_h x\right)\right||\zeta(x)|\Delta_h(x,\gamma)^{-1/2} d\lambda^{\rho_h(r(x))}(\gamma)d\nu_t(x) \\
&\leq \left(\int\int |f(\gamma)|\left|\xi\left(\gamma^{-1} \cdot_h x\right)\right|^2 \Delta_h(x,\gamma)^{-1} d\lambda^{\rho_h(r(x))}(\gamma)d\nu_t(x)\right)^{1/2} \cdot \\
&\qquad\qquad \cdot \left(\int\int |f(\gamma)||\zeta(x)|^2 d\lambda^{\rho_h(r(x))}(\gamma)d\nu_t(x)\right)^{1/2} \\
&= \left(\int\int \left|f\left(\gamma^{-1}\right)\right||\xi(x)|^2 d\lambda^{\rho_h(r(x))}(\gamma)d\nu_t(x)\right)^{1/2} \cdot \\
&\qquad\qquad \cdot \left(\int\int |f(\gamma)| d\lambda^{\rho_h(r(x))}(\gamma)|\zeta(x)|^2 d\nu_t(x)\right)^{1/2} \\
&\leq \|f\|_I \|\xi\|_2 \|\zeta\|_2 .
\end{aligned}$$

Thus $\|\pi_{h,t}(f)\| \leq \|f\|_I$ for any $f \in C_c(\Gamma)$. Let us prove that $\pi_{h,t} : C_c(\Gamma) \to B\left(L^2(G, \nu_t)\right)$ is a $*$-homomorphism. Let $f \in C_c(\Gamma)$, $\xi, \zeta \in L^2(G, \nu_t)$. We have

$$\begin{aligned}
&\langle \pi_{h,t}(f)\xi, \zeta\rangle = \\
&= \int\int f(\gamma)\xi\left(\gamma^{-1} \cdot_h x\right)\Delta_h(x,\gamma)^{-1/2} d\lambda^{\rho_h(r(x))}(\gamma)\overline{\zeta(x)}d\nu_t(x) \\
&= \int\int f\left(\gamma^{-1}\right)\xi(x)\overline{\zeta\left(\gamma^{-1} \cdot_h x\right)}\Delta_h(x,\gamma)^{-1/2} d\lambda^{\rho_h(r(x))}(\gamma)d\nu_t(x) \\
&= \overline{\int\int f^*(\gamma)\zeta\left(\gamma^{-1} \cdot_h x\right)\Delta_h(x,\gamma)^{-1/2}\overline{\xi(x)}d\lambda^{\rho_h(r(x))}(\gamma)d\nu_t(x)} \\
&= \overline{\langle \pi_{h,t}(f^*)\zeta, \xi\rangle}.
\end{aligned}$$

Hence $\pi_{h,t}(f)^* = \pi_{h,t}(f^*)$ for all $f \in C_c(\Gamma)$. If $f, g \in C_c(\Gamma)$ and $\xi \in L^2(G, \nu_t)$, we have

$$\begin{aligned}
&\pi_{h,t}(f * g)\xi(x) = \\
&= \int f * g(\gamma)\,\xi\left(\gamma^{-1} \cdot_h x\right)\Delta_h(x,\gamma)^{-1/2} d\lambda^{\rho_h(r(x))}(\gamma) \\
&= \int\int f(\gamma_1)\,g\left(\gamma_1^{-1}\gamma\right) d\lambda^{r(\gamma)}(\gamma_1)\,\xi\left(\gamma^{-1} \cdot_h x\right)\Delta_h(x,\gamma)^{-1/2} d\lambda^{\rho_h(r(x))}(\gamma) \\
&= \int\int f(\gamma_1)\,g\left(\gamma_1^{-1}\gamma\right)\xi\left(\gamma^{-1} \cdot_h x\right)\Delta_h(x,\gamma)^{-1/2} d\lambda^{\rho_h(r(x))}(\gamma_1)\,d\lambda^{\rho_h(r(x))}(\gamma) \\
&= \int\int f(\gamma_1)\,g(\gamma)\,\xi\left((\gamma_1\gamma)^{-1} \cdot_h x\right)\Delta_h(x,\gamma_1\gamma)^{-1/2} d\lambda^{d(\gamma_1)}(\gamma)\,d\lambda^{\rho_h(r(x))}(\gamma_1) \\
&= \int f(\gamma_1)\int g(\gamma)\,\xi\left(\gamma^{-1} \cdot_h \left(\gamma_1^{-1} \cdot_h x\right)\right)\Delta_h\left(\gamma_1^{-1} \cdot_h x, \gamma\right)^{-1/2} d\lambda^{d(\gamma_1)}(\gamma)\,\cdot \\
&\qquad\qquad \cdot\Delta_h(x,\gamma_1)^{-1/2} d\lambda^{\rho_h(r(x))}(\gamma_1) \\
&= \pi_{h,t}(f)\left(\pi_{h,t}(g)\xi\right)(x),
\end{aligned}$$

for all $x \in G$. Thus $\pi_{h,t}(f * g) = \pi_{h,t}(f)\,\pi_{h,t}(g)$ for all $f, g \in C_c(\Gamma)$. □

5. A C^*-algebra associated to a locally compact Hausdorff second countable groupoid

In this section we define a C^*-algebra associated to a locally compact Hausdorff second countable groupoid. The construction is similar to those made in Section 5 [8] and [2].

Definition 17. *Let* Γ *be a* locally compact Hausdorff second countable-*groupoid endowed with a Haar system* $\lambda = \left\{\lambda^u, u \in \Gamma^{(0)}\right\}$ *and* $f \in C_c(\Gamma)$. *For any* locally compact Hausdorff second countable-*groupoid* G *endowed with a Haar system* $\nu = \left\{\nu^t, t \in G^{(0)}\right\}$ *and any Borel morphism* h *from* Γ *to* G *compatible with* (λ, ν) *let us define*

$$\|f\|_h = \sup_t \|\pi_{h,t}(f)\|$$

where $\pi_{h,t}$ *is the representation associated to* h *and* t *defined in prop. 16. Let us also define*

$$\|f\| = \sup_h \|f\|_h$$

where h *runs over all morphism defined on* (Γ, λ).

Remark 18. *Clearly* $\|\cdot\| = \sup_h \|\cdot\|_h$ *defined above is a* C^**-semi-norm. Let us define a morphism* l_Γ *from* Γ *to* Γ *by* $\rho_{l_\Gamma} = id_{\Gamma^{(0)}}$ *and* $\gamma \cdot_{l_\Gamma} x = \gamma x$ *(multiplication on* Γ*). It is easy to check that the conditions in the Definition 3 are satisfied. Also a straightforward computation shows that* l_Γ *is compatible with* (λ, λ) *and that the modular function of* $(l_\Gamma, \lambda, \lambda)$ *is* $\Delta_{l_\Gamma} \equiv 1$. *Let Then the representation associated to* l *and* $u \in \Gamma^{(0)}$, $\pi_{h,u} : C_c(\Gamma) \to \mathrm{B}\left(L^2(G, \lambda_u)\right)$, *is defined by*

$$\pi_{l,u}(f)\,\xi(x) = \int f(\gamma)\,\xi\left(\gamma^{-1}x\right) d\lambda^{r(x)}(\gamma) = f * \xi(x)$$

by for all $f \in C_c(\Gamma)$ *and* $\xi \in L^2(G,\ \lambda_u)$. *Therefore for all* $f \in C_c(\Gamma)$

$$\|f\|_{l_\Gamma} = \|f\|_{red} ,$$

the reduced norm of f (Definition 2.36/p. 50 [6], *or Definition II.2.8./p. 82* [7]*). According to Proposition. II.1.11/p. 58* [7], $\{\pi_{l,u}\}_u$ *is a faithful family of representations of* $C_c(\Gamma,\lambda)$, *so* $\|\cdot\|_{red}$ *is a norm. Hence* $\|\cdot\| = sup_h \|\cdot\|_h$ *(where h runs over all morphism defined on* (Γ,λ)*) is a norm on* $C_c(\Gamma,\lambda)$.

Definition 19. *Let* Γ *be a* locally compact Hausdorff second countable-*groupoid endowed with a Haar system* $\lambda = \{\lambda^u, u \in \Gamma^{(0)}\}$. *The* C^**-algebra* $C^*(\Gamma,\lambda)$ *is defined to be the completion of* $C_c(\Gamma,\lambda)$ *in the norm* $\|\cdot\| = sup_h \|\cdot\|_h$, *where h runs over all morphism defined on* (Γ,λ).

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