Spectral Sets For Locally Bounded Operators

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Abstract

For a quotient bounded operator T on a locally convex space we define the set $\sigma(Q,T)$. If T is a locally bounded operators on a sequentially complete locally convex space then the equalities

$$r_{lb}(T) = \mid \sigma(T) \mid = \mid \sigma_{lb}(T) \mid = \mid \sigma(Q,T) \mid$$

holds and the spectral set $\sigma(T)$ is compact.

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1 Introduction

Let X be a locally convex Hausdorff space and $L(X)$ ($\mathcal{L}(X)$) denote the algebra of linear operators (continuous operators) from X into inself.

A family $\mathcal{P}$ of seminorms who generate the topology of a locally convex space X (in the sense that the topology of X is the coarsest whit respect to which all seminorms of $\mathcal{P}$ are continuous) will be called *a calibration on* X. A calibration on X will be principal if it is directed. The set of calibrations for X is denoted by $\mathcal{C}(X)$ and the set of principal calibration is denoted by $\mathcal{C}_0(X)$.

Any family of seminorms on a linear space is partially ordered by relation ,, $\leq$", where

$$p \leq q \Leftrightarrow p(x) \leq q(x), (\forall)\, x \in X.$$

A family of seminorms is preordered by the relation „ $\prec$", where

$$p \prec q \Leftrightarrow \text{ there exists some } r > 0 \text{ such that } p(x) \leq rq(x), \forall x \in X.$$

If $p \prec q$ and $q \prec p$, we write $p \approx q$.

Definition 1.1 *Two families $\mathcal{P}_1$ and $\mathcal{P}_2$ of seminorms on a linear space are called Q-equivalent (denoted $\mathcal{P}_1 \approx \mathcal{P}_2$) provided:*

1. *for each $p_1 \in \mathcal{P}_1$ there exists $p_2 \in \mathcal{P}_2$ such that $p_1 \approx p_2$;*

2. *For each $p_2 \in \mathcal{P}_2$ there exists $p_1 \in \mathcal{P}_1$ such that $p_2 \approx p_1$.*

It is obviously that two Q-equivalent and separating families of seminorms on a linear space generate the same locally convex topology.

If $(X, \mathcal{P}), (Y, \mathcal{Q})$ are locally convex spaces, then for all seminorms $p, q \in \mathcal{P}$ the application $m_{pq} : L(X, Y) \to \mathbf{R} \cup \{\infty\}$, defined by

$$m_{pq}(T) = \sup_{p(x)\neq 0} \frac{q\,(Tx)}{p\,(x)},$$

is called the *mixed operator seminorm* of T associated with p and q. When $X = Y$ and $p = q$ we use notation $\hat{p} = m_{pp}$.

Lemma 1.1 ([21]) *If $(X, \mathcal{P})$ is a locally convex spaces and $T \in L(X)$, then*

1. $m_{pq}(T) = \sup_{p(x)=1} q\,(Tx) = \sup_{p(x)\leq 1} q\,(Tx)\,, (\forall)\, p \in P, (\forall)\, q \in Q;$

2. $q\,(Tx) \leq m_{pq}\,(T)\, p\,(x),\ (\forall)\, x \in X,$ *whenever* $m_{pq}(T) < \infty$.

3. $m_{pq}(T) = \inf\{\ M > 0 \ \mid q\,(Tx) \leq Mp\,(\ x\ )\,, (\forall)\, x \in X\},$
whenever $m_{pq}(T) < \infty$.

Definition 1.2 *Let X be a locally convex space. An operator $T \in L(X)$ is:*

1. *quontient bounded operator with respect to a calibration $\mathcal{P} \in \mathcal{C}(X)$ if for every seminorm $p \in \mathcal{P}$ there exists $c_p > 0$ such that*

$$p\,(Tx) \leq c_p p\,(x)\,, (\forall)\, x \in X.$$

2. *universally bounded with respect to the calibration $\mathcal{P} \in \mathcal{C}(X)$ if there exists $c_0 > 0$ such that*

$$p(Tx) \leq c_0 p\,(x)\,, (\forall)\, x \in X, (\forall)\, p \in \mathcal{P}.$$

3. *locally bounded if maps some zero neighborhood in a bounded set.*

Definition 1.3 *If X is a locally convex space, then we denote by:*

1. *$Q_{\mathcal{P}}(X)$ the class of quontient bounded operators with respect to some calibration $\mathcal{P} \in \mathcal{C}(X)$;*

2. *$B_{\mathcal{P}}(X)$ the class of universally bounded operators with respect to some calibration $\mathcal{P} \in \mathcal{C}(X)$,*

3. *$\mathcal{LB}(X)$ the class of the locally bounded operators on X:*

4. *$\mathcal{LB}_0(X)$ the subalgebra of $\mathcal{L}(X)$ genrated by $\mathcal{LB}(X)$ and the identity operator I.*

Remark 1.1 *1. Let X be a locally convex space spaces. If $T \in \mathcal{LB}(X)$, then there exists some calibration $\mathcal{P} \in \mathcal{C}(X)$ such that*

$$T \in B_{\mathcal{P}}(X) \subset Q_{\mathcal{P}}(X) \subset \mathcal{L}(X).$$

2. If X is a seminormed space then $Q_{\mathcal{P}}(X) = B_{\mathcal{P}}(X) = \mathcal{LB}(X)$

The class of locally bounded operators is an algebra and it will be usually equipped with the topology of uniform convergence on a zero neighborhood. We say that a sequence (S_n) converge uniformly to zero on a zero neighborhood U if for each zero neighborhood V there exists a positive index $n_0 \in \mathbf{N}$ such that

$$S_n(U) \subset V, (\forall)\, n \geq n_0.$$

In terms of operator seminorms we have the following definition:

Definition 1.4 *Let X be a locally convex space. A sequence $(S_n)_n \subset \mathcal{LB}(X)$ converges uniformly to zero on some zero-neighborhood if for each principal calibration $\mathcal{P} \in \mathcal{C}_0(X)$ there exists some seminorm $p \in \mathcal{P}$ such that for every $q \in \mathcal{P}$ and every $\epsilon > 0$ there exists an index $n_{q,\epsilon} \in \mathbf{N}$, with the property*

$$m_{pq}(S_n) < \epsilon, (\forall)\, n \geq n_{q,\epsilon}.$$

A family $G \subset \mathcal{LB}(X)$ is uniformly bounded on some zero-neighborhood if there exists some seminorm $p \in \mathcal{P}$ such that for every $q \in \mathcal{P}$ there exists $\epsilon_q > 0$ with the property

$$m_{pq}(S) < \epsilon_q, (\forall)\, S \in G.$$

If X is a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$, then for every $p \in \mathcal{P}$ the application $\hat{p} : Q_{\mathcal{P}}(X) \to \mathbf{R}$ defined by

$$\hat{p}(T) = \inf\{r > 0 \mid p(Tx) \leq r\, p(x)\,, (\forall)\, x \in X\ \},$$

is a submultiplicative seminorm on $Q_{\mathcal{P}}(X)$, satisfying $\hat{p}(I) = 1$. We denote by $\hat{\mathcal{P}}$ the family $\{\ \hat{\mathcal{P}} \mid p \in \mathcal{P}\ \}$.

Proposition 1.1 ([10]) *Let X is a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$.*

1. *$Q_{\mathcal{P}}(X)$ is a unital subalgebra of the algebra of continuous linear operators on X*

2. *$Q_{\mathcal{P}}(X)$ is a unitary l.m.c.-algebra with respect to the topology determined by $\hat{\mathcal{P}}$*

3. *If $\mathcal{P}' \in \mathcal{C}(X)$ such that $\mathcal{P} \approx \mathcal{P}'$, then $Q_{\mathcal{P}'}(X) = Q_{\mathcal{P}}(X)$ and $\hat{\mathcal{P}} \approx \hat{\mathcal{P}}\ '$*

4. *The topology generated by $\hat{\mathcal{P}}$ on $Q_{\mathcal{P}}(X)$ is finer than the topology of uniform convergence on bounded subsets of X*

Definition 1.5 *If $(X, \mathcal{P})$ is a locally convex space and $T \in Q_{\mathcal{P}}(X)$ we denote by $r_{\mathcal{P}}(T)$ the radius of boundness of operator T in $Q_{\mathcal{P}}(X)$, i.e.*

$$r_{\mathcal{P}}(T) = \inf\{\alpha > 0 \ \mid \ \alpha^{-1}T \text{ generates a bounded semigroup in } Q_{\mathcal{P}}(X)\}.$$

We have said that $r_{\mathcal{P}}(T)$ is the $\mathcal{P}$-spectral radius of the operator T.

Proposition (1.1) implies that for each $\mathcal{P}' \in \mathcal{C}(X)$, $\mathcal{P} \approx \mathcal{P}'$, we have $Q_{\mathcal{P}\ '}(X) = Q_{\mathcal{P}}(X)$, so if H is a Q-equivalence class in $\mathcal{C}(X)$, then

$$r_{\mathcal{P}}(T) = r_{\mathcal{P}'}(T), (\forall)\, \mathcal{P}, \mathcal{P}' \in H.$$

Proposition 1.2 ([20]) *If X is a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$, then for each $T \in Q_{\mathcal{P}}(X)$ we have:*

$$r_{\mathcal{P}}(T) = \inf\left\{\lambda > 0 \mid \lim_{n\to\infty} \frac{T^n}{\lambda^n} = 0 \right\}.$$

Proposition 1.3 ([20]) *Let X be a sequentially complete locally convex space and $\mathcal{P} \in \mathcal{C}(X)$. If $T \in Q_{\mathcal{P}}(X)$, then $\mid \sigma(Q_{\mathcal{P}}, T) \mid = r_{\mathcal{P}}(T)$.*

Definition 1.6 *Given a linear operator T on a topological vector space X, we consider*

$$r_{lb}(T) = \inf\left\{\nu > 0 \left| \frac{T^n}{\nu^n} \to 0 \text{ uniformly on some zero neigborhood} \right.\right\}$$

Lemma 1.2 *If $\mathcal{P}$ a calibration on X, then $B_{\mathcal{P}}(X)$ is a unital normed algebra with respect to the norm $\|\bullet\|_{\mathcal{P}}$ defined by*

$$\|T\|_{\mathcal{P}} = \inf\{M > 0 \mid \ p\,(Tx) \leq Mp\,(x)\,, (\forall)\, x \in X, (\forall)\ p \in \mathcal{P}\}.$$

Corollary 1.1 *If $\mathcal{P} \in \mathcal{C}(X)$, then for each $T \in B_{\mathcal{P}}(X)$ we have*

$$\|T\|_{\mathcal{P}} = \sup\{\hat{p}(T) \mid p \in \mathcal{P}\}, (\forall)\, T \in B_{\mathcal{P}}(X).$$

Proof. Let be $T \in B_{\mathcal{P}}(X)$ and $\alpha = \sup\{\ \hat{p}(T) \mid p \in \mathcal{P}\}$. From the definition of the norm $\|\bullet\|_{\mathcal{P}}$ and of the operatorial seminorm $\hat{p}$, $p \in \mathcal{P}$, results that $\|T\|_{\mathcal{P}} \leq \alpha$.

If $\|T\|_{\mathcal{P}} < \alpha$, then there exists some seminorm $p \in \mathcal{P}$ such that

$$\|T\|_{\mathcal{P}} < \hat{p}(T) \leq \alpha.$$

which implies that there exists some element $x_0 \in X$ for which we have

$$p(Tx_0) > \|T\|_{\mathcal{P}}\, p\,(x_0)\,.$$

Since, this relation contradicts the definition of the norm $\|T\|_{\mathcal{P}}$, results that $\alpha \leq \|T\|_{\mathcal{P}}$.

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Proposition 1.4 ([10]) *Let X be a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$. Then:*

1. *$B_{\mathcal{P}}(X)$ is a subalgebra of $\mathcal{L}(X)$;*

2. *$(B_{\mathcal{P}}(X), \|\bullet\|_{\mathcal{P}})$ is unitary normed algebra;*

3. *for each $\mathcal{P}' \in \mathcal{C}(X)$, with the property $\mathcal{P} \approx \mathcal{P}'$, we have*

$$B_{\mathcal{P}}(X) = B_{\mathcal{P}'}(X) \text{ and } \|\bullet\|_{\mathcal{P}} = \|\bullet\|_{\mathcal{P}'}\,.$$

Proposition 1.5 ([4]) *Let X be a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$. Then:*

1. *the topology given by the norm $\|\bullet\|_{\ \mathcal{P}}$ on the algebra $B_{\mathcal{P}}(X)$ is finer than the topology of uniform convergence;*

2. *if $(T_n)_n$ is a Cauchy sequences in $(B_{\mathcal{P}}(X), \|\bullet\|_{\mathcal{P}})$ which converges to an operator T, we have $T \in B_{\mathcal{P}}(X)$;*

3. *the algebra $(B_{\mathcal{P}}(X), \|\bullet\|_{\mathcal{P}})$ is complete if X is sequentially complete.*

Proposition 1.6 ([10]) *Let $(X, \mathcal{P})$ be a locally convex space. An operator $T \in Q_{\mathcal{P}}(X)$ is bounded in the algebra $Q_{\mathcal{P}}(X)$ if and only if there exists some calibration $\mathcal{P}' \in \mathcal{C}(X)$ such that $\mathcal{P} \approx \mathcal{P}'$ and $T \in B_{\mathcal{P}'}(X)$.*

Definition 1.7 *Let $(X, \mathcal{P})$ be a locally convex space.*

1. *If $T \in Q_{\mathcal{P}}(X)$ $(T \in B_{\mathcal{P}}(X))$ we said that $\alpha \in \mathbf{C}$ is in the resolvent set $\rho(Q_{\mathcal{P}}, T)$ $(\rho(B_{\mathcal{P}}, T))$ if there exists $(\alpha I - T)^{-1} \in Q_{\mathcal{P}}(X)$ $((\alpha I - T)^{-1} \in B_{\mathcal{P}}(X))$. The spectral set $\sigma(Q_{\mathcal{P}}, T)$ $(\sigma(B_{\mathcal{P}}, T))$will be the complementary set of $\rho(Q_{\mathcal{P}}, T)$ $(\rho(B_{\mathcal{P}}, T))$.*

2. *Let T be a locally bounded operator on locally convex space X. We say that $\lambda \in \rho_{lb}(T)$ if there exists a scalar α and a locally bounded operator S on X such that $(\lambda I - T)^{-1} = \alpha I + S$.*

Remark 1.2 *1.*

2. *The set $\rho_{lb}(T)$ will be the spectrum of T in the algebra $\mathcal{LB}_0(X)$. The spectral set $\sigma_{lb}(T)$ is defined to be the complement of the resolvent set $\rho_{lb}(T)$.*

3. *It is obvious that if $T \in B_{\mathcal{P}}(X)$, then we have the following inclusions*

$$\sigma(T) \subset \sigma(Q_{\mathcal{P}}, T) \subset \sigma(B_{\mathcal{P}}, T).$$

Proposition 1.7 ([4]) *Proposition If $(X, \mathcal{P})$ is a locally convex space and $T \in B_{\mathcal{P}}(X)$, then the set $\sigma(B_{\mathcal{P}}, T)$ is compact.*

Corollary 1.2 *If $(X, \mathcal{P})$ is a locally convex space and $T \in (Q_{\mathcal{P}}(X))_0$ then there exist $c_T > 0$ such that*

$$\hat{p}(T) \leq c_T, (\forall)\, p \in \mathcal{P}.$$

Given $(X, \mathcal{P})$, for each $p \in \mathcal{P}$ denote by N^p the null space and by X_p the quotient space X/N^p. For each $p \in \mathcal{P}$ consider the natural mapping

$$x \to x_p \equiv x + N^p$$

(from X to X_p). It is obvious that X_p is a normed space, for each $p \in \mathcal{P}$, with norm $|| \bullet ||_p$ defined by

$$\|x_p\|_p = p\,(x)\,, (\forall)\, x \in X.$$

Consider the algebra homomorphism $T \to T^p$ of $Q_{\mathcal{P}}(X)$ into $\mathcal{L}(X_p)$ defined by

$$T^p\,(x_p) = (Tx)_p, (\forall)\, x \in X.$$

This operators are well defined because $T(N^p) \subset N^p$. Moreover, for each $p \in \mathcal{P}, \mathcal{L}(X_p)$ is a unital normed algebra and we have

$$\begin{aligned} \|T_p\|_p &= \sup \left\{ \|T_p x_p\|_p \;\middle|\; \|x_p\|_p \leq 1 \text{ for } x_p \in X_p \right\} \\ &= \sup \left\{ p\left(Tx\right) \;\middle|\; p\left(x\right) \leq 1 \text{ for } x \in X \right\} \end{aligned}$$

For $p \in \mathcal{P}$ consider that the normed space $(\tilde{X}_p, \|\bullet\|_p)$ is the completition of $(X_p, \|\bullet\|_p)$. If $T \in Q_{\mathcal{P}}(X)$, then the operator T^p has a unique continuous linear extension $\tilde{T}^p$ on $(\tilde{X}_p, \|\bullet\|_p)$.

Definition 1.8 *If X is a locally convex space and $T \in Q_{\mathcal{P}}(X)$, we denote by $\sigma(Q,T)$ the set*

$$\cap\{\sigma(Q_{\mathcal{P}}, T) \mid \mathcal{P} \in \mathcal{C}(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\}.$$

Lemma 1.3 *If X is a locally convex space and $T \in Q_{\mathcal{P}}(X)$ then*

$$\mid \sigma(Q,T) \mid \leq \inf\{r_{\mathcal{P}}(T) \mid \mathcal{P} \in \mathcal{C}(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\}.$$

Proof. This is a direct consequence of the proposition (1.3). ■

Definition 1.9 *An operator T is quotient bounded operator on a locally convex space X if there exists some calibration $\mathcal{P}$ on X such that $T \in Q_{\mathcal{P}}(X)$.*

Remark 1.3 *An operator T is quotient bounded on a locally convex space X if and only if there exists some calibration $\mathcal{P} \in \mathcal{C}(X)$ such that $\hat{p}(T)$ is finit for each $p \in \mathcal{P}$.*

Lemma 1.4 *If T is a quotient bounded operator on a locally convex space X, then there exists some principal calibration $\mathcal{P}' \in \mathcal{C}_0(X)$ such that $T \in Q_{\mathcal{P}'}(X)$.*

Proof. Let $\mathcal{P}$ be a calibration on X such that $T \in Q_{\mathcal{P}}(X)$ and denote by $\mathcal{P}'$ the set of all seminorms given by the relations

$$p'\left(x\right) = \max_{i=\overline{1,n}} p_i\left(x\right), (\forall)\, x \in X,$$

where $p_i \in \mathcal{P}', i = \overline{1,n}$, and $n \in \mathbf{N}$.

Let $p' \in \mathcal{P}'$ be arbitrary chosen. Since $T \in Q_{\mathcal{P}}(X)$, from previous remark and lemma (1.1) results that

$$p_i(Tx) \leq \hat{p}_i(T) p_i\left(x\right), (\forall)\, x \in X, i = \overline{1,n},$$

If $c_{p'} = \max_{i=\overline{1,n}} \hat{p}_i(T)$, then

$$p_i(Tx) \leq c_{p'} p_i(x) \leq c_{p'} p'(x), (\forall)\, x \in X, i = \overline{1,n},$$

so

$$p'(Tx) \leq c_{p'} p'(x), (\forall)\, x \in X,$$

Therefore, $T \in Q_{\mathcal{P}'}(X)$.

■

Lemma 1.5 *If X is a locally convex space and $T \in Q_{\mathcal{P}}(X)$ then*

$$\inf\{r_{\mathcal{P}}(T) \mid \mathcal{P} \in \mathcal{C}_0(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\} =$$

$$= \inf\{r_{\mathcal{P}}(T) \mid \mathcal{P} \in \mathcal{C}(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\}.$$

Proof. Assume that $\mathcal{P} \in \mathcal{C}(X)$ such that $T \in Q_{\mathcal{P}}(X)$.

If $\mid \lambda \mid > r_{\mathcal{P}}(T)$, then the family $\left(\frac{T^n}{\lambda^n}\right)_{n\geq 0}$ is bounded in $Q_{\mathcal{P}}(X)$, i.e. for every $p \in \mathcal{P}$ there exists $\epsilon_{\mathcal{P}} > 0$ such that

$$\hat{p}\left(\frac{T^n}{\lambda^n}\right) \leq \epsilon_p, (\forall)\, n \geq 0.$$

Let $\mathcal{P}'$ be the principal calibration associated with the calibration $\mathcal{P}$, i.e. for each $p' \in \mathcal{P}'$ there exists $p_1, ..., p_n \in \mathcal{P}$ such that $p' = \max\{p_1, ..., p_n\}$.

If $\epsilon_{p'} = \max\{\epsilon_{p_1}, ..., \epsilon_{p_n}\}$, then

$$\hat{p}'\left(\frac{T^n}{\lambda^n}\right) \leq \epsilon_{p'}, (\forall)\, n \geq 0.$$

so $\mid \lambda \mid > r_{\mathcal{P}'}(T)$(proposition (1.2)). Since λ is arbitrary chosen results $r_{\mathcal{P}'}(T) \leq r_{\mathcal{P}}(T)$.

Therefore,

$$\inf\{r_{\mathcal{P}}(T) \mid \mathcal{P} \in \mathcal{C}_0(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\} \leq$$

$$\leq \inf\{r_{\mathcal{P}}(T) \mid \mathcal{P} \in \mathcal{C}(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\}.$$

The reverse inequality is obvious.

■

Lemma 1.6 *If X is a locally convex space and $T \in Q_{\mathcal{P}}(X)$ then*

$$\sigma(Q, T) = \cap\{\sigma(Q_{\mathcal{P}}, T) | \mathcal{P} \in \mathcal{C}_0(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\}.$$

Proof. From definition of the set $\sigma(Q,T)$ results the inclusion

$$\sigma(Q,T) \subset \cap\{\ \sigma(Q_{\mathcal{P}},T) \mid \mathcal{P} \in \mathcal{C}_0(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\ \}.$$

If $\lambda \notin \sigma(Q,T)$, then there exists some calibration $\mathcal{P} \in \mathcal{C}(X)$ such that $\lambda \in \rho(Q_{\mathcal{P}},T)$, so for every $p \in \mathcal{P}$ we have $\hat{p}\,(R\,(\lambda,T)) < \infty$.

Denote by $\mathcal{P}'$ the principal calibration of all seminorms

$$p'\,(x) = \max_{i=\overline{1,n}} p_i\,(x)\,, (\forall)\, x \in X,$$

where $p_i \in \mathcal{P}', i = \overline{1,n}$, and $n \in$N.

Let $p' \in \mathcal{P}$' be such seminorm. Since $R\,(\lambda,T) \in Q_{\mathcal{P}}(X)$, the lemma (1.1) implies that

$$p_i(R\,(\lambda,T)\,x) \le \hat{p}_i(R\,(\lambda,T))p_i\,(x)\,, (\forall)\, x \in X, i = \overline{1,n},$$

If $c_{p'} = \max_{i=\overline{1,n}} \hat{p}_i(R\,(\lambda,T))$, then

$$p_i(R\,(\lambda,T)\,x) \le c_{p'}p_i\,(x) \le c_{p'}p'\,(x)\,, (\forall)\, x \in X, i = \overline{1,n},$$

so we have

$$p'(R\,(\lambda,T)\,x) \le c_{p'}p'\,(x)\,, (\forall)\, x \in X,$$

Therefore, $R\,(\lambda,T) \in Q_{\mathcal{P}'}(X)$ and $\lambda \notin \sigma(Q_{\mathcal{P}'},T)$, which implies that $\cap\{\sigma(Q_{\mathcal{P}},T)|\mathcal{P} \in \mathcal{C}_0(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\} \subset \sigma(Q,T)$.

∎

2 Locally bounded operators

Lemma 2.1 ([12]) *If T_1 and T_2 are locally bounded operators on X, then there exists a calibration $\mathcal{P}'$ on X such that $T_1, T_2 \in B_{\mathcal{P}'}(X)$.*

Proposition 2.1 ([3]) *Let T be a locally bounded operator on a sequentially complete locally convex space X and $\mathcal{P} \in \mathcal{C}(X)$ such that $T \in Q_{\mathcal{P}}(X)$. If $p \in \mathcal{P}$ such that*

$$m_{pq}(T) < \infty, (\forall)\, q \in \mathcal{P},$$

then $\rho(X_p, T^p) = \rho(\tilde{X}_p, \tilde{T}^p)$.

Proposition 2.2 ([21]) *Let X be a sequentially complete locally convex space and $\mathcal{P} \in \mathcal{LB}(X)$. If $\mid \lambda \mid > r_{\mathcal{P}}(T)$, then the Neumann series $\sum\limits_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ converges to $R\,(\lambda,T)$ on a zero neighborhood. Moreover, $\mid \sigma_{lb}(T) \mid \le r_{lb}(T)$.*

Lemma 2.2 *If X is a locally convex space and $T \in \mathcal{LB}(X)$, then*

$$r_{lb}(T) = \inf \left\{ \nu > 0 | \left(\frac{T^n}{\nu^n} \right)_n \textit{ is bounded on a zero neighborhood} \right\}.$$

Proof. If $\mathcal{P} \in \mathcal{C}_0(X)$ we consider

$$r'(T) = \inf \left\{ \nu > 0 \mid \left(\frac{T^n}{\nu^n} \right)_n \quad \text{is uniformlly bounded on a zero neighborhood} \right\}.$$

Assume that $\nu > r_{lb}(T)$, i.e.$\left(\frac{T^n}{\nu^n}\right)_n$is converges uniformly to zero on a zero neighborhood. Then there exists $p_1 \in \mathcal{P}$ such that for each $q \in \mathcal{P}$ and every $\epsilon > 0$ there exists $n_{q,\epsilon} \in \mathbf{N}$, with the property

$$m_{p_1 q} \left(\frac{T^n}{\nu^n} \right) < \epsilon, (\forall)\, n \geq n_{q,\epsilon}.$$

Since the operator T is locally bounded there exists $p_2 \in \mathcal{P}$ such that for every $q \in \mathcal{P}$ we have $m_{p_2 q}(T) < \infty$. But the calibration $\mathcal{P}$ is principal, so there exists $p_0 \in \mathcal{P}$ such that $p_1 \leq p_0$ and $p_2 \leq p_0$.

Let $\epsilon > 0$ be arbitrary fixed. For each $q \in \mathcal{P}$ we have

$$m_{p_0 q} \left(\frac{T^n}{\nu^n} \right) = \sup \left\{ q \left(\frac{T^n}{\nu^n} x \right) \mid \; p_0 \left(x \right) \leq 1 \right\} \leq$$

$$\leq \sup \left\{ q \left(\frac{T^n}{\nu^n} x \right) \mid \; p_1 \left(x \right) \leq 1 \right\} = m_{p_1 q} \left(\frac{T^n}{\nu^n} \right), (\forall)\, n \geq n_{q,\epsilon}$$

$$m_{p_0 q} \left(\frac{T}{\nu} \right) = \sup \left\{ q \left(\frac{T}{\nu} x \right) \mid \; p_0 \left(x \right) \leq 1 \right\} \leq$$

$$\leq \sup \left\{ q \left(\frac{T}{\nu} x \right) \mid \; p_2 \left(x \right) \leq 1 \right\} = m_{p_2 q} \left(\frac{T}{\nu} \right).$$

Moreover, from lemma (1.1) we have

$$q \left(\frac{T^k}{\nu^k} x \right) \leq m_{p_2 q} \left(\frac{T}{\nu} \right) p_2 \left(\frac{T^{k-1}}{\nu^{k-1}} x \right) \leq ... \leq m_{p_2 q} \left(\frac{T}{\nu} \right) m_{p_2 p_2} \left(\frac{T}{\nu} \right)^{k-1} p_2 \left(x \right),$$

for every, $x \in X$ where $k = \overline{1, n_{q,\epsilon}}$, so lemma (1.1) implies that

$$m_{p_2 q} \left(\frac{T^k}{\nu^k} \right) \leq m_{p_2 q} \left(\frac{T}{\nu} \right) m_{p_2 p_2} \left(\frac{T}{\nu} \right)^{k-1}, k = \overline{1, n_{q,\epsilon}}.$$

So, if $n_{q,\epsilon} \geq 1$ and

$$\alpha_q = \max \left\{ \epsilon, m_{p_2 q}\left(\frac{T}{\nu}\right), ..., m_{p_2 q}\left(\frac{T}{\nu}\right) m_{p_2 p_2}\left(\frac{T}{\nu}\right)^{n_{q,\epsilon}-1} \right\},$$

then

$$m_{p_0 q}\left(\frac{T^n}{\nu^n}\right) \leq \alpha_q, (\forall)\, n \in \mathbf{N},$$

This means that the family $\left(\frac{T^n}{\nu^n}\right)_n$is uniformly bounded on a zero neighborhood, so $r'(T) \leq \nu$. Since $\nu > r_{lb}(T)$ is arbitrary chosen results that $r'(T) \leq r_{lb}(T)$.

Now we prove the opposite inequality. If $\alpha > r'(T)$, then there exists $\beta \in (r'(T), \alpha)$, such that $\left(\frac{T^n}{\beta^n}\right)_n$ is uniformly bounded on a zero neighborhood, i.e. there exists $p_0 \in \mathcal{P}$ such that for every $q \in \mathcal{P}$ there exists $\beta_q > 0$ with he property

$$m_{p_0 q}\left(\frac{T^n}{\beta^n}\right) <\beta_q, (\forall)\, n \in \mathbf{N}.$$

Therefore,

$$m_{p_0 q}\left(\frac{T^n}{\alpha^n}\right) = \left(\frac{\beta}{\alpha}\right)^n m_{p_0 q}\left(\frac{T^n}{\beta^n}\right) < \left(\frac{\beta}{\alpha}\right)^n \beta_q, (\forall)\, n \in \mathbf{N}.$$

Since $\frac{\beta}{\alpha} < 1$ results that for each $q \in \mathcal{P}$ and every $\epsilon > 0$ there exists $n_{q,\epsilon} \in \mathbf{N}$, with the property

$$\left(\frac{\beta}{\alpha}\right)^n \beta_q < \epsilon, (\forall)\, n \geq n_{q,\epsilon},$$

Therefore

$$m_{p_0 q}\left(\frac{T^n}{\alpha^n}\right) < \epsilon, (\forall)\, n \geq n_{q,\epsilon}$$

and $\left(\frac{T^n}{\alpha^n}\right)_n$ is uniformly bounded on a zero neighborhood and $r_{lb}(T) \leq \alpha$. But $\alpha > r'(T)$ is arbitrary chosen, so $r_{lb}(T) \leq r'(T)$.

∎

Proposition 2.3 *If X is a locally convex space and $T \in \mathcal{LB}(X)$, then*

$$\sigma_{lb}(T) = \sigma(Q, T) = \sigma(T).$$

Proof. The inclusion $\sigma(T) \subset \sigma(Q,T)$ is obvious. If $\lambda \notin \sigma_{lb}(T)$, then there exists $\alpha \in$C and $S \in \mathcal{LB}(X)$ such that $(\lambda I - T)^{-1} = \alpha I + S$. From lemma (2.1) there exists $\mathcal{P} \in \mathcal{C}(X)$ such that $T, S \in B_{\mathcal{P}}(X)$ and $(\lambda I - T)^{-1} \in B_{\mathcal{P}}(X) \subset Q_{\mathcal{P}}(X)$, Therefore, $\lambda \notin \sigma(Q_{\mathcal{P}}, T)$ and $\sigma(Q,T) \subset \sigma_{lb}(T)$.

Now we prove that $\sigma_{lb}(T) \subset \sigma(T)$. If X is a locally bounded space, then the X is senimormed space, so the remark (1.1) implies that $\sigma_{lb}(T) = \sigma(T)$.

Assume that X is not locally bounded. If $\lambda \notin \sigma(T), \lambda \neq 0$, then $R(\lambda, T)$ is continuous. From equality $R(\lambda,T)(\lambda I - T) = I$ results

$$R(\lambda,T) = \frac{1}{\lambda} R(\lambda,T)T + \frac{1}{\lambda} I$$

which implies that the operator $R(\lambda,T)T$ is locally bounded, so $\lambda \notin \sigma_{lb}(T)$.

If $\lambda = 0 \notin \sigma(T)$, then the operator $R(0,T) = (-T)^{-1}$ is continuous. Since $I = (-T)^{-1}(-T)$, the identity operator is locally bounded, which is not true because X is not a locally bounded space. Therefore, $\lambda = 0$ belongs to each sets ($\sigma(T), \sigma(Q,T)$, respectively $\sigma_{lb}(T)$) in the case when X is not a locally bounded space. Moreover $\sigma_{lb}(T) \subset \sigma(T)$.

∎

Proposition 2.4 *If T is a locally bounded operator on a locally convex space X, then*

$$r_{lb}(T) = \inf\{r_{\mathcal{P}}(T) \mid \mathcal{P} \in \mathcal{C}(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\}.$$

Proof. Denote

$$r'(T) = \inf\{r_{\mathcal{P}}(T) \mid \mathcal{P} \in \mathcal{C}(X) \text{ such that } T \in Q_{\mathcal{P}}(X)\}$$

If $\lambda > r_{lb}(T)$ and $\mathcal{P} \in \mathcal{C}_0(X)$, such that $T \in Q_{\mathcal{P}}(X)$, then there exists $\mu \in (r_{lb}(T), \lambda)$ such that the sequences $\left(\frac{T^n}{\mu^n}\right)_{n \geq 1}$ converges to zero on a zero neighborhood, i.e. there exists $p \in \mathcal{P}$ such that for each $q \in \mathcal{P}$ and every $\epsilon > 0$ there exists $n_{q,\epsilon} \in$N, with the property

$$m_{pq}\left(\frac{T^n}{\mu^n}\right) < \epsilon,\ (\forall)\, n \geq n_{q,\epsilon}.$$

If we consider the family $Q = \{q_m \mid\ m \geq 1, q \in \mathcal{P}\ \}$, where

$$q_m\,(x) = \max\left\{mp\,(x)\,, q\,(x)\right\}, (\forall)\, x \in X.$$

then $\mathcal{Q} \in \mathcal{C}(X)$ and T is quontient bounded with respect to the calibration $\mathcal{Q}$. If $q_m \in \mathcal{Q}$, then

$$q_m\left(\frac{T^n}{\mu^n}x\right) = \max\left\{mp\left(\frac{T^n}{\mu^n}x\right), q\left(\frac{T^n}{\mu^n}x\right)\right\} \leq$$

$$\leq \max\left\{m\hat{p}\left(\frac{T^n}{\mu^n}\right)p\left(x\right), m_{pq}\left(\frac{T^n}{\mu^n}\right)p\left(x\right)\right\} \leq$$

$$\leq \max\left\{m\varepsilon p\left(x\right), \varepsilon p\left(x\right)\right\} = m\varepsilon p\left(x\right) \leq \varepsilon q_m\left(x\right),$$

for every

$$n \geq n_{q_m,\epsilon} = \max\left\{n_{p,\epsilon}, n_{q,\epsilon}\right\}.$$

Therefore

$$\hat{q}_m\left(\frac{T^n}{\mu^n}\right) \leq \epsilon, (\forall)\, n \geq n_{q_m,\epsilon},$$

and since $\frac{\mu}{\lambda} < 1$, results

$$\hat{q}_m\left(\frac{T^n}{\lambda^n}\right) = \left(\frac{\mu}{\lambda}\right)^n \hat{q}_m\left(\frac{T^n}{\mu^n}\right) < \epsilon, (\forall)\, n \geq n_{q_m,\epsilon}.$$

Then $r_{\mathcal{Q}}(T) \leq \lambda$ (proposition (1.2)). This means that $r'(T) \leq \lambda$, for every $\mid \lambda \mid > r_{lb}(T)$, so $r'(T) \leq r_{lb}(T)$.

Now we will prove the reverse inequality. If $\lambda > r'(T)$, then lemma (1.5) and proposition (1.2) implies that there exists $\mu \in [r'(T), \lambda)$ and $\mathcal{P} \in \mathcal{C}_0(X)$ such that the sequence $\left(\frac{T^n}{\mu^n}\right)_{n\geq 1}$ converges to zero in $Q_{\mathcal{P}}(X)$, i.e. for every $\epsilon > 0$ and each $p \in \mathcal{P}$ there exists $n_{p,\epsilon} \in \mathbf{N}$, with the property

$$\hat{p}\left(\frac{T^n}{\mu^n}\right) < \epsilon, (\forall)\, n \geq n_{p,\epsilon}.$$

But T is locally bounded, so there exists $q \in \mathcal{P}$ with the property that for each $p \in \mathcal{P}$ there exists $c_p > 0$ such that $m_{qp}\left(\frac{T}{\mu}\right) < c_p$. Therefore, for every $\epsilon > 0$ and each $p \in \mathcal{P}$ there exists $n_{p,\epsilon} \in \mathbf{N}$, such that

$$p\left(\frac{T^{n+1}}{\lambda^{n+1}}x\right) = \left(\frac{\mu}{\lambda}\right)^{n+1} p\left(\frac{T}{\mu}\left(\frac{T^n}{\mu^n}x\right)\right) \leq \left(\frac{\mu}{\lambda}\right)^{n+1} m_{qp}\left(\frac{T}{\mu}\right) q\left(\frac{T^n}{\mu^n}x\right) \leq$$

$$< \left(\frac{\mu}{\lambda}\right)^{n+1} c_p \epsilon q\left(x\right) < \epsilon q\left(x\right), (\forall)\, x \in X,$$

for every $n \geq n_{p,\varepsilon}$. Lemma (1.1) implies that for every $\epsilon > 0$ and each $p \in \mathcal{P}$ there exists $n_{p,\epsilon} \in \mathbf{N}$, such that

$$m_{qp}\left(\frac{T^n}{\lambda^n}\right) < \epsilon, (\forall)\, n \geq n_{p,\epsilon},$$

so the sequence $\left(\frac{T^n}{\lambda^n}\right)_{n\geq 1}$converges to zero on a zero neighborhood and $r_{lb}(T) \leq \lambda$. Moreover, $r_{lb}(T) \leq r'(T)$.

■

Lemma 2.3 *Let T be a locally bounded operator on a locally convex space X and $\mathcal{P} \in C(X)$, such that $T \in Q_{\mathcal{P}}(X)$. If $p \in \mathcal{P}$ such that*

$$m_{pq}(T) < \infty, (\forall)\, q \in \mathcal{P},$$

and $\lambda \in \rho(T)$ has the property $\hat{p}\left(R\left(\lambda, T\right)\right) < \infty$, then $\lambda \in \rho(X_p, T^p)$ and

$$R\left(\lambda, T^p\right)(x + N_p) = R\left(\lambda, T\right)(x) + N^p, (\forall)\, x \in X.$$

Proof. Denote by I^p the identity operator on the space X_p. From the definition of T^p results that $\lambda \in \rho(X_p, T^p)$ and the $S : X^p \to X^p$, given by

$$S(x + N_p) = R\left(\lambda, T\right)(x) + N_p, (\forall)\, x \in X,$$

is a continuous linear operator on X_p. Moreover,

$$S(\lambda I^p - T^p)(x + N_p) = S((\lambda I - T)^p\,(x + N_p)) = S((\lambda I - T)(x) + N^p) =$$

$$= R\left(\lambda, T\right)(\lambda I - T)(x) + N^p = x + N^p.$$

$$(\lambda I^p - T^p)S\,(x + N_p) = (\lambda I^p - T^p)S\,(x + N_p) =$$

$$= (\lambda I^p - T^p)(R\left(\lambda, T\right)(x) + N_p) =$$

$$= (\lambda I - T)^p(R\left(\lambda, T\right)(x) + N_p) = (\lambda I - T)R\left(\lambda, T\right)(x) + N^p = x + N^p,$$

for every $x \in X$, so $S = R\left(\lambda, T^p\right)$.

■

Proposition 2.5 *If X is a sequentially complete locally convex space and $T \in \mathcal{LB}(X)$, then*

$$r_{lb}(T) = \mid \sigma(T) \mid = \mid \sigma_{lb}(T) \mid = \mid \sigma(Q, T) \mid .$$

Proof. Let remind to us that $\sigma_{lb}(T) = \sigma(Q, T) = \sigma(T)$ and

$$\mid \sigma(T) \mid = \mid \sigma_{lb}(T) \mid = \mid \sigma(Q, T) \mid \leq r_{lb}(T).$$

Let $\mathcal{P} \in \mathcal{C}_0(X)$ be arbitrary chosen such that $T \in Q_{\mathcal{P}}(X)$. Since T is locally bounded there exists $p \in \mathcal{P}$ such that

$$m_{pq}(T) < c_q < \infty, (\forall)\, q \in \mathcal{P}.$$

We prove that $\sigma(X_p, T^p) \subset \sigma_{lb}(T)$ and $r_{lb}(T) \leq r(X_p, \tilde{T}^p)$, where $r(\tilde{X}_p, \tilde{T}^p)$ is the spectral radius of $\tilde{T}^p$ in algebra $\mathcal{L}(\tilde{X}_p)$.

If $p \leq p_1$, then

$$m_{p_1 q}(T) \leq m_{pq}(T) < c_q < \infty, (\forall)\, q \in \mathcal{P}.$$

First we prove that $\sigma(\tilde{X}_p, \tilde{T}^p) \subset \sigma_{lb}(T)$. If $\lambda \in \rho_{lb}(T)$, then $\lambda \in \rho(T)$ (proposition (2.3)).

Assume that $r_{lb}(T) < 1$.

Case I. If $r_{lb}(T) <\mid \lambda \mid$, then the series $\sum\limits_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ converges uniformly on a zero neighborhood to $R(\lambda, T)$ (proposition (2.2), i.e. there exists some seminorm $q_0 \in \mathcal{P}$ such that for each $q \in \mathcal{P}$ and every $\epsilon > 0$ there exists $n_{q,\epsilon} \in \mathbf{N}$, with the property

$$m_{q_0 q}\left(R(\lambda, T) - \sum_{k=0}^{n} \frac{T^k}{\lambda^{k+1}}\right) < \epsilon, (\forall)\, n \geq n_{q,\epsilon}.$$

From lemma (2.2) results that $\left(\frac{T^n}{\lambda^n}\right)_n$ is uniformly bounded on a zero neighborhood, i.e. there exists some seminorm $q_1 \in \mathcal{P}$ such that for each $q \in \mathcal{P}$ there exists $\beta_q > 0$ with the property

$$m_{q_1 q}\left(\frac{T^n}{\lambda^n}\right) < \beta_q, (\forall)\, n \in \mathbf{N}.$$

Since $\mathcal{P}$ is a principal calibration there exists $q_2 \in \mathcal{P}$ such that $q_1 \leq q_2$ and $q_0 \leq q_2$. Therefore, for each $q \in \mathcal{P}$ we have

$$q(R(\lambda, T)x) \leq q\left(\left(R(\lambda, T) - \sum_{k=0}^{n_{q\epsilon}} \frac{T^k}{\lambda^{k+1}}\right)x\right) + q\left(\sum_{k=0}^{n_{q\epsilon}} \frac{T^k}{\lambda^{k+1}}x\right) \leq$$

$$\leq m_{q_0 q}\left(R(\lambda, T) - \sum_{k=0}^{n_{q\epsilon}} \frac{T^k}{\lambda^{k+1}}\right) q_0(x) + \sum_{k=0}^{n_{q\epsilon}} q\left(\frac{T^k}{\lambda^{k+1}}x\right) \leq$$

$$\leq m_{q_0 q}\left(R(\lambda, T) - \sum_{k=0}^{n_{q\epsilon}} \frac{T^k}{\lambda^{k+1}}\right) q_0(x) + \left(\lambda^{-1} + \sum_{k=1}^{n_{q\epsilon}} m_{q_1 q}\left(\frac{T^k}{\lambda^{k+1}}\right)\right) q_1(x) \leq$$

$$\leq \left(\epsilon + \lambda^{-1} n_{q\epsilon} \beta_q + \lambda^{-1}\right) q_2(x), (\forall)\, x \in X.$$

so, for each $q \in \mathcal{P}$ there exists

$$\gamma_q = \epsilon + \lambda^{-1} n_{q,\epsilon}\, \beta_q + \lambda^{-1} > 0$$

such that

$$q\left(R\left(\lambda,T\right)x\right)\leq\gamma_{q}q_{2}\left(x\right),\left(\forall\right)x\in X.$$

Let $p_1 \in \mathcal{P}$ such that $q_2 \leq p_1$ and $p \leq p_1$. Then.

$$q\left(R\left(\lambda,T\right)x\right)\leq\gamma_{q}p_{1}\left(x\right),\left(\forall\right)x\in X,$$

and

$$m_{p_{1}q}\left(R\left(\lambda,T\right)\right)<\infty,\left(\forall\right)q\in\mathcal{P}, \tag{1}$$

Moreover, $\hat{p}\left(R\left(\lambda,T\right)\right)<\infty$. From the remark we made at the beginning of the proof results that

$$m_{p_1q}(T)\leq m_{pq}(T)<c_q<\infty.$$

Without the lost of generality of the proof we can replace the seminorm p initially used with p_1. Therefore, the condition of lemma (2.3) are fulfilled, so $\lambda\in\rho(X_p,T^p)$ and $\lambda\in\rho(\tilde{X}_p,\tilde{T}^p)$ (propostion (2.1)).

If $\mid\lambda\mid\leq r_{lb}(T)$, then $r_{lb}(T)<1<\mid\lambda^{-1}\mid$, so

$\lambda^{-1}\in\rho(X_p,T^p)=\rho(\tilde{X}_p,\tilde{T}^p)$.

Since $\lambda,\lambda^{-1}\in\rho(T)$, from equalities

$$\left(\lambda^{-1}-\lambda\right)R\left(\lambda,T\right)\left(\lambda^{-1}I-T\right)\left[\left(\lambda^{-1}-\lambda\right)^{-1}I-\left(\lambda^{-1}I-T\right)^{-1}\right]=$$

$$=R\left(\lambda,T\right)\ \left[\left(\lambda^{-1}I-T\right)-\left(\lambda^{-1}-\lambda\right)I\right]=R\left(\lambda,T\right)\left(\lambda I-T\right)=I$$

results that

$$\left[\left(\lambda^{-1}-\lambda\right)^{-1}I-\left(\lambda^{-1}I-T\right)^{-1}\right]^{-1}=\left(\lambda^{-1}-\lambda\right)R\left(\lambda,T\right)\left(\lambda^{-1}I-T\right). \tag{2}$$

Therefore, $\left(\lambda-\lambda^{-1}\right)^{-1}\in\rho\left(R\left(\lambda^{-1},T\right)\right)$. From relation 1 results that

$$m_{pq}(R\left(\lambda^{-1},T\right))<\infty,\left(\forall\right)q\in\mathcal{P},$$

so from lemma (2.3) and proposition (2.1) results that the operator

$$\left(\lambda^{-1}-\lambda\right)^{-1}\tilde{I}^p-\left(\lambda^{-1}\tilde{I}^p-\tilde{T}^p\right)^{-1}$$

is invertible and continuous on the Banach space $\tilde{X}_p$. Therefore

$$\left[\left(\lambda^{-1}-\lambda\right)^{-1}\tilde{I}^p-\left(\lambda^{-1}\tilde{I}^p-\tilde{T}^p\right)^{-1}\right]^{-1}\in L(\tilde{X}_p).$$

If we put the relation (2) under the form

$$R(\lambda, T) = \left(\lambda^{-1} - \lambda\right)^{-1} \left[\left(\lambda^{-1} - \lambda\right)^{-1} I - \left(\lambda^{-1} I - T\right)^{-1}\right]^{-1} \left(\lambda^{-1} I - T\right)^{-1}$$

will results that $\lambda \in \rho(\tilde{X}_p, \tilde{T}^p)$.

Case II. Assume that $1 \leq r_{lb}(T) < c < \infty$ (since T is locally bounded). Then, the operator $T_1 = c^{-1}T$ is locally bounded and has the property requested at the beginning of the proof with respect to the seminorm p and $r_{lb}(T_1)<1$.

Therefore, we are in the conditions of the case I, so

$$c^{-1}\rho_{lb}(T) = \rho_{lb}(T_1) \subset \rho(\tilde{X}_p, \tilde{T}_1^p) = c^{-1}\rho(\tilde{X}_p, \tilde{T}^p),$$

This will show that

$$\sigma(\tilde{X}_p, \tilde{T}^p) \subset \sigma_{lb}(T).$$

Now we prove that $r_{lb}(T) \leq r(\tilde{X}_p, \tilde{T}^p)$. If $\lambda > r(\tilde{X}_p, \tilde{T}^p)$, then there exists $\mu \in (r(\tilde{X}_p, \tilde{T}^p), \lambda)$ such that the sequence $\left(\frac{(\tilde{T}^p)^n}{\mu^n}\right)_{n\geq 1}$ converges to zero in $\mathcal{L}(\tilde{X}_p)$, i.e. for every $\epsilon > 0$ there exists some index $n_{p,\epsilon} \in \mathbf{N}$ such that

$$\hat{p}\left(\frac{T^n}{\mu^n}\right) = \left\|\frac{\left(\tilde{T}^p\right)^n}{\mu^n}\right\|_p < \epsilon, (\forall)\, n \geq n_{p,\epsilon}.$$

Therefore, for each $q \in \mathcal{P}$, every $\epsilon > 0$ and every $x \in$X we have

$$q\left(\frac{T^{n+1}}{\lambda^{n+1}}x\right) = \left(\frac{\mu}{\lambda}\right)^{n+1} q\left(\frac{T^{n+1}}{\mu^{n+1}}x\right) \leq \left(\frac{\mu}{\lambda}\right)^{n+1} m_{pq}(T)\, p\left(\frac{T^n}{\mu^{n+1}}x\right) \leq$$

$$\leq \lambda^{-1}\left(\frac{\mu}{\lambda}\right)^n m_{pq}(T)\, p\left(\frac{T^n}{\mu^n}x\right) \leq \lambda^{-1}\left(\frac{\mu}{\lambda}\right)^n m_{pq}(T)\, \hat{p}\left(\frac{T^n}{\lambda^n}\right) p(x) \leq$$

$$\leq \lambda^{-1} c_q \left(\frac{\mu}{\lambda}\right)^n \epsilon p(x),$$

But $\frac{\mu}{\lambda} < 1$, so there exists some index $n_1 \in \mathbf{N}$ such that

$$\lambda^{-1} c_q \left(\frac{\mu}{\lambda}\right)^n < 1, (\forall)\, n \geq n_1,$$

wich implies that

$$q\left(\frac{T^{n+1}}{\lambda^{n+1}}x\right) < \epsilon p(x)$$

for all natural number n (sufficiently large) and every $x \in X$. This show that for each $q \in \mathcal{P}$ and every $\epsilon > 0$ there exists some index $n_{q,\epsilon} \in$N such that

$$m_{pq}\left(\frac{T^n}{\lambda^n}\right) < \epsilon, (\forall)\, n \geq n_{q,\epsilon},$$

i.e. the sequences $\left(\frac{T^n}{\lambda^n}\right)_n$converges uniformly to zero on a zero neigborhood, so $\lambda > r_{lb}(T)$. Since $\lambda > r(\tilde{X}_p, \tilde{T}^p)$ is arbitrary chosen results $r_{lb}(T) \leq r(\tilde{X}_p, \tilde{T}^p)$. Therefore, we have

$$\mid \sigma(\tilde{X}_p, \tilde{T}^p) \mid \leq \mid \sigma(Q, T) \mid = \mid \sigma_{lb}(T) \mid \leq r_{lb}(T) \leq r(\tilde{X}_p, \tilde{T}^p),$$

Since $\tilde{T}^p$ is a bounded operator on the Banach space $\tilde{X}_p$, so
$\mid \sigma(\tilde{X}_p, \tilde{T}^p) \mid = r(\tilde{X}_p, \tilde{T}^p)$
and
$r_{lb}(T) = \mid \sigma(T) \mid = \mid \sigma_{lb}(T) \mid = \mid \sigma(Q, T) \mid$.

∎

Remark 2.1 *Maeda [13] prove that if X is a quasi-complete locally convex space and $T \in \mathcal{LB}(X)$, then the Waelbroeck spectrum in $\mathcal{LB}_0(X)$ and classical spectrum of T are equal and $\sigma(T)$ is compact.*

Proposition 2.6 *If X is a sequentially complete locally convex space and $T \in \mathcal{LB}(X)$, then $\sigma(T)$ is compact.*

Proof. Assume that $\mathcal{P} \in \mathcal{C}(X)$, satisfies the condition $(\lambda I - T)^{-1}, T \in B_{\mathcal{P}}(X)$, and $\mu \in \mathbf{C}$.

Let remind to us that $\sigma(T) = \sigma_{lb}(T)$. If $\lambda \in \rho_{lb}(T)$, then from definition of ρ_{lb} result that there exists some scalar $\alpha \in \mathbf{C}$ and some locally bounded operator S such that $(\lambda I - T)^{-1} = \alpha I + S$. The lemma (2.1) implies that there exists some calibration $\mathcal{P} \in \mathcal{C}(X)$ such that $T, S \in B_{\mathcal{P}}(X)$. Moreover, we have $(\lambda I - T)^{-1} \in B_{\mathcal{P}}(X)$. Let $\mu \in \mathbf{C}$ such that $|\mu| < \left\|(\lambda I - T)^{-1}\right\|_{\mathcal{P}}^{-1}$.

We will show that $\lambda + \mu \in \rho(T)$. Since $\left\|\mu\,(\lambda I - T)^{-1}\right\|_{\mathcal{P}} < 1$, the operatorial series $\sum\limits_{k=0}^{\infty} (-\mu)^k\,(\lambda I - T)^{-(k+1)}$ converges in operatorial norm of the space $B_{\mathcal{P}}(X)$. By proposition (1.5) the algebra $B_{\mathcal{P}}(X)$ is complete, so there exists an operator $R\,(\mu) \in B_{\mathcal{P}}(X)$ such that

$$\sum_{k=0}^{\infty} (-\mu)^k\,(\lambda I - T)^{-(k+1)} = R\,(\mu)\,.$$

Using the equalities

$$((\lambda + \mu) I - T) R(\mu) = (\lambda I - T) R(\mu) + \mu R(\mu) =$$

$$= (\lambda I - T) \left(\sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-(k+1)} \right) + \mu \left(\sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-(k+1)} \right) =$$

$$= \sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-k} - \sum_{k=0}^{\infty} (-\mu)^{k+1} (\lambda I - T)^{-(k+1)} =$$

$$= I R(\mu) ((\lambda + \mu) I - T) = \mu R(\mu) + R(\mu)(\lambda I - T) =$$

$$= \mu \left(\sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-(k+1)} \right) + \left(\sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-(k+1)} \right) + \lambda I - T =$$

$$= - \sum_{k=0}^{\infty} (-\mu)^{k+1} (\lambda I - T)^{-(k+1)} + \sum_{k=0}^{\infty} (-\mu)^k (\lambda I - T)^{-k} = I$$

we prove that $\lambda + \mu \in \rho(B_{\mathcal{P}}, T) \subset \rho(T)$. Therefore

$$\{\beta \mid\mid \beta - \lambda \mid < \left\| (\lambda I - T)^{-1} \right\|_{\mathcal{P}}^{-1} \} \subset \rho(T),$$

Since $\lambda \in \rho_{lb}(T)$ is an arbitrary chosen result that the set $\rho(T)$ is open, so the spectral set $\sigma(T)$ is closed. The spectral set $\sigma(B_{\mathcal{P}}, T)$ is compact, so from the inclusion $\sigma(T) \subset \sigma(B_{\mathcal{P}}, T)$ results that the set $\sigma(T)$ is compact. ∎

Remark 2.2 *The function* $\mu \rightarrow R(\mu) = ((\lambda + \mu) I\text{-}T)^{-1}$ *is analytic at the point* $\mu = 0$.

Corollary 2.1 *Let* X *be a sequentially complete locally convex space and* $T \in \mathcal{LB}(X)$. *If* $\lambda \in \rho(T)$ *and* $d(\lambda)$ *is the distance from* λ *to the set* $\sigma(T)$, *then*

$$\left\| (\lambda I - T)^{-1} \right\|_{\mathcal{P}} \geq \frac{1}{d(\lambda)},$$

whenever $\mathcal{P} \in \mathcal{C}(X)$, *such that* $(\lambda I - T)^{-1}, T \in B_{\mathcal{P}}(X)$.

Proof. Assume that $\mathcal{P} \in \mathcal{C}(X)$, satisfies the condition $(\lambda I - T)^{-1}, T \in B_{\mathcal{P}}(X)$, and $\mu \in \mathbf{C}$ such that $|\mu| < \left\| (\lambda I - T)^{-1} \right\|_{\mathcal{P}}^{-1}$. Then from the proof of previous proposition results that

$$\lambda + \mu \in \rho(B_{\mathcal{P}}, T) \subset \rho(T),$$

so $\left\| (\lambda I - T)^{-1} \right\|_{\mathcal{P}}^{-1} \leq d(\lambda)$.

■

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