Quasi-Nilpotent Operators on Locally Convex Spaces *

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Abstract

In this article we extend the notion of quasi-nilpotent equivalent operators, introduced by Colojoara and Foias [4] for Banach spaces, to the class of bounded operators on sequentially complete locally convex spaces.

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1 Introduction

The class of quasi-nilpotent equivalent operators on a Banach space was introduced by Colojoara and Foias [4]. The aim of this paper is to search if we can extend this theory to the class of bounded operators on sequentially complete locally convex spaces.

Any family $\mathcal{P}$ of seminorms which generate the topology of a locally convex space X (in the sense that the topology of X is the coarsest with respect to which all seminorms of $\mathcal{P}$ are continuous) will be called a calibration on X. The set of all calibrations for X is denoted by $\mathcal{C}(X)$ and the set of all principal calibration by $\mathcal{C}_0(X)$.

An operator T on a locally convex space X is quotient bounded with respect to a calibration $\mathcal{P} \in \mathcal{C}(X)$ if for every seminorm $p \in \mathcal{P}$ there exists some $c_p > 0$ such that

$$p\,(Tx) \leq c_p p\,(x)\,,(\forall)\,x \in X.$$

The class of quotient bounded operators with respect to a calibration $\mathcal{P} \in \mathcal{C}(X)$ is denoted by $Q_{\mathcal{P}}(X)$. For every $p \in \mathcal{P}$ the application $\hat{p} : Q_{\mathcal{P}}(X) \to \mathbf{R}$ defined by

$$\hat{p}(T) = \inf\{\, r > 0 \mid p(Tx) \leq rp\,(x)\,,(\forall)\,x \in X\},$$

is a submultiplicative seminorm on $Q_{\mathcal{P}}(X)$, satisfying the relation $\hat{p}(I) = 1$, and has the following properties

1. $\hat{p}(T)= \sup\limits_{p(x)=1} p\,(Tx) = \sup\limits_{p(x)\leq 1} p\,(Tx)$, $(\forall)\,p \in \mathcal{P}$, $(\forall)\,q \in \mathcal{Q}$;
2. $p\,(Tx) \leq \hat{p}\,(T)\,p\,(x)$, $(\forall)\,x \in X$.

We denote by $\hat{\mathcal{P}}$ the family $\{\hat{p} \mid p \in \mathcal{P}\}$. If $T \in Q_{\mathcal{P}}(X)$ we said that $\alpha \in \mathbb{C}$ is in the resolvent set $\rho(Q_{\mathcal{P}}, T)$ if there exists $(\alpha I - T)^{-1} \in Q_{\mathcal{P}}(X)$. The spectral set $\sigma(Q_{\mathcal{P}}, T)$ will be the complement set of $\rho(Q_{\mathcal{P}}, T)$.

An operator $T \in Q_{\mathcal{P}}(X)$ is a bounded element of the algebra $Q_{\mathcal{P}}(X)$ if it is bounded element in the sense of G.R.Allan [1], i.e some scalar multiple of it generates a bounded semigroup. The class of the bounded elements of $Q_{\mathcal{P}}(X)$ is denoted by $(Q_{\mathcal{P}}(X))_0$. If $r_{\mathcal{P}}(T)$ is the radius of boundness of the operator T in $Q_{\mathcal{P}}(X)$, i.e.

$$r_{\mathcal{P}}(T) = \inf\{\alpha > 0 \;\mid\; \alpha^{-1}T \text{ generates a bounded semigroup in } Q_{\mathcal{P}}(X)\},$$

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then in [1] is proved that

$$r_{\mathcal{P}}(T) = \sup\{ \limsup_{n\to\infty} (\hat{p}(T^n))^{1/n} \mid p \in \mathcal{P}\}.$$

The Waelbroeck resolvent set $\rho_W(Q_{\mathcal{P}}, T)$ of an operator $T \in (Q_{\mathcal{P}}(X))_0$ is the subset of elements of $\lambda_0 \in \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, for which there exists a neighborhood $V \in \mathcal{V}_{(\lambda_0)}$ such that:

1. the operator $\lambda I - T$ is invertible in $Q_{\mathcal{P}}(X)$ for all $\lambda \in V \backslash \{\infty\}$
2. the set $\{ \ (\ \lambda I - T\ )^{-1} \mid \lambda \in V \backslash \{\infty\} \ \}$ is bounded in $Q_{\mathcal{P}}(X)$.

The Waelbroeck spectrum of T, denoted by $\sigma_W(Q_{\mathcal{P}}, T)$, is the complement of the set $\rho_W(Q_{\mathcal{P}}, T)$ in $\mathbb{C}_\infty$. It is obvious that $\sigma(Q_{\mathcal{P}}, T) \subset \sigma_W(Q_{\mathcal{P}}, T)$. An operator $T \in Q_{\mathcal{P}}(X)$ is regular if $\infty \notin \sigma_W(Q_{\mathcal{P}}, T)$, i.e. there exists some $t > 0$ such that:

1. the operator $\lambda I - T$ is invertible in $Q_{\mathcal{P}}(X)$, for all $\mid \lambda \mid > t$
2. the set $\{R(\lambda, T) \mid\mid \lambda \mid > t\}$ is bounded in $Q_{\mathcal{P}}(X)$.

Given $(X, \mathcal{P})$ a locally convex space, for each $p \in \mathcal{P}$ we denote by N^p the null space and by X^p the quotient space X/N^p. For each $p \in \mathcal{P}$ consider the canonical quotient map $\pi_p : X \to X/N^p$ given by relation

$$\pi_p(x) = x_p \equiv x + N^p, (\forall)\, x \in X,$$

(from X to X^p) which is an onto morphism. It is obvious that X_p is a normed space, for each $p \in \mathcal{P}$, with norm $|| \bullet ||_p$ defined by

$$\|x_p\|_p = p\,(x)\,, (\forall)\, x \in X.$$

Consider the algebra homomorphism $T \to T^p$ of $Q_{\mathcal{P}}(X)$ into $\mathcal{L}(X^p)$ defined by

$$T^p\,(x_p) = (Tx)_p, (\forall)\, x \in X.$$

This operators are well defined because $T(N^p) \subset N^p$. Moreover, for each $p \in \mathcal{P}$, $\mathcal{L}(X_p)$ is a unital normed algebra and we have

$$\|T^p\|_p = \sup\left\{ \|T^p x_p\|_p \;\middle|\; \|x_p\|_p \leq 1 \text{ for } x_p \in X_p \right\}$$

$$= \sup\{p\,(Tx)\ \mid p\,(x) \leq 1 \text{ for } x \in X\,\} = \hat{p}(T)$$

For every $p \in \mathcal{P}$ consider the normed space $(\tilde{X}^p, \|\bullet\|_p)$ the completition of $(X_p, \|\bullet\|_p)$. If $T \in Q_{\mathcal{P}}(X)$, then the operator T^p has an unique continuous linear extension $\tilde{T}^p$ on $(\tilde{X}^p, \|\bullet\|_p)$ and

$$\sigma(Q_{\mathcal{P}}, T) = \underset{p\in\mathcal{P}}{\cup} \sigma(\tilde{T}_p) = \underset{p\in\mathcal{P}}{\cup} \sigma(T_p).$$

2 Bounded Operators with SVEP

Lemma 2.1 *If $(X, \mathcal{P})$ is a sequentially complete locally convex space and $T \in (Q_{\mathcal{P}}(X))_0$, then*

$$\overset{\circ}{\rho}(Q_{\mathcal{P}}, T) = \rho_W(Q_{\mathcal{P}}, T).$$

Proof. Assume that there exists $\lambda_0 \in \rho(Q_{\mathcal{P}}, T) \backslash \rho_W(Q_{\mathcal{P}}, T)$ such that $\lambda_0 \in \overset{\circ}{\rho}(Q_{\mathcal{P}}, T)$. Since $\lambda_0 \notin \rho_W(Q_{\mathcal{P}}, T)$, then for each neighborhood U of λ_0 the set

$$\{ \ (\ \lambda I - T \ )^{-1} \mid \lambda \in U \ \}$$

is not bounded in $Q_{\mathcal{P}}(X)$. Let $U \in \rho(Q_{\mathcal{P}}, T)$ an open set such that $\lambda_0 \in U$. This implies that there exists $\lambda_1 \in U$ and $p \in \mathcal{P}$ such that for every $n \in N$ there exists $x_n \in X$ ($p(x_n) \neq 0$) with the property

$$p\left(R(\lambda_1, T)x_n\right) > np\left(x_n\right),$$

Therefore, for $y_n = R(\lambda_1, T)x_n$ we have

$$p\left(y_n\right) > np\left((\lambda_1 I - T)y_n\right),$$

which implies that $\lambda_1 \in \sigma_a(Q_{\mathcal{P}}, T) \subset \sigma(Q_{\mathcal{P}}, T)$ (see [11]). This contradicts the supposition we made, so lemma is proved. ∎

Definition 2.2 *If $(X, \mathcal{P})$ is a sequentially complete locally convex space we say that the operator $T \in (Q_{\mathcal{P}}(X))_0$ has the single-valued extension property (we will write SVEP) if for any analytic function $f : D_f \to X$, where $D_f \subset \mathbb{C}$ is an open set, with the property*

$$(\lambda I - T)f(\lambda) \equiv 0_X, (\forall)\, \lambda \in D_f,$$

results that $f \equiv 0, (\forall)\, \lambda \in D_f$.

Definition 2.3 *Let $(X, \mathcal{P})$ be a sequentially complete locally convex space and $T \in (Q_{\mathcal{P}}(X))_0$. For every $x \in X$ we say that the analytic function $f_x : D_x \to X$ is an analytic extension of the function $\lambda \to R(\lambda, T)$ if D_x is an open set such that $\rho_W(Q_{\mathcal{P}}, T) \subset D_x$ and*

$$(\lambda I - T)f(\lambda) \equiv x, (\forall)\, \lambda \in D_x.$$

Denote by $\rho_T(x)$ the set of all complex number λ_0 for which there exists an open set D_{λ_0}, such that $\lambda_0 \in D_{\lambda_0}$, and an analytic function $f_x : D_{\lambda_0} \to X$ which has the property

$$(\lambda I - T)f_x(\lambda) \equiv x, (\forall)\, \lambda \in D_x.$$

The set $\sigma_T(x)$ will be the complement of the set $\rho_T(x)$.

Remark 2.4
1. *In the case of bounded operators on a Banach space we have the condition $\rho(T) \subset D_x$, but the lemma 2.1 implies that this conditions in the case of quotient bounded operators on sequentially complete locally convex space is naturally replaced by the condition $\rho_W(Q_{\mathcal{P}}, T) \subset D_x$.*

2. *It is known that for a locally bounded operator $T \in Q_{\mathcal{P}}(X)$ we have the equalities*

$$\rho(Q_{\mathcal{P}}, T) = \rho_W(Q_{\mathcal{P}}, T) = \rho(T),$$

so in this case we can use $\rho(T)$ instead of $\rho_W(Q_{\mathcal{P}}, T)$ in all definitions we presented above.

Remark 2.5 *If $T \in (Q_{\mathcal{P}}(X))_0$ has SVEP then for each $x \in X$ there exists an unique maximal analytic extension of the application $\lambda \to R(\lambda, T)$, which will be denoted by $\tilde{x}$. Since $T \in (Q_{\mathcal{P}}(X))_0$ has SVEP the set $\rho_T(x)$ is correctly defined and is unique. Moreover, $\rho_T(x)$ is open and $\sigma_T(x)$ is closed.*

Remark 2.6 *If $T \in (Q_{\mathcal{P}}(X))_0$ has SVEP and $x \in X$, then*

1. *$\rho_T(x)$ is an open set;*

2. $\rho_T(x)$ *is the domain of definition for* $\tilde{x}$*;*

3. $\rho_W(Q_{\mathcal{P}}, T) \subset \rho_T(x)$.

Lemma 2.7 *Let* $(X, \mathcal{P})$ *be a sequentially complete locally convex space. If* $T \in (Q_{\mathcal{P}}(X))_0$ *then*

1. *the application* $\lambda \to R(\lambda, T)$ *is holomorphic on* $\rho_W(Q_{\mathcal{P}}, T)$*;*

2. $\frac{d^n}{d\lambda^n} R(\lambda, T) = (-1)^n n! R(\lambda, T)^{n+1}$*, for every* $n \in \mathbb{N}$*;*

3. $\lim\limits_{|\lambda|\to\infty} R(\lambda, T) = 0$ *and* $\lim\limits_{|\lambda|\to\infty} R(1, \lambda^{-1}T) = \lim\limits_{|\lambda|\to\infty} \lambda R(1, T) = I$.

Proof. 1) If $\lambda_0 \in \rho_W(Q_{\mathcal{P}}, T)$ then there exists $V \in \mathcal{V}_{(\lambda_0)}$ with the properties (1) and (2) from definition of Walebroeck resolvent set. Since for every $\lambda \in V \backslash \{\infty\}$ we have

$$R(\lambda, T) - R(\lambda_0, T) = (\lambda_0 - \lambda) R(\lambda, T) R(\lambda_0, T)$$

and the set $\{R(\lambda, T)|\ \lambda \in V\backslash\{\infty\}\}$ is bounded in $Q_{\mathcal{P}}(X)$ results that the application $\lambda \to R(\lambda, T)$ is continuous in λ_0, so

$$\lim_{\lambda\to\lambda_0} \frac{R(\lambda, T) - R(\lambda_0, T)}{\lambda - \lambda_0} = -R^2(\lambda_0, T)$$

If $\lambda_0 = \infty$ then, there exists some neighborhood $V \in \mathcal{V}_{(\infty)}$ such that the application $\lambda \to R(\lambda, T)$ is defined and bounded on $V\backslash\{\infty\}$. Moreover, this application it is holomorphic and bounded on $V\backslash\{\infty\}$, which implies that it is holomorphic at ∞.

Therefore, the application $\lambda \to R(\lambda, T)$ is holomorphic on $\rho_W(Q_{\mathcal{P}}, T)$.

2) Results from the proof of (1).

3) For each $\lambda \in \rho_W(Q_{\mathcal{P}}, T)$ we have

$$\lambda^{-1}(I + TR(\lambda, T))(\lambda I - T) = I,$$

so

$$R(\lambda, T) = \lambda^{-1}(I + TR(\lambda, T)). \tag{1}$$

If $V \in \mathcal{V}_{(\lambda_0)}$ satisfies the conditions of the definition of Walebroeck resolvent set, then the set

$$\{TR(\lambda, T)|\ \lambda \in V\backslash\{\infty\}\}$$

is bounded, so from relation (1) results that $\lim\limits_{|\lambda|\to\infty} R(\lambda, T) = 0$.

From equality $R(\lambda, T) = \lambda^{-1} R(1, \lambda^{-1}T), \lambda \neq 0$, and relation (1) results that

$$R(1, \lambda^{-1}T) = I + TR(\lambda, T),$$

so

$$\lim_{|\lambda|\to\infty} R(1, \lambda^{-1}T) = \lim_{|\lambda|\to\infty} (I + TR(\lambda, T)) = I$$

∎

Lemma 2.8 *If* $T \in (Q_{\mathcal{P}}(X))_0$ *has SVEP, then* $\sigma_T(x) = \varnothing$ *if and only if* $x = 0_X$.

Proof. If $\sigma_T(x) = \varnothing$, then $\tilde{x}$ is an entire function. Since $|\sigma_W(Q_{\mathcal{P}}, T)| = r_{\mathcal{P}}(T)$, results that

$$(\lambda I - T)\tilde{x}(\lambda) = x,\ (\forall)|\lambda| > r_{\mathcal{P}}(T), \tag{2}$$

so by lemma 2.7 we have

$$\lim_{|\lambda|\to\infty} \tilde{x}(\lambda) = \lim_{|\lambda|\to\infty} R(\lambda, T)x = 0.$$

Therefore, from Liouville's theorem results that $\tilde{x}(\lambda) \equiv 0$. Using the properties of functional calculus presented in [17] and (2) we have

$$x = \frac{1}{2\pi i} \int_{r_{\mathcal{P}}(T)+1} R(\lambda, T) x d\lambda = \frac{1}{2\pi i} \int_{r_{\mathcal{P}}(T)+1} x(\lambda) d\lambda = 0$$

It is obvious that if $x = 0_X$, then $\sigma_T(x) = \varnothing$. ∎

3 Quasi-nilpotent Equivalent Operators

For a pair of operators $T, S \in (Q_{\mathcal{P}}(X))_0$, not necessarily permutable, we consider the following notation

$$(T-S)^{[n]} = \sum_{k=0}^{n} (-1)^{n-k} C_n^k T^k S^{n-k},$$

where $C_n^k = \frac{n!}{(n-k)!k!}$, for all $n \geq 1$ and $k = \overline{1,n}$.

Remark 3.1 *[4] If $T, S, P \in (Q_{\mathcal{P}}(X))_0$ then for all $n \geq 1$ we have:*

1. $(T-S)^{[n+1]} = T(T-S)^{[n]} - (T-S)^{[n]}S$.
2. $\sum\limits_{k=0}^{n} (-1)^{n-k} C_n^k (T-S)^{[k]}(S-P)^{[n-k]} = (T-P)^{[n]}$.

Definition 3.2 *We say that two operators $T, S \in (Q_{\mathcal{P}}(X))_0$ are quasi-nilpotent equivalent operators if for every $p \in \mathcal{P}$ we have*

$$\lim_{n\to\infty} \left(\hat{p}\left((T-S)^{[n]}\right)\right)^{1/n} = 0 \text{ and } \lim_{n\to\infty} \left(\hat{p}\left((T-S)^{[n]}\right)\right)^{1/n} = 0.$$

In this case we write $T \overset{q}{\backsim} S$.

Remark 3.3 *If $T, S \in (Q_{\mathcal{P}}(X))_0$, then $(T-S)^{[n]} \in Q_{\mathcal{P}}(X)$.*

Lemma 3.4 *Let $(X, \mathcal{P})$ be a locally convex space and $T, S \in (Q_{\mathcal{P}}(X))_0$, such that $T \overset{q}{\backsim} S$. Then the series $\sum\limits_{n=0}^{\infty} (T-S)^{[n]}$ and $\sum\limits_{n=0}^{\infty} (S-T)^{[n]}$ converges in $Q_{\mathcal{P}}(X)$.*

Proof. If $T \overset{q}{\backsim} S$, then

$$\lim_{n\to\infty} \hat{p}\left((T-S)^{[n]}\right)^{1/n} = 0, (\forall) p \in \mathcal{P},$$

so by root test the series $\sum\limits_{n=0}^{\infty} \hat{p}((T-S)^{[n]})$ converges. Moreover, for each $\varepsilon \in (0,1)$ and every $p \in \mathcal{P}$ there exists some index $n_{\varepsilon,p} \in \mathbb{N}$ such that

$$\hat{p}\left((T_1 - T_2)^{[n]}\right) \leq \varepsilon^n, (\forall)\, n \geq n_{\varepsilon,p}$$

which implies that

$$\sum_{k=n}^{m} \hat{p}\left((T_1 - T_2)^{[k]}\right) < \sum_{k=n}^{m} \varepsilon^k < \frac{\varepsilon^n}{1-\varepsilon}, (\forall)\ m > n \geq n_{\varepsilon,p},$$

so $\left(\sum\limits_{k=0}^{n} (T_1 - T_2)^{[k]}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since the algebra $Q_{\mathcal{P}}(X)$ is sequentially complete results that the series $\sum\limits_{n=0}^{\infty} (T_1 - T_2)^{[n]}$ converges in $Q_{\mathcal{P}}(X)$.

Analogously, we can prove that the series $\sum\limits_{n=0}^{\infty} (S-T)^{[n]}$ converges in $Q_{\mathcal{P}}(X)$ ∎

Lemma 3.5 *The relation $\overset{q}{\backsim}$ defined above is a equivalence relation on $(Q_{\mathcal{P}}(X))_0$.*

Proof. It is obvious that $\overset{q}{\backsim}$ is simetric and reflexive. Now will prove that $\overset{q}{\backsim}$ is transitive. Let $T_1, T_2, T_3 \in (Q_{\mathcal{P}}(X))_0$ such that $T_1 \overset{q}{\backsim} T_2$ and $T_2 \overset{q}{\backsim} T_3$. Then for every $\varepsilon > 0$ and every $p \in \mathcal{P}$ there exists $n_{\varepsilon,p} \in \mathbb{N}$ such that

$$\hat{p}\left((T_1 - T_2)^{[n]}\right) \leq \varepsilon^n \text{ and } \hat{p}\left((T_2 - T_3)^{[n]}\right) \leq \varepsilon^n,$$

for every $n \geq n_{\varepsilon,p}$. If

$$M_{\varepsilon,p} = \max_{k=\overline{1,n_{\varepsilon,p}-1}} \left\{ \frac{\hat{p}\left((T_1 - T_2)^{[n]}\right)}{\varepsilon^k}, \frac{\hat{p}\left((T_2 - T_3)^{[n]}\right)}{\varepsilon^k}, 1 \right\}, (\forall) p \in \mathcal{P},$$

then for every $n \in \mathbb{N}$ we have

$$\hat{p}\left((T_1 - T_2)^{[n]}\right) \leq M_{\varepsilon,p}\varepsilon^n \text{ and } \hat{p}\left((T_2 - T_3)^{[n]}\right) \leq M_{\varepsilon,p}\varepsilon^n, (\forall) p \in \mathcal{P}\ .$$

The previous relation implies that

$$\hat{p}\left((T_1 - T_3)^{[n]}\right) = \hat{p}\left(\sum_{k=0}^{n} (-1)^{n-k} C_n^k (T_1 - T_2)^{[k]} (T_2 - T_3)^{[n-k]}\right) \leq$$

$$\leq \sum_{k=0}^{n} (-1)^{n-k} C_n^k \hat{p}\left((T_1 - T_2)^{[k]}\right) \hat{p}\left((T_2 - T_3)^{[n-k]}\right) \leq \sum_{k=0}^{n} (-1)^{n-k} C_n^k M_{\varepsilon,p}^2 \varepsilon^k \varepsilon^{n-k} = (2\varepsilon)^n M_{\varepsilon,p}^2$$

for all $n \in \mathbb{N}$ and every $p \in \mathcal{P}$, so

$$\hat{p}\left((T_1 - T_3)^{[n]}\right)^{1/n} \leq 2\varepsilon \sqrt[n]{M_{\varepsilon,p}^2}, (\forall)\, n \in \mathbb{N}, (\forall) p \in \mathcal{P}.$$

Therefore,

$$\lim_{n\to\infty} \hat{p}\left((T_1 - T_3)^{[n]}\right)^{1/n} = 0, (\forall) p \in \mathcal{P}$$

Analogously, we can prove that

$$\lim_{n\to\infty} \hat{p}\left((T_3 - T_1)^{[n]}\right)^{1/n} = 0, (\forall) p \in \mathcal{P},$$

so $T_1 \overset{q}{\backsim} T_3$. ■

Lemma 3.6 *If $(X, \mathcal{P})$ is a locally convex space then $T, S \in (Q_{\mathcal{P}}(X))_0$ are then quasi-nilpotent equivalent operators if and only if $\tilde{T}_p, \tilde{S}_p \in \mathcal{L}(\tilde{X}^p)$ are quasi-nilpotent equivalent operators on the Banach space $\tilde{X}^p$, for every $p \in \mathcal{P}$.*

Proof. For every $p \in \mathcal{P}$ the subspace N^p is invariant for T_p and T_p, so

$$\left(\tilde{T}_p\right)^k \left(\tilde{S}_p\right)^l = \left(T^k S^l\right)_p, \ (\forall) k, l \in \mathbb{N}.$$

Hence

$$(\tilde{T}_p - \tilde{S}_p)^{[n]} = \left((T - S)^{[n]}\right)_p, \ (\forall) m \in \mathbb{N}$$

If $T \overset{q}{\backsim} S$, then from definition results that

$$\lim_{n\to\infty} \left(\hat{p}\left((T - S)^{[n]}\right)\right)^{1/n} = 0 \text{ and } \lim_{n\to\infty} \left(\hat{p}\left((T - S)^{[n]}\right)\right)^{1/n} = 0. \tag{3}$$

so

$$\lim_{n\to\infty}\left\|(\tilde{T}_p-\tilde{S}_p)^{[n]}\right\|_p^{1/n}=\lim_{n\to\infty}\hat{p}\left((T-S)^{[n]}\right)=0 \tag{4}$$

$$\lim_{n\to\infty}\left\|(\tilde{S}_p-\tilde{T}_p)^{[n]}\right\|_p^{1/n}=\lim_{n\to\infty}\hat{p}\left((S-T)^{[n]}\right)=0 \tag{5}$$

Therefore, $\tilde{T}_p,\tilde{S}_p\in\mathcal{L}(\tilde{X}^p))_0$ are quasi-nilpotent equivalent operators, for every $p\in\mathcal{P}$.

Conversely, if $\tilde{T}_p\overset{q}{\sim}\tilde{S}_p$, for every $p\in\mathcal{P}$, then the relation (4) and (5) holds, so condition (3) is verified.

∎

Lemma 3.7 *If $(X,\mathcal{P})$ is a locally convex space and $T,S\in(Q_{\mathcal{P}}(X))_0$ are then quasi-nilpotent equivalent operators, then $\sigma(Q_{\mathcal{P}},T)=\sigma(Q_{\mathcal{P}},S)$.*

Proof. From previous lemma results that $\tilde{T}_p,\tilde{S}_p\in\mathcal{L}(\tilde{X}^p))_0$ are quasi-nilpotent equivalent operators, for every $p\in\mathcal{P}$, hence by theorem 2.2 ([4]) we have $\sigma(\tilde{T}_p)=\sigma(\tilde{S}_p)$. Moreover, $\sigma(Q_{\mathcal{P}},T)=\underset{p}{\cup}\sigma(\tilde{T}_p)$ and $\sigma(Q_{\mathcal{P}},S)=\underset{p}{\cup}\sigma(\tilde{S}_p)$, so the corollary is proved. ∎

Theorem 3.8 *Let $(X,\mathcal{P})$ be a locally convex space. If $T,S\in(Q_{\mathcal{P}}(X))_0$ are quasi-nilpotent equivalent operators, then $\sigma_W(Q_{\mathcal{P}},T)=\sigma_W(Q_{\mathcal{P}},S)$.*

Proof. From lemma 2.7 results that the functions $\lambda\to R(\lambda,T)$ and $\lambda\to R(\lambda,S)$ are holomorphic on the set $\rho_W(Q_{\mathcal{P}},T)$, respectively $\rho_W(Q_{\mathcal{P}},S)$.

Let $\lambda_0\in\sigma_W(Q_{\mathcal{P}},T)$ arbitrary fixed. Since $\sigma_W(Q_{\mathcal{P}},T)$ is an open set there exists $0<r_1<r_2$ such that $D_i(\lambda_0)\subset\sigma_W(Q_{\mathcal{P}},T)$, $i=\overline{1,2}$, where

$$D_i(\lambda_0)=\{\mu\in\mathbb{C}||\mu-\lambda_0|<r_i\},\ i=\overline{1,2},$$

and the set $\{\ R(\lambda,T)|\ \lambda\in D_1(\lambda_0)\ \}$ is bounded in $Q_{\mathcal{P}}(X)$. For each $p\in\mathcal{P}$ we consider that

$$M_p=\sup\{\ \hat{p}(R(\lambda,T))|\ \lambda\in D_1(\lambda_0)\ \}.$$

We denote by $R(\mu,T)=\sum\limits_{k=0}^{n}R_n(\lambda)(\mu-\lambda)^n$ the Taylor expansion of the resolvent around each point λ of $D_1(\lambda_0)$. From complex analysis we have the formula

$$R_n(\lambda)=\frac{1}{n!}\frac{d^n}{d\lambda^n}R(\mu,T)=\frac{1}{2\pi i}\int\limits_{|\omega-\lambda|=r_2}\frac{R(\omega,T)}{(\omega-\lambda)^{n+1}}d\omega,\ (\forall)\lambda\in D_1(\lambda_0),(\forall)n\geq 0,$$

so,

$$\hat{p}(R_n(\lambda))=\hat{p}\left(\frac{1}{2\pi i}\int\limits_{|\omega-\lambda|=r_2}\frac{R(\omega,T)}{(\omega-\lambda)^{n+1}}d\omega\right)\leq$$

$$\leq\hat{p}(R_n(\lambda))=r_1\sup\{\ \hat{p}(R(\lambda,T))|\ \lambda\in D_1(\lambda_0)\ \}\sup\{\ \frac{1}{(\omega-\lambda)^{n+1}}|\ \lambda\in D_1(\lambda_0)\ \}\leq$$

$$\leq r_2M_p(r_2-r_1)^{-(n+1)},$$

for all $\lambda\in D_1(\lambda_0)$ and every $n\geq 0$. Since $T\overset{q}{\sim}S$ results that for every $\varepsilon>0$ and every $p\in\mathcal{P}$ there exists $n_{\varepsilon,p}\in\mathbb{N}$ such that

$$\hat{p}\left((T-S)^{[n]}\right)\leq\varepsilon^n \text{ and } \hat{p}\left((S-T)^{[n]}\right)\leq\varepsilon^n,(\forall)n\geq n_{\varepsilon,p}.$$

Assume that $\varepsilon < r_1 - r_0$. Then for every $p \in \mathcal{P}$ there exists $n_{\varepsilon,p} \in \mathbb{N}$ such that

$$\hat{p}\left((S-T)^{[n]}R_n(\lambda)\right) \leq \varepsilon^n r_1 M_p (r_1 - r_0)^{-(n+1)} =$$

$$= r_1(r_1 - r_0)^{-1} M_p \left(\frac{\varepsilon}{r_1 - r_0}\right)^n,$$

for every $n \geq n_{\varepsilon,p}$ and every $\lambda \in D_1(\lambda_0)$, so $\left(\sum\limits_{n=0}^{m}(-1)^n(S-T)^{[n]}R_n(\lambda)\right)_m$ is a Cauchy sequences. Since $Q_{\mathcal{P}}(X)$ is sequentially complete results that the series

$$R(\lambda) = \sum_{n=0}^{\infty}(-1)^n(S-T)^{[n]}R_n(\lambda)$$

converges uniformly in D_0. Therefore, the function $\lambda \to R(\lambda)$ is analytic in $\rho_W(Q_{\mathcal{P}}, T)$.

Using lemma 2.7 by induction it can be prove that if we differentiate $n \geq 1$ times the equalities

$$(\lambda I - T)R(\lambda, T) = R(\lambda, T)(\lambda I - T) = I,$$

then for every $n \geq 1$ we obtain

$$(\lambda I - T)\frac{d^n}{d\lambda^n}R(\lambda, T) = \frac{d^n}{d\lambda^n}R(\lambda, T)(\lambda I - T) =$$

$$= -n\frac{d^{n-1}}{d\lambda^{n-1}}R(\lambda, T), (\forall)\lambda \in D_0(\lambda_0)$$

so

$$(\lambda I - T)R_n(\lambda) = (\lambda I - T)\frac{1}{n!}\frac{d^n}{d\lambda^n}R(\mu, T) =$$

$$= -n\frac{1}{n!}\frac{d^{n-1}}{d\lambda^{n-1}}R(\lambda, T) = -R_{n-1}(\lambda), \tag{6}$$

for every $\lambda \in D_1(\lambda_0)$ and every $n \geq 1$.

From lemma 3.1 and relation (6) results the following equalities

$$(\lambda I - S)R(\lambda) = \sum_{n=0}^{\infty}(-1)^n(\lambda I - S)(S-T)^{[n]}R_n(\lambda) =$$

$$= \sum_{n=0}^{\infty}(\lambda I - S)\left((\lambda I - S) - (\lambda I - T)\right)^{[n]} SR_n(\lambda) =$$

$$= \sum_{n=0}^{\infty}\{((\lambda I - S) - (\lambda I - T))^{[n+1]}R_n(\lambda) + ((\lambda I - S) - (\lambda I - T))^{[n]}(\lambda I - T)R_n(\lambda)\} =$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}(S-T)^{[n+1]}R_n(\lambda) + (\lambda I - T)R_0(\lambda) + \sum_{n=1}^{\infty}(-1)^n(S-T)^{[n]}(\lambda I - T)R_n(\lambda) =$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}(S-T)^{[n+1]}R_n(\lambda) + (\lambda I - T)R_0(\lambda) + \sum_{n=0}^{\infty}(-1)^{n+1}(S-T)^{[n+1]}(\lambda I - T)R_{n+1}(\lambda) =$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}(S-T)^{[n+1]}R_n(\lambda) + (\lambda I - T)R(\lambda, T) + \sum_{n=0}^{\infty}(-1)^{n+1}(S-T)^{[n+1]}(-R_n(\lambda)) = I$$

Analogously we prove that $R(\lambda)(\lambda I - S) = I$, so $\rho_W(Q_{\mathcal{P}}, T) \subset \rho_W(Q_{\mathcal{P}}, S)$. The inclusion $\rho_W(Q_{\mathcal{P}}, S) \subset \rho_W(Q_{\mathcal{P}}, T)$ can be proved in the same way.

■

Lemma 3.9 *Let $(X,\mathcal{P})$ be a locally convex space and $T \in (Q_{\mathcal{P}}(X))_0$ such that $r_{\mathcal{P}}(T) < 1$. Then the operator $I-T$ is invertible and $(I-T)^{-1} = \sum\limits_{n=0}^{\infty} T^n$.*

Proof. Assume that $r_{\mathcal{P}}(T) < t < 1$. Hence results that

$$\limsup_{n\to\infty} \left(\hat{p}\left(T^n\right)\right)^{1/n} < t, (\forall)\ p \in \mathcal{P},$$

so for each $p \in \mathcal{P}$ there exists $n_p \in \mathbb{N}$ such that

$$\left(\hat{p}\left(T^n\right)\right)^{1/n} \leq \sup_{n\geq n_p} \left(\hat{p}\left(T^n\right)\right)^{1/n} < t, (\forall)\ n \geq n_p.$$

This relation implies that the series $\sum\limits_{n=0}^{\infty} \hat{p}\left(T^n\right)$ converges, so

$$\lim_{n\to\infty} \hat{p}\left(T^n\right) = 0, (\forall)\ p \in \mathcal{P},$$

therefore $\lim\limits_{n\to\infty} T^n = 0$. Since the algebra $Q_{\mathcal{P}}(X)$ is sequentially complete results that the series $\sum\limits_{n=0}^{\infty} T^n$ converges. Moreover,

$$(I-T)\sum_{n=0}^{m} T^n = \sum_{n=0}^{m} T^n (I-T) = I - T^{m+1},$$

so

$$(I-T)\sum_{n=0}^{\infty} T^n = \sum_{n=0}^{\infty} T^n (I-T) = I,$$

which implies that $I-T$ is invertible and $(I-T)^{-1} = \sum\limits_{n=0}^{\infty} T^n$.

■

Theorem 3.10 *Let $(X,\mathcal{P})$ be a locally convex space. If $T,S \in (Q_{\mathcal{P}}(X))_0$ are quasi-nilpotent equivalent operators, then T has SVEP if and only if S has SVEP.*

Proof. Assume that T has SVEP. Let $D_f \subset \mathbb{C}$ be an open set such that $\rho_W(Q_{\mathcal{P}}, S) \subset D_f$ and $f : D_f \to X$ be an analytic function on D_f which satisfies the property

$$(\lambda I - S) f(\lambda) = 0,\ \lambda \in D_f.$$

Then, for every $n \geq 0$ we have

$$(T-S)^{[n]} f(\lambda) = \sum_{k=0}^{n} (-1)^{n-k} C_n^k T^k S^{n-k} f(\lambda) =$$

$$= \sum_{k=0}^{n} (-1)^{n-k} C_n^k T^k \lambda^{n-k} f(\lambda) = (T-\lambda I)^n f(\lambda) \tag{7}$$

Since $T \overset{q}{\sim} S$ results that for every $\varepsilon > 0$ and every $p \in \mathcal{P}$ there exists $n_{\varepsilon,p} \in \mathbb{N}$ such that

$$\hat{p}\left((T-S)^{[n]}\right) \leq \varepsilon^n \text{ and } \hat{p}\left((S-T)^{[n]}\right) \leq \varepsilon^n, (\forall) n \geq n_{\varepsilon,p}.$$

Let $\mu \neq \lambda$. Then for every $\varepsilon \in (0, |\mu - \lambda|)$ and for every $p \in \mathcal{P}$ there exists $n_{\varepsilon,p} \in \mathbb{N}$

$$\hat{p}\left(\frac{(T-S)^{[n]}}{(\mu-\lambda)^{n+1}}\right) \leq \frac{\varepsilon^n}{|\mu-\lambda|^{n+1}}, (\forall) n \geq n_{\varepsilon,p},$$

so the $\left(\sum\limits_{n=0}^{m} \frac{(T-S)^{[n]}}{(\mu-\lambda)^{n+1}}\right)_m$ is a Cauchy sequences. Since $Q_{\mathcal{P}}(X)$ is sequentially complete results that the series $\sum\limits_{n=0}^{\infty} \frac{(T-S)^{[n]}}{(\mu-\lambda)^{n+1}}$ is absolutely convergent in the topology of $Q_{\mathcal{P}}(X)$ for every $\mu \neq \lambda$. Moreover, if $r_{\mathcal{P}}(T - \lambda I) < |\mu - \lambda|$, then $r_{\mathcal{P}}(\frac{T-\lambda I}{\mu-\lambda}) < 1$ and from lemma 3.9 results that

$$\sum_{n=0}^{\infty} \frac{(T-\lambda I)^n}{(\mu-\lambda)^{n+1}} = (\mu - \lambda)I - (T - \lambda I) = R(\mu, T). \tag{8}$$

From the relations (7) and (8) results that

$$(\mu I - T)\left(\sum_{n=0}^{\infty} \frac{(T-S)^{[n]}}{(\mu-\lambda)^{n+1}}\right) f(\lambda) = (\mu I - T)\left(\sum_{n=0}^{\infty} \frac{(T-\lambda I)^n}{(\mu-\lambda)^{n+1}}\right) f(\lambda) =$$

$$= (\mu I - T)R(\mu, T)f(\lambda) = f(\lambda),$$

for every μ with the property $r_{\mathcal{P}}(T - \lambda I) < |\mu - \lambda|$. Therefore,

$$g_\lambda(\mu) = \sum_{n=0}^{\infty} \frac{(T-S)^{[n]}}{(\mu-\lambda)^{n+1}} f(\lambda)$$

is an analytic function on $\mathbb{C}\backslash\{\lambda\}$ which verifies the relation

$$(\mu I - T)g_\lambda(\mu) = f(\lambda) \tag{9}$$

on the open set $\{\mu \in \mathbb{C} \,| r_{\mathcal{P}}(T - \lambda I) < |\mu - \lambda|\} \subset \mathbb{C}\backslash\{\lambda\}$. Since T has SVEP results that the function $g_\lambda(\mu)$ verifies the relation (9) for all $\mu \neq \lambda$. This implies that $\mathbb{C}\backslash\{\lambda\} \subset \rho_T(f(\lambda))$, i.e. $\sigma_T(f(\lambda)) \subset \{\lambda\}$.

Let $\lambda_0 \in D_f$ arbitrary fixed and $r > 0$ such that $D_0 = \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_0| \leq r_0\} \subset D_f$. Since $g_\lambda(\mu)$ is analytic on $\mathbb{C}\backslash\{\lambda\}$ from relation (9) results that

$$(\mu I - T)\frac{1}{2\pi i} \int\limits_{|\xi-\lambda|=r_0} \frac{g_\xi(\mu)}{\xi - \lambda_0} d\xi = \frac{1}{2\pi i} \int\limits_{|\xi-\lambda|=r_0} \frac{f(\mu)}{\xi - \lambda_0} d\xi = f(\lambda_0) \tag{10}$$

for all $\mu \in D_0$, so $\mu \in \rho_T(f(\lambda_0))$, for every $\mu \in D_0$. Hence $\lambda_0 \in \rho_T(f(\lambda_0))$ and since we already proved above that $\sigma_T(f(\lambda_0)) \subset \{\lambda_0\}$ results that $\sigma_T(f(\lambda)) = \dot{\varnothing}$. Lemma 2.8 implies that $f(\lambda) \equiv 0$ on D_0 and since $\lambda_0 \in D_f$ is arbitrary chosen, results that $f(\lambda) \equiv 0$ on D_f. Therefore, S has SVEP. Analogously we can prove that if S has SVEP then T has SVEP. ∎

Theorem 3.11 *Let $(X, \mathcal{P})$ be a locally convex space. If $T, S \in (Q_{\mathcal{P}}(X))_0$ are quasi-nilpotent equivalent operators and T has SVEP, then $\sigma_T(x) = \sigma_S(x)$, for every $x \in X$.*

Proof. First we remark that from previous theorem results that S has SVEP. Let $x(\lambda)$ be the analytic function on $\rho_T(x)$ which verify the condition

$$(\lambda I - T)x(\lambda) = x, \ \lambda \in \rho_T(x). \tag{11}$$

Let $\lambda_0 \in \rho_T(x)$ arbitrary fixed. Since $\rho_T(x)$ is an open set there exists $0 < r_1 < r_2$ such that $D_i(\lambda_0) \subset \sigma_W(Q_{\mathcal{P}}, T)$, $i = \overline{1,2}$, where

$$\bar{D}_i(\lambda_0) = \{\mu \in \mathbb{C} |\ |\mu - \lambda_0| \leq r_i\},\ i = \overline{1,2}.$$

For every $p \in \mathcal{P}$ denote by M_p^1 the maximum of $x(\lambda)$ on $\bar{D}_2(\lambda_0)$. Hence, for $\lambda \in \bar{D}_2(\lambda_0)$ we have

$$p\left(\frac{x^{(n)}(\lambda)}{n!}\right) = p\left(\frac{1}{2\pi i}\int\limits_{|\xi-\lambda_0|=r_2} \frac{x(\xi)}{(\xi-\lambda)^{n+1}} d\xi\right) \leq \frac{M_p^1 r_2}{(r_2 - r_1)^{n+1}},\ (\forall) n \geq 0. \tag{12}$$

In the proof of lemma 3.5 we proved that for every $\varepsilon > 0$ and each $p \in \mathcal{P}$ there exists $M_{\varepsilon,p} > 0$ such that

$$\hat{p}\left((T-S)^{[n]}\right) \leq M_{\varepsilon,p}\varepsilon^n,\ (\forall) n \geq 0. \tag{13}$$

Therefore, the relations (12) and (13) implies that

$$p\left((T-S)^{[n]}\frac{x^{(n)}(\lambda)}{n!}\right) \leq \hat{p}\left((T-S)^{[n]}\right) p\left(\frac{x^{(n)}(\lambda)}{n!}\right) < \frac{M_{\varepsilon,p}M_p^1 r_2}{r_2 - r_1}\left(\frac{\varepsilon}{r_2 - r_1}\right)^n,$$

for all $n \geq 0$. Taking $\varepsilon = \frac{r_2 - r_1}{2}$ results that for each $p \in \mathcal{P}$ there exists $M_{\varepsilon,p} > 0$ such that

$$p\left((T-S)^{[n]}\frac{x^{(n)}(\lambda)}{n!}\right) \leq \frac{M_p}{2^n},\ (\forall) n \geq 0, \tag{14}$$

where $M_p = \frac{M_{\varepsilon,p}M_p^1 r_2}{r_2 - r_1}$ does not depend on $\lambda \in \bar{D}_2(\lambda_0)$. The relation (14) shows that the series

$$\sum_{n=0}^{\infty} p\left((-1)^n (T-S)^{[n]}\frac{x^{(n)}(\lambda)}{n!}\right)$$

converges for every $\lambda \in \bar{D}_2(\lambda_0)$ and every $p \in \mathcal{P}$, so since X is sequentially complete results that the series $\sum\limits_{n=0}^{\infty} (-1)^n (T-S)^{[n]}\frac{x^{(n)}(\lambda)}{n!}$ converges absolutely and uniformly on $\bar{D}_2(\lambda_0)$. But $\lambda_0 \in \rho_T(x)$ is arbitrary fixed, hence this series converges absolutely and uniformly on every compact $K \subset \rho_T(x)$, which implies that

$$x_1(\lambda) = \sum_{n=0}^{\infty} (-1)^n (T-S)^{[n]}\frac{x^{(n)}(\lambda)}{n!} \tag{15}$$

is analytic on $\rho_T(x)$. Now we prove that

$$(\lambda I - S)x_1(\lambda) = x,\ \lambda \in \rho_T(x).$$

If we differentiate $n \geq 1$ times the equality (11), then we have

$$(\lambda I - T)x^{(n)}(\lambda) = -nx^{(n-1)}(\lambda),\ \lambda \in \rho_T(x).$$

From previous relations and remark 3.1 results

$$(\lambda I - S)x_1(\lambda) = \sum_{n=0}^{\infty}(-1)^n(\lambda I - S)(S-T)^{[n]}\frac{x^{(n)}(\lambda)}{n!} =$$

$$= \sum_{n=0}^{\infty}(\lambda I - S)\left((\lambda I - S) - (\lambda I - T)\right)^{[n]}\frac{x^{(n)}(\lambda)}{n!} =$$

$$= \sum_{n=0}^{\infty} \left\{ ((\lambda I - S) - (\lambda I - T))^{[n+1]} + ((\lambda I - S) - (\lambda I - T))^{[n]} (\lambda I - T) \right\} \frac{x^{(n)}(\lambda)}{n!} =$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} (S - T)^{[n+1]} \frac{x^{(n)}(\lambda)}{n!} + \sum_{n=0}^{\infty} (-1)^{n} (S - T)^{[n]} (\lambda I - T) \frac{x^{(n)}(\lambda)}{n!} =$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} (S - T)^{[n+1]} \frac{x^{(n)}(\lambda)}{n!} + (\lambda I - T) x(\lambda) + \sum_{n=1}^{\infty} (-1)^{n} (S - T)^{[n]} \frac{x^{(n-1)}(\lambda)}{(n-1)!} =$$

$$= (\lambda I - T) x(\lambda) = x$$

for all $\lambda \in \rho_T(x)$. This shows that $\rho_T(x) \subset \rho_S(x)$, so $\sigma_S(x) \subset \sigma_T(x)$. Analogously it can be proved that $\sigma_T(x) \subset \sigma_S(x)$.

∎

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