A Functional Calculus for Quotient Bounded Operators *

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Abstract

If $(X, \mathcal{P})$ is a sequentially complete locally convex space, then a quotient bounded operator is regular (in the sense of Waelbroeck) if and only if it is a bounded element (in the sense of Allan) of algebra $Q_{\mathcal{P}}(X)$. Using germs of analytic functions from a open set of $\mathbb{C}$ to the locally convex algebra $Q_{\mathcal{P}}(X)$, the classic functional calculus for the bounded operators on a Banach space is naturally generalized for bounded elements of the algebra $Q_{\mathcal{P}}(X)$.

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1 Introduction

In this paper we define a functional calculus for bounded operators on sequentially complete spaces which is inspired by Waelbroeck's functional calculus presented in [20]. It is well-known that if X is a Banach space and $L(X)$ is Banach algebra of bounded operators on X, then the formula

$$f(T) = \frac{1}{2\pi i} \int_{\Gamma} f(z) (zI - T)^{-1} dz,$$

(where f is an analytic function on some neighborhood of $\sigma(T)$, Γ is a closed rectifiable Jordan curve whose interior domain D is such that $\sigma(T) \subset D$, and f is analytic on D and continuous on $D \cup \Gamma$) defines a homomorphism $f \to f(T)$ from the set of all analytic functions on some neighborhood of $\sigma(T)$ into $L(X)$, with very useful properties. If we want to generalize this theory for sequentially complete locally convex spaces we need new notions which are related to this spaces. Through this paper all the locally convex spaces will be assumed sequentially complete Hausdorff space, over the complex field $\mathbb{C}$, and all the operators will be linear.

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The collection of all families of seminorms $\mathcal{P}$ which generate the topology of a locally convex space X (in the sense that the topology of X is the coarsest with respect to which all seminorms of $\mathcal{P}$ are continuous) will be denoted by $\mathcal{C}(X)$. The set of all directed families $\mathcal{P} \in \mathcal{C}(X)$ is denoted by $\mathcal{C}_0(X)$. On a family of seminorms on a linear space X we define the relation ,, $\leq$" by

$$p \leq q \Leftrightarrow p(x) \leq q(x), \, (\forall) \, x \in X.$$

A family of seminorms is preordered by the relation " $\prec$", where

$$p \prec q \Leftrightarrow \text{there exists some } r > 0 \text{ such that } p(x) \leq rq(x), \text{ for all } x \in X.$$

If $p \prec q$ and $q \prec p$, we write $p \approx q$. Two families $\mathcal{P}_1$ and $\mathcal{P}_2$ of seminorms on a linear space are called Q-equivalent (denoted $\mathcal{P}_1 \approx \mathcal{P}_2$) if:

1. for each $p_1 \in P_1$ there exists $p_2 \in \mathcal{P}_2$ such that $p_1 \approx p_2$;
2. for each $p_2 \in P_2$ there exists $p_1 \in \mathcal{P}_1$ such that $p_2 \approx p_1$.

Two Q-equivalent and separating families of seminorms on a linear space generate the same locally convex topology.

An operator $T \in L(X)$ is:

1. a quotient bounded operator with respect to $\mathcal{P} \in \mathcal{C}(X)$ if for every seminorm $p \in \mathcal{P}$ there exists $c_p > 0$ such that
$$p(Tx) \leq c_p p(x), (\forall) \, x \in X.$$
2. an universally bounded with respect to $\mathcal{P} \in \mathcal{C}(X)$ if there exists $c_0 > 0$ such that
$$p(Tx) \leq c_0 p(x), (\forall) \, x \in X, (\forall) \, p \in \mathcal{P}.$$

Example 1.1 *1. Let X be the vectorial space of the complex function and $\mathcal{F}$ be the set of the finite parts from $\mathbb{C}$. We consider on X the topology generated by the family of the seminorms $\mathcal{P} = \{p_F \mid F \in \mathcal{F}\}$, where*

$$p_F(f) = \max\{ \mid f(x) \mid \mid \ x \in F\}, (\forall) \, F \in \mathcal{F}, (\forall) \, f \in X.$$

Then, the operator $T : X \to X$ given by

$$(Tf)(x) = xf(x), \ (\forall) \, x \in \mathbf{C}.$$

is a quotient bounded operator with respect to $\mathcal{P}$.

2. Every locally bounded operator on a locally convex space X (i.e. an operator which maps some zero neighborhood into a bounded set) is quotient bounded with respect to some family of seminorms $\mathcal{P} \in \mathcal{C}(X)$.

The class of the quotient bounded operators (universally bounded operators) with respect to $\mathcal{P} \in \mathcal{C}(X)$ is denoted by $Q_{\mathcal{P}}(X)$ (respectively $B_{\mathcal{P}}(X)$). It is obvious that $B_{\mathcal{P}}(X) \subset Q_{\mathcal{P}}(X)$. For every $p \in \mathcal{P}$ the application $\hat{p} : Q_{\mathcal{P}}(X) \to \mathbf{R}$ defined by

$$\hat{p}(T) = \inf\{ r > 0 \mid p(Tx) \leq rp\,(x)\,, (\forall)\, x \in X\},$$

is a submultiplicative seminorm on $Q_{\mathcal{P}}(X)$, satisfying the relation $\hat{p}(I) = 1$, and has the following properties

1. $\hat{p}(T) = \sup\limits_{p(x)=1} p\,(Tx) = \sup\limits_{p(x)\leq 1} p\,(Tx)$, $(\forall)\, p \in \mathcal{P}$;
2. $p\,(Tx) \leq \hat{p}\,(T)\, p\,(x)$, $(\forall)\, x \in X$.

For a calibration $\mathcal{P} \in \mathcal{C}(X)$, we denote by $\hat{\mathcal{P}}$ the family $\{\hat{p} \mid p \in \mathcal{P}\}$. $(Q_{\mathcal{P}}(X), \hat{\mathcal{P}})$ is a sequentially complete locally multiplicatively convex algebra for all $\mathcal{P} \in \mathcal{C}(X)$, and $B_{\mathcal{P}}(X)$ is a unitary normed algebra with respect to the norm $\|\bullet\|_{\mathcal{P}}$ defined by

$$\|T\|_{\mathcal{P}} = \sup\{\hat{p}(T) \mid p \in \mathcal{P}\}, (\forall)\, T \in B_{\mathcal{P}}(X).$$

and

$$\|T\|_{\mathcal{P}} = \inf\{M > 0 \mid \; p\,(Tx) \leq Mp\,(x)\,, (\forall)\, x \in X, (\forall)\;\, p \in \mathcal{P}\}.$$

If $T \in Q_{\mathcal{P}}(X)$ we said that $\alpha \in \mathbb{C}$ is in the resolvent set $\rho(Q_{\mathcal{P}}, T)$ if there exists $(\alpha I - T)^{-1} \in Q_{\mathcal{P}}(X)$. The spectral set $\sigma(Q_{\mathcal{P}}, T)$ will be the complement of $\rho(Q_{\mathcal{P}}, T)$.

An operator $T \in Q_{\mathcal{P}}(X)$ is a bounded element of the algebra $Q_{\mathcal{P}}(X)$ if it is a bounded element in the sense of G.R. Allan [1], i.e some scalar multiple of it generates a bounded semigroup. The class of the bounded elements of $Q_{\mathcal{P}}(X)$ is denoted by $(Q_{\mathcal{P}}(X))_0$. An operator $T \in Q_{\mathcal{P}}(X)$ is bounded in the algebra $Q_{\mathcal{P}}(X)$ if and only if there is $\mathcal{P}' \in \mathcal{C}(X)$ such that $\mathcal{P} \approx \mathcal{P}'$ and $T \in B_{\mathcal{P}'}(X)$ [9]. If $r_{\mathcal{P}}(T)$ is the $\mathcal{P}$-spectral radius of the operator T, i.e. it is the radius of boundness of the operator T in $Q_{\mathcal{P}}(X)$ given by

$$r_{\mathcal{P}}(T) = \inf\{\alpha > 0 \mid \alpha^{-1}T \text{ generates a bounded semigroup in } Q_{\mathcal{P}}(X)\},$$

then in [1] and [18] was proved that the following relation hold

$$r_{\mathcal{P}}(T) = \sup\{ \limsup_{n\to\infty} (\hat{p}\,(T^n))^{1/n} \mid \; p \in \mathcal{P}\} =$$

$$= \sup\{ \lim_{n\to\infty} (\hat{p}\,(T^n))^{1/n} \mid \; p \in \mathcal{P}\} = \sup\{ \inf_{n\geq 1} (\hat{p}\,(T^n))^{1/n} \mid \; p \in \mathcal{P}\} \tag{1}$$

$$r_{\mathcal{P}}(T) < +\infty \text{ if and only if } T \in (Q_{\mathcal{P}}(X))_0; \tag{2}$$

$$r_{\mathcal{P}}(T) = \inf\left\{\lambda > 0 \mid \; \lim_{n\to\infty} \frac{T^n}{\lambda^n} = 0 \;\right\}; \tag{3}$$

If $(X, \mathcal{P})$ is a locally convex space and $T \in (Q_{\mathcal{P}}(X))_0$, then the Neumann series $\sum\limits_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ converges to $R(\lambda, T)$(in $Q_{\mathcal{P}}(X)$), for every $|\lambda| > r_{\mathcal{P}}(T)$, and $R(\lambda, T) \in Q_{\mathcal{P}}(X)$ [18]. Moreover,

$$|\sigma(Q_{\mathcal{P}}, T)| = r_{\mathcal{P}}(T). \tag{4}$$

If $T \in (Q_{\mathcal{P}}, (X))$ has the spectrum $\sigma(Q_{\mathcal{P}}, T)$ bounded, then $T \in (Q_{\mathcal{P}}(X))_0$ [9]. Hence, $T \in (Q_{\mathcal{P}}, (X))_0$ if and only if the spectrum $\sigma(Q_{\mathcal{P}}, T)$ is bounded. If $(X, \mathcal{P})$ is a locally convex space and $T \in (Q_{\mathcal{P}}(X))_0$ we denote by $r^0_{\mathcal{P}}(T)$ the radius of boundness of the operator T in $(Q_{\mathcal{P}}(X))_0$. We say that $r^0_{\mathcal{P}}(T)$ is the $\mathcal{P}$-spectral radius of the operator T in the algebra $(Q_{\mathcal{P}}(X))_0$. From definition it follows that $r^0_{\mathcal{P}}(T) = r_{\mathcal{P}}(T)$ and $r^0_{\mathcal{P}}(T)$ has all the properties of the spectral radius $r_{\mathcal{P}}(T)$ presented above. We denote by $\rho(Q^0_{\mathcal{P}}, T)$ the resolvent set of T in $(Q_{\mathcal{P}}(X))_0$. The spectral set $\sigma(Q^0_{\mathcal{P}}, T)$ will be the complement of $\rho(Q^0_{\mathcal{P}}, T)$.

Definition 1.2 *Let $(X, \mathcal{P})$ be a locally convex space. The Waelbroeck resolvent set of an operator $T \in Q_{\mathcal{P}}(X)$, denoted by $\rho_W(Q_{\mathcal{P}}, T)$, is the subset of elements $\lambda_0 \in \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, for which there exists a neighborhood $V \in \mathcal{V}_{(\lambda_0)}$ such that:*

1. *the operator $\lambda I - T$ is invertible in $Q_{\mathcal{P}}(X)$ for all $\lambda \in V \backslash \{\infty\}$*
2. *the set $\{ \ (\ \lambda I - T\ )^{-1} \mid \lambda \in V \backslash \{\infty\} \ \}$ is bounded in $Q_{\mathcal{P}}(X)$.*

The Waelbroeck spectrum of T, denoted by $\sigma_W(Q_{\mathcal{P}}, T)$, is the complement of the set $\rho_W(Q_{\mathcal{P}}, T)$ in $\mathbb{C}_\infty$. It is obvious that $\sigma(Q_{\mathcal{P}}, T) \subset \sigma_W(Q_{\mathcal{P}}, T)$.

Definition 1.3 *Let $(X, \mathcal{P})$ be a locally convex space. An operator $T \in Q_{\mathcal{P}}(X)$ is regular if $\infty \notin \sigma_W(Q_{\mathcal{P}}, T)$, i.e. there exists some $t > 0$ such that:*

1. *the operator $\lambda I - T$ is invertible in $Q_{\mathcal{P}}(X)$, for all $\mid \lambda \mid > t$;*
2. *the set $\{R(\lambda, T) \mid \mid \lambda \mid > t\}$ is bounded in $Q_{\mathcal{P}}(X)$.*

2 Bounded operators in $Q_{\mathcal{P}}(X)$

Lemma 2.1 *Let $(X, \mathcal{P})$ be a locally convex space and $T \in (Q_{\mathcal{P}}(X))_0$ such that $r_{\mathcal{P}}(T) < 1$. Then the operator $I - T$ is invertible and $(I - T)^{-1} = \sum\limits_{n=0}^{\infty} T^n$.*

Proof. Assume that $r_{\mathcal{P}}(T) < t < 1$. From relation (3) it follows that

$$\limsup_{n \to \infty} \left(\hat{p}\left(T^n\right)\right)^{1/n} < t, (\forall)\ p \in \mathcal{P},$$

so for each $p \in \mathcal{P}$ there exists $n_p \in \mathbb{N}$ such that

$$\left(\hat{p}\left(T^n\right)\right)^{1/n} \leq \sup_{n \geq n_p} \left(\hat{p}\left(T^n\right)\right)^{1/n} < t, (\forall)\ n \geq n_p.$$

This relation implies that the series $\sum_{n=0}^{\infty} \hat{p}\left(T^n\right)$ converges, so

$$\lim_{n\to\infty} \hat{p}\left(T^n\right) = 0, (\forall)\ p \in \mathcal{P},$$

therefore $\lim_{n\to\infty} T^n = 0$. Since the algebra $Q_{\mathcal{P}}(X)$ is sequentially complete it results that the series $\sum_{n=0}^{\infty} T^n$ converges. Moreover,

$$(I-T)\sum_{n=0}^{m} T^n = \sum_{n=0}^{m} T^n(I-T) = I - T^{m+1},$$

so

$$(I-T)\sum_{n=0}^{\infty} T^n = \sum_{n=0}^{\infty} T^n(I-T) = I,$$

which implies that $I-T$ is invertible and $(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$. ∎

Lemma 2.2 *Let $(X, \mathcal{P})$ be a locally convex space. If $T \in (Q_{\mathcal{P}}(X))_0$ then*

1. *the mapping $\lambda \to R(\lambda, T)$ is holomorphic on $\rho_W(Q_{\mathcal{P}}, T)$;*
2. *$\frac{d^n}{d\lambda^n} R(\lambda, T) = (-1)^n n! R(\lambda, T)^{n+1}$, for every $n \in \mathbb{N}$;*
3. *$\lim_{|\lambda|\to\infty} R(\lambda, T) = 0$ and $\lim_{|\lambda|\to\infty} R(1, \lambda^{-1}T) = \lim_{|\lambda|\to\infty} \lambda R(1, T) = I$;*
4. *$\sigma_W(Q_{\mathcal{P}}, T) \neq \varnothing$.*

Proof. 1) If $\lambda_0 \in \rho_W(Q_{\mathcal{P}}, T)$ then there exists $V \in \mathcal{V}_{(\lambda_0)}$ with the properties (1) and (2) from definition (1.2). For every $\lambda \in V\backslash\{\infty\}$ we have

$$R(\lambda, T) - R(\lambda_0, T) = (\lambda_0 - \lambda) R(\lambda, T) R(\lambda_0, T)$$

and since the set $\{R(\lambda, T) |\ \lambda \in V\backslash\{\infty\}\}$ is bounded in $Q_{\mathcal{P}}(X)$ results that the application $\lambda \to R(\lambda, T)$ is continuous in λ_0, so

$$\lim_{\lambda\to\lambda_0} \frac{R(\lambda, T) - R(\lambda_0, T)}{\lambda - \lambda_0} = -R^2(\lambda_0, T)$$

If $\lambda_0 = \infty$ then, there exists some neighborhood $V \in \mathcal{V}_{(\infty)}$ such that the application $\lambda \to R(\lambda, T)$ is defined and bounded on $V\backslash\{\infty\}$. Moreover, this application is holomorphic and bounded on $V\backslash\{\infty\}$, which implies that it is holomorphic at ∞.

Therefore, the application $\lambda \to R(\lambda, T)$ is holomorphic on $\rho_W(Q_{\mathcal{P}}, T)$.

2) Results from the proof of (1).

3) For each $\lambda \in \rho_W(Q_{\mathcal{P}}, T)$, $\lambda \neq 0$, we have

$$\lambda^{-1}(I + TR(\lambda, T))(\lambda I - T) = I,$$

so

$$R(\lambda,T)=\lambda^{-1}(I+TR(\lambda,T)). \tag{5}$$

If $V\in\mathcal{V}_{(\lambda_0)}$ satisfies the conditions of the definition 1.2, then the set

$$\{TR(\lambda,T)|\ \lambda\in V\backslash\{\infty\}\}$$

is bounded, so from relation (5) it results that $\lim\limits_{|\lambda|\to\infty}R(\lambda,T)=0$.

From the equality

$$R(\lambda,T)=\lambda^{-1}R(1,\lambda^{-1}T),\lambda\neq 0,$$

and relation (5) results that

$$R(1,\lambda^{-1}T)=I+TR(\lambda,T),$$

so

$$\lim_{|\lambda|\to\infty}R(1,\lambda^{-1}T)=\lim_{|\lambda|\to\infty}(I+TR(\lambda,T))=I.$$

4) Assume that $\sigma_W(Q_{\mathcal{P}},T)=\varnothing$. Then the application $\lambda\to R(\lambda,T)$ is holomorphic on $\mathbb{C}$ and converges to 0 at infinity. From Liouville Theorem results that $R(\lambda,T)=0$, for all $\lambda\in\mathbb{C}$, hence $I=(\lambda I-T)R(\lambda,T)=0$, which is not true. ∎

Proposition 2.3 *Let $(X,\mathcal{P})$ be a locally convex space. An operator $T\in Q_{\mathcal{P}}(X)$ is regular if and only if $T\in(Q_{\mathcal{P}}(X))_0$.*

Proof. Assume that $T\in(Q_{\mathcal{P}}(X))_0$. It follows that there is $\mathcal{P}'\in\mathcal{C}(X)$ such that $\mathcal{P}\approx\mathcal{P}'$ and $T\in B_{\mathcal{P}'}(X)$. Moreover, $Q_{\mathcal{P}}(X)=Q_{\mathcal{P}'}(X)$.

If $|\ \lambda\ |>2\left\|T\right\|_{\mathcal{P}'}$, then the Neumann series $\sum\limits_{n=0}^{\infty}\frac{T^n}{\lambda^{n+1}}$ converges in $B_{\mathcal{P}'}(X)$ and its sum is $R\left(\lambda,T\right)$. This means that the operator $\lambda I-T$ is invertible in $Q_{\mathcal{P}}(X)$ for all $|\ \lambda\ |>2\left\|T\ \right\|_{\mathcal{P}'}$. Moreover, for each $\epsilon>0$ there exists an index $n_\epsilon\in\mathbb{N}$ such that

$$\left\|R\left(\lambda,T\right)-\sum_{k=0}^{n}\frac{T^k}{\lambda^{k+1}}\right\|_{\mathcal{P}'}<\epsilon,(\forall)\,n\geq n_\epsilon,$$

which implies that for each $n\geq n_\epsilon$ we have

$$\left\|R\left(\lambda,T\right)\right\|_{\mathcal{P}'}\leq\left\|R\left(\lambda,T\right)-\sum_{k=0}^{n_\epsilon}\frac{T^k}{\lambda^{k+1}}\right\|_{\mathcal{P}'}+\left\|\sum_{k=0}^{n_\epsilon}\frac{T^k}{\lambda^{k+1}}\right\|_{P'}<$$

$$<\epsilon+|\ \lambda\ |^{-1}\sum_{k=0}^{n_\epsilon}\left\|\frac{T^k}{\lambda^k}\right\|_{\mathcal{P}'}<\epsilon+(2\left\|T\right\|_{\mathcal{P}'})^{-1}\sum_{k=0}^{n_\epsilon}2^{-k}<\epsilon+(\left\|T\right\|_{\mathcal{P}'})^{-1}.$$

Since $\epsilon>0$ was arbitrarily chosen, we have that

$$\left\|R\left(\lambda,T\right)\right\|_{\mathcal{P}'}<(\left\|T\right\|_{\mathcal{P}'})^{-1},(\forall)\ |\ \lambda\ |>2\left\|T\right\|_{\mathcal{P}'}$$

From the definition of the norm $\| \|_{\mathcal{P}'}$ it follows that

$$\hat{p'}\,(R\,(\lambda, T)) < (\|T\|_{\mathcal{P}'})^{-1},$$

for any $p \in \mathcal{P}'$ and for each $|\,\lambda\,| > 2\,\|T\|_{\mathcal{P}'}$, which means that the set

$$\{R\,(\lambda, T)\,|\ \ |\,\lambda\,| > 2\,\|T\|_{\mathcal{P}'}\}$$

is bounded in $Q_{\mathcal{P}}(X) = Q_{\mathcal{P}'}(X)$. Hence T is regular.

Now suppose that $T \in Q_{\mathcal{P}}(X)$ is regular, but it is not bounded in $Q_{\mathcal{P}}(X)$. Since $\sigma(Q_{\mathcal{P}}, T) \subset \sigma_W(Q_{\mathcal{P}}, T)$ from (2) it follows that

$$|\,\sigma_W(Q_{\mathcal{P}}, T)\,| = |\,\sigma(Q_{\mathcal{P}}, T)\,| = r_{\mathcal{P}}(T) = \infty,$$

which contradicts the assumption we have made. Thus, T is a bounded element of $Q_{\mathcal{P}}(X)$. ∎

Proposition 2.4 *Let $(X, \mathcal{P})$ be a locally convex space. If $T \in (Q_{\mathcal{P}}(X))_0$, then*

$$\rho_W(Q_{\mathcal{P}}, T) = \rho(Q^0_{\mathcal{P}}, T).$$

Proof. If $\lambda_0 \in \rho(Q^0_{\mathcal{P}}, T)$ then from the previous proposition it follows that $R(\lambda_0, T)$ is a regular element of the algebra $Q_{\mathcal{P}}(X)$, so there exists $t > 0$ for which the condition (1) and (2) of the definition 1.3 are fulfilled. Those conditions are equivalent with

1') $(\lambda-\lambda_0)^{-1}I - R(\lambda_0, T)$ is invertible in $Q_{\mathcal{P}}(X)$ for all $|\lambda-\lambda_0| < t^{-1}, \lambda \neq \lambda_0$;

2') the set
$$\{R((\lambda-\lambda_0)^{-1}, R(\lambda_0, T))|\ |\lambda-\lambda_0| < t^{-1}, \lambda \neq \lambda_0\}$$

is bounded in $Q_{\mathcal{P}}(X)$.

From the condition (2') and lemma 2.2 it results that the set

$$\{(\lambda-\lambda_0)^{-1}R((\lambda-\lambda_0)^{-1}, R(\lambda_0, T))|\ |\lambda-\lambda_0| < t^{-1}, \lambda \neq \lambda_0\}$$

is bounded in $Q_{\mathcal{P}}(X)$. Moreover, each seminorm $\hat{p}$, $p \in \mathcal{P}$, is submultiplicative, so the set

$$\{(\lambda-\lambda_0)^{-1}R(\lambda_0, T)R((\lambda-\lambda_0)^{-1}, R(\lambda_0, T))|\ |\lambda-\lambda_0| < t^{-1}, \lambda \neq \lambda_0\}$$

is also bounded in $Q_{\mathcal{P}}(X)$. Since

$$(\lambda I - T)(\lambda_0-\lambda)^{-1}R(\lambda_0, T)R((\lambda-\lambda_0)^{-1}, R(\lambda_0, T)) =$$

$$= ((\lambda_0 I - T) + (\lambda-\lambda_0)I)(\lambda_0-\lambda)^{-1}R(\lambda_0, T)R((\lambda-\lambda_0)^{-1}, R(\lambda_0, T)) =$$

$$= (\lambda_0-\lambda)^{-1}R((\lambda-\lambda_0)^{-1}, R(\lambda_0, T)) - R(\lambda_0, T)R((\lambda-\lambda_0)^{-1}, R(\lambda_0, T)) =$$

$$= ((\lambda_0-\lambda)^{-1}I - R(\lambda_0, T))R((\lambda-\lambda_0)^{-1}, R(\lambda_0, T)) = I,$$

results that

$$R(\lambda, T) = (\lambda_0 - \lambda)^{-1} R(\lambda_0, T) R((\lambda - \lambda_0)^{-1}, R(\lambda_0, T)). \tag{6}$$

Therefore, the conditions
1) $\lambda I - T$ is invertible for all $|\lambda - \lambda_0| < t^{-1}$;
2) $\{R(\lambda, T) | \ |\lambda - \lambda_0| < t^{-1}\}$ is bounded in $Q_{\mathcal{P}}(X)$,
of definition (1.2) are fulfilled, so $\lambda_0 \in \rho_W(Q_{\mathcal{P}}, T)$ and $\rho(Q^0_{\mathcal{P}}, T) \subset \rho_W(Q_{\mathcal{P}}, T)$.

Conversely, if $\lambda_0 \in \rho_W(Q_{\mathcal{P}}, T)$ there exists $K > 0$ such that
1") $\lambda I - T$ is invertible for all $|\lambda - \lambda_0| < K$;
2") $\{R(\lambda, T) | \ |\lambda - \lambda_0| < K\}$ is bounded in $Q_{\mathcal{P}}(X)$.

From the equality

$$(\lambda_0 - \lambda) R(\lambda, T)(\lambda_0 I - T)((\lambda_0 - \lambda)^{-1} I - R(\lambda_0, T)) = I$$

it follows that

$$R((\lambda_0 - \lambda)^{-1}, R(\lambda_0, T)) = (\lambda_0 - \lambda) R(\lambda, T)(\lambda_0 I - T)$$

Hence property (2") implies that the set

$$\{(\lambda - \lambda_0)^{-1} R((\lambda - \lambda_0)^{-1}, R(\lambda_0, T)) | \ |\lambda - \lambda_0|^{-1} > K^{-1}, \lambda \neq \lambda_0\}$$

is bounded in $Q_{\mathcal{P}}(X)$, so $R(\lambda_0, T)$ is regular in $Q_{\mathcal{P}}(X)$. From the previous proposition results that $R(\lambda_0, T) \in (Q_{\mathcal{P}}(X))_0$ and $\lambda_0 \in \rho(Q^0_{\mathcal{P}}, T)$. ∎

Proposition 2.5 *Let $(X, \mathcal{P})$ be a locally convex space. If $T \in (Q_{\mathcal{P}}(X))_0$ and $|\lambda_0| > r_{\mathcal{P}}(T)$, then $\lambda_0 \in \rho(Q^0_{\mathcal{P}}, T)$.*

Proof. The series $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ converges to $R(\lambda_0, T) \in Q_{\mathcal{P}}(X)$, for all $|\lambda| > r_{\mathcal{P}}(T)$, hence it results that there exists $\epsilon > 0$ such that

$$D(\lambda_0, \varepsilon) = \{\lambda \mid |\lambda - \lambda_0| < \varepsilon\} \subset \{\mu \mid |\mu| > r_{\mathcal{P}}(T)\},$$

so the operator $\lambda I - T$ is invertible, for every $\lambda \in D(\lambda_0, \epsilon)$, and $(\lambda I - T)^{-1} \in Q_{\mathcal{P}}(X)$.

Now we will prove that the set $\sigma(Q_{\mathcal{P}}, R(\lambda_0, T))$ is bounded. If $|\mu| > \epsilon^{-1}$, then $|\mu|^{-1} < \epsilon$ and $\lambda_0 - \mu^{-1} \in D(\lambda_0, \epsilon)$. From the previous observations results that $(\lambda_0 - \mu^{-1})I - T$ is invertible and $((\lambda_0 - \mu^{-1})I - T)^{-1} \in Q_{\mathcal{P}}(X)$.

Since

$$\mu^{-1} R\left(\lambda_0 - \mu^{-1}, T\right)(\lambda_0 I - T)(\mu I - R(\lambda_0, T)) =$$
$$= R\left(\lambda_0 - \mu^{-1}, T\right)(\lambda_0 I - T) - \mu^{-1} R\left(\lambda_0 - \mu^{-1}, T\right) =$$
$$= R\left(\lambda_0 - \mu^{-1}, T\right)(((\lambda_0 - \mu^{-1})I - T) + \mu^{-1} I) - \mu^{-1} R\left(\lambda_0 - \mu^{-1}, T\right) =$$
$$= I + \mu^{-1} R\left(\lambda_0 - \mu^{-1}, T\right) - \mu^{-1} R\left(\lambda_0 - \mu^{-1}, T\right) = I.$$

it results that

$$R(\mu, R(\lambda_0, T)) = \mu^{-1} R\left(\lambda_0 - \mu^{-1}, T\right)(\lambda_0 I - T)$$

But

$$R\left(\lambda_0 - \mu^{-1}, T\right), (\lambda_0 I - T) \in Q_{\mathcal{P}}(X),$$

so $R(\mu, R(\lambda_0, T)) \in Q_{\mathcal{P}}(X)$, for all $|\ \mu\ | > \epsilon^{-1}$.

This implies that $\sigma(Q_{\mathcal{P}}, R(\lambda_0, T)) \subset D\left(0, \epsilon^{-1}\right)$, so the set $\sigma(Q_{\mathcal{P}}, R(\lambda_0, T))$ is bounded. Thus $R(\lambda_0, T) \in (Q_{\mathcal{P}}(X))_0$. ∎

Corollary 2.6 *Let X be a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$. If $T \in (Q_{\mathcal{P}}(X))_0$ then*

$$|\sigma(Q_{\mathcal{P}}, T)| = |\sigma_W(Q_{\mathcal{P}}, T)| = r_{\mathcal{P}}(T)$$

Proof. It is a direct consequence of propositions 2.4, 2.5 and relation (4). ∎

Definition 2.7 *Let $(X, \mathcal{P})$ be a locally convex space. An operator $T \in Q_{\mathcal{P}}(X)$ is said to be $\mathcal{P}$-quasinilpotent if $r_{\mathcal{P}}(T) = 0$.*

Remark 2.8

1. *If $T \in Q_{\mathcal{P}}(X)$ is $\mathcal{P}$-quasinilpotent, then $T \in (Q_{\mathcal{P}}(X))_0$ and $\sigma_W(Q_{\mathcal{P}}, T) = \{0\}$.*
2. *$T \in Q_{\mathcal{P}}(X)$ is $\mathcal{P}$-quasinilpotent if and only if $\sigma(Q_{\mathcal{P}}, T) = \{0\}$.*

3 A functional calculus for bounded operators

A functional calculus for regular operator on quasi-complete locally convex space is presented by L.Waelbroeck in [20]. In this section using some ideas from I. Colojoara [4] and L.Waelbroeck [20] we prove that we can construct a functional calculus for the bounded elements of the algebra $Q_{\mathcal{P}}(X)$ (which by previous section are regular elements of this algebra), when $(X, \mathcal{P})$ ($\mathcal{P} \in \mathcal{C}(X)$) is a sequentially complete locally convex space. For the theory of holomorphic functions on locally convex spaces can see [2] or [5].

Let $\mathcal{P} \in C(X)$ be arbitrary chosen and $D \subset \mathbb{C}$ a relatively compact open set. Denote by $\mathcal{O}(D, Q_{\mathcal{P}}(X))$ the unitary algebra of the functions $f : D \to Q_{\mathcal{P}}(X)$ which are holomorphic on D and continuous on $\overline{D}$.

Lemma 3.1 *If $p \in \mathcal{P}$, then the mapping $|\bullet|_{p,D}: \mathcal{O}(D, Q_{\mathcal{P}}(X)) \to \mathbb{R}$ given by*

$$|\ f\ |_{p,D} = \sup_{z \in D} p(f(z)), (\forall)\, f \in \mathcal{O}(D, Q_{\mathcal{P}}(X)),$$

is a submultiplicative seminorm on $\mathcal{O}(D, Q_{\mathcal{P}}(X))$.

If we denote by $\tau_{\mathcal{P},D}$ the topology defined by the family of seminorms $\{| \cdot |_{p,D} \,| p \in \mathcal{P}\}$ on $\mathcal{O}(D, Q_{\mathcal{P}}(X))$, then $(\mathcal{O}(D, Q_{\mathcal{P}}(X)), \tau_{\mathcal{P},D})$ is a l.m.c.-algebra.

Let $K \subset \mathbb{C}$ be a compact set, arbitrarlly chosen. We define the set
$\mathcal{O}(K, Q_{\mathcal{P}}(X)) = \cup\{\mathcal{O}(D, Q_{\mathcal{P}}(X))|\, D \text{ is relatively compact open set and } K \subset D\}$

If $D_1, D_2 \subset \mathbb{C}$ are relatively compact open sets such that $K \subset D_i$, $i = 1, 2$, and $f_i \in \mathcal{O}(D_i, Q_{\mathcal{P}}(X))$, $i = 1, 2$, we say that $f_1 \backsim f_2$ if and only if there exists an open set D such that $K \subset D \subset D_1 \cap D_2$ and $f_1|_D = f_2|_D$. Denote by $\mathcal{A}(K, Q_{\mathcal{P}}(X))$ the set of the equivalence classes of $\mathcal{O}(K, Q_{\mathcal{P}}(X))$ respect to this equivalence relation. It is easy to see that $\mathcal{A}(K, Q_{\mathcal{P}}(X))$ is a unitary algebra and the elements of this algebra are usually called germs of holomorphic functions from K to $Q_{\mathcal{P}}(X)$.

Remark 3.2 *We consider the following notations:*

1. *$\tilde{f}$ is the germ of the holomorphic function $f \in \mathcal{O}(D, Q_{\mathcal{P}}(X))$.*
2. *φ is the canonical morphism $\mathcal{O}(K, Q_{\mathcal{P}}(X)) \to \mathcal{A}(K, Q_{\mathcal{P}}(X))$;*
3. *φ_D is the restriction of φ to $\mathcal{O}(D, Q_{\mathcal{P}}(X))$.*

Remark 3.3
1. *Since we can identify $\mathbb{C}$ with $\mathbb{C}I = \{\lambda I \mid \lambda \in \mathbb{C}\}$, the algebras $\mathcal{O}(K, \mathbb{C})$ and $\mathcal{A}(K, \mathbb{C})$ can be considered subalgebras of $\mathcal{O}(K, Q_{\mathcal{P}}(X))$, respectively $\mathcal{A}(K, Q_{\mathcal{P}}(X))$. Therefore, we write $\mathcal{O}(K)$ and $\mathcal{A}(K)$ instead of $\mathcal{O}(K, \mathbb{C})$ and $\mathcal{A}(K, \mathbb{C})$.*
2. *If $\tau_{\mathcal{P},ind} = \lim\limits_{\to D} \tau_{\mathcal{P},D}$ (inductive limit), then $(\mathcal{A}(K, Q_{\mathcal{P}}(X)), \tau_{\mathcal{P},ind})$ is a l.m.c.-algebra.*

We need the following lemma from complex analysis.

Lemma 3.4 *For each compact set $K \subset \mathbb{C}$ and each relatively compact open set $D \supset K$ there exists some open set G such that:*

1. *$K \subset G \subset \overline{G} \subset D$;*
2. *G has a finite number of conex components $(G_i)_{i=\overline{1,n}}$, the closure of which are pairwise disjoint;*
3. *the boundary ∂G_i of $G_i, i = \overline{1,n}$, consists of a finite positive number of closed rectifiable Jordan curves $(\Gamma_{ij})_{j=\overline{1,m_i}}$, no two of which intersect;*
4. *$K \cap \Gamma_{ij} = \varnothing$, for each $i = \overline{1,n}$ and every $j = \overline{1,m_i}$.*

Definition 3.5 *If the sets K and D are like in the previous lemma, then an open set G is called Cauchy domain for the pair (K, D) if satisfies the properties (1)-(4). The boundary*

$$\Gamma = \cup_{i=\overline{1,n}} \cup_{j=\overline{1,m_i}} \Gamma_{ij}$$

of G is called Cauchy boundary for the pair (K, D).

Theorem 3.6 *If* $\mathcal{P} \in \mathcal{C}_0(X)$ *and* $T \in (Q_{\mathcal{P}}(X))_0$, *then for each relatively compact open set* $D \supset \sigma_W(Q_{\mathcal{P}}, T)$ *there exists an application*

$$\mathcal{F}_{T,D} : \mathcal{O}(D, Q_{\mathcal{P}}(X)) \to Q_{\mathcal{P}}(X)$$

with the properties:

1. $\mathcal{F}_{T,D}$ *is continuous and linear;*
2. $\mathcal{F}_{T,D}(k_S) = S$, *where* $k_S \equiv S$;
3. $\mathcal{F}_{T,D}(id_I) = T$, *where* $id_I(\lambda) = \lambda I$, *for every* $\lambda \in \mathbb{C}$.

Proof. Let Γ be a Cauchy boundary for the pair $(\sigma_W(Q_{\mathcal{P}}, T), D)$. Then the integral

$$\frac{1}{2\pi i}\int\limits_{\Gamma} f(\lambda) R(\lambda, T)\, d\lambda, (\forall) f \in \mathcal{O}(D, Q_{\mathcal{P}}(X)),$$

exists like Stieltjes integral, since $Q_{\mathcal{P}}(X)$ is a sequentially complete l.m.c.-algebra and the applications $t \rightsquigarrow f(\omega(t))R(\omega(t), T)$ are continuous on $[0, 1]$ for a continuous parametrization ω of Γ.

Moreover, if Γ_1 and Γ_2 are Cauchy boundaries for the pair $(\sigma_W(Q_{\mathcal{P}}, T), D)$ then

$$\frac{1}{2\pi i}\int\limits_{\Gamma_1} f(z) R(\lambda, T)\, d\lambda = \frac{1}{2\pi i}\int\limits_{\Gamma_2} f(\lambda) R(\lambda, T)\, d\lambda, (\forall) f \in \mathcal{O}(D, Q_{\mathcal{P}}(X)),$$

hence the application $\mathcal{F}_{T,D} : \mathcal{O}(K, Q_{\mathcal{P}}(X)) \to Q_{\mathcal{P}}(X)$ given by formula

$$\mathcal{F}_{T,D}(f) = \frac{1}{2\pi i}\int\limits_{\Gamma} f(\lambda) R(\lambda, T)\, dz, (\forall) f \in \mathcal{O}(D, Q_{\mathcal{P}}(X)),$$

is well defined. Now we prove that $\mathcal{F}_{T,D}$ has the properties (1)-(3).

The linearity is obvious. For every $p \in \mathcal{P}$ and every $f \in \mathcal{O}(D, Q_{\mathcal{P}}(X))$ we have

$$\hat{p}(\mathcal{F}_{T,D}(f)) \leq \frac{L(\Gamma)}{2\pi}\sup_{\lambda\in\Gamma}\hat{p}(R(\lambda, T))\sup_{\lambda\in\Gamma}\hat{p}(f(\lambda)) \leq \frac{L(\Gamma)}{2\pi}\sup_{\lambda\in\Gamma}\hat{p}(R(\lambda, T)) \mid f \mid_{p,D},$$

where $L(\Gamma)$ is the length of Γ, which implies the continuity of application $\mathcal{F}_{T,D}$.

Let $r > r_{\mathcal{P}}(T)$ and $\Gamma_r = \{z \in C |\ |z| = r\}$. For each $\lambda \in \Gamma_r$ we have $r_{\mathcal{P}}(\frac{T}{\lambda}) < 1$, so from lemma 2.1 results that

$$R(\lambda, T) = \lambda^{-1}\left(I - \frac{T}{\lambda}\right)^{-1} = \lambda^{-1}\sum_{n\in\mathbb{N}}\left(\frac{T}{\lambda}\right)^{n} = \sum_{n\in\mathbb{N}}\frac{T^n}{\lambda^{n+1}}$$

This observation implies that

$$\mathcal{F}_{T,D}(k_S) = \frac{1}{2\pi i}\int\limits_{\Gamma} k_S(\lambda) R(\lambda, T)\, d\lambda = \frac{S}{2\pi i}\sum_{n\in\mathbb{N}} T^n \int\limits_{\Gamma}\frac{d\lambda}{\lambda^{n+1}} = S$$

$$\mathcal{F}_{T,D}\left(id_I\right) = \frac{1}{2\pi i}\int\limits_{\Gamma} \lambda R\left(\lambda, T\right) d\lambda = \frac{1}{2\pi i}\sum_{n\in\mathbb{N}} T^n \int\limits_{\Gamma} \frac{d\lambda}{\lambda^n} = T.$$

■

Corollary 3.7 *If $\mathcal{P} \in \mathcal{C}_0(X)$ and $T \in (Q_{\mathcal{P}}(X))_0$, then there exists an application $\mathcal{F}_T : \mathcal{A}(\sigma_W(Q_{\mathcal{P}}, T), Q_{\mathcal{P}}(X)) \to Q_{\mathcal{P}}(X)$ which satisfies the conditions:*

1. *$\mathcal{F}_T$ is continuous and linear;*
2. *$\mathcal{F}_T\left(\tilde{k}_S\right) = S$, where $\tilde{k}_T$ is the germ of the function $k_S \equiv S$;*
3. *$\mathcal{F}_T\left(\widetilde{id}_I\right) = T$, where $\widetilde{id}_I$ is the germ of the function $id_I(\lambda) = \lambda I$, for all $\lambda \in \mathbb{C}$*

Proof. If $\tilde{f} \in \mathcal{A}(\sigma_W(Q_{\mathcal{P}}, T))$, then we consider

$$\mathcal{F}_T(\tilde{f}) = \mathcal{F}_{T,D}\left(f\right), (\forall)\, \tilde{f} \in \mathcal{A}(\sigma_W(Q_{\mathcal{P}}, T)),$$

where $f \in \mathcal{O}(D, Q_{\mathcal{P}}, T))$ is an element of the equivalence class $\tilde{f}$. It is obvious that the definition of $\mathcal{F}_T\left(\tilde{f}\right)$ is independent of the function f and $\mathcal{F}_T\left(\tilde{f}\right)$ is linear. Since $\mathcal{F}_{T,D} = \mathcal{F}_T \circ \varphi_D$ and $\mathcal{F}_{T,D}$ is continuous results that $\mathcal{F}_T$ is continuous.

The properties (2) and (3) results directly from the previous theorem. ■

Corollary 3.8 *If $\mathcal{P} \in \mathcal{C}_0(X)$ and $T \in (Q_{\mathcal{P}}(X))_0$, then there exists an unique unitary continuous morphism $F_T : \mathcal{A}(\sigma_W(Q_{\mathcal{P}}, T)) \to Q_{\mathcal{P}}(X)$ which satisfies the condition $F_T\left(\widetilde{id}\right) = T$, where id is the identity function on $\mathbb{C}$.*

Proof. The application F_T and $F_{T,D}$ are defined in the same way like the applications $\mathcal{F}_T$ and $\mathcal{F}_{T,D}$. It is easily to see that F_T and $F_{T,D}$ are linear and continuous. Moreover, F_T is unitary and $F_T\left(\widetilde{id}\right) = T$.

Now, we prove that F_T is multiplicative. Let $\tilde{f}, \tilde{g} \in \mathcal{A}(\sigma_W(Q_{\mathcal{P}}, T))$ and $f \in \tilde{f}$, respectively $g \in \tilde{g}$. We consider that G and G' are two Cauchy domains with the property $\overline{G'} \subset G$. If Γ and Γ' are the boundaries of G and G' then

$$F_T(\tilde{f})F_T\left(\tilde{g}\right) = -\frac{1}{(2\pi i)^2}\int_{\Gamma}\int_{\Gamma'} f(\lambda)g(\omega)R(\lambda, T)R(\omega, T)d\lambda d\omega$$

Since $\overline{G'} \subset G$, results that $\Gamma \cap \Gamma' = \Phi$, so

$$\omega - \lambda \neq 0,\ (\forall)\lambda \in \Gamma, (\forall)\omega \in \Gamma'.$$

From the equality

$$R(\lambda, T) - R(\omega, T) = (\omega - \lambda)R(\lambda, T)R(\omega, T)$$

it follows

$$F_T\left(\tilde{f}\right)F_T\left(\tilde{g}\right)=\frac{1}{(2\pi i)^2}\int_{\Gamma}f(\lambda)R(\lambda,T)\left(\int_{\Gamma'}\frac{g(\omega)}{\omega-\lambda}d\omega\right)d\lambda+$$

$$+\frac{1}{(2\pi i)^2}\int_{\Gamma'}g(\omega)R(\omega,T)\left(\int_{\Gamma}\frac{f(\lambda)}{\lambda-\omega}d\lambda\right)d\omega=$$

$$=\frac{1}{2\pi i}\int_{\Gamma}f(\lambda)g\left(\lambda\right)R(\lambda,T)d\omega=F_T\left(\tilde{f}\tilde{g}\right)$$

Assume that $F:\mathcal{A}(\sigma_W(Q_{\mathcal{P}},T))\to Q_{\mathcal{P}}(X)$ is an unitary continuous morphism which satisfies the condition $F\left(\widetilde{id}\right)=T$. We prove that $F_T=F$.

Let $\tilde{f}\in\mathcal{A}(\sigma_W(Q_{\mathcal{P}},T))$, $D\supset\sigma_W(Q_{\mathcal{P}},T)$ a relatively compact open set, $f\in\mathcal{O}(D)$, such that $f\in\tilde{f}$, and G a Cauchy domain for $(\sigma_W(Q_{\mathcal{P}},T),D)$ with the boundary Γ. For every $n\in\mathbb{N}\setminus\{0\}$ and $z_1,...,z_n\in\Gamma$ we consider the function $f_n:G\to\mathbb{C}$ given by the relation

$$f_n\left(\omega\right)=\frac{1}{2\pi i}\sum_{j=1}^{n}\frac{f\left(z_j\right)\left(z_{j+1}-z_j\right)}{z_j-\omega},(\forall)\,\omega\in G. \tag{7}$$

Then,

$$\lim_{n\to\infty}f_n\left(\omega\right)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{z-\omega}dz=f\left(\omega\right)$$

and since this convergence is uniformly on each compact set $K\subset G$, results that $\lim_{\tau_{ind}}\tilde{f}_n=\tilde{f}$. Using the continuity of F it results that

$$\lim_{n\to\infty}F\left(\tilde{f}_n\right)=F\left(\tilde{f}\right) \tag{8}$$

Since F is a unitary morphism with the property $F\left(\widetilde{id}\right)=T$, then from relation (7) results that

$$F\left(\tilde{f}_n\right)=\frac{1}{2\pi i}\sum_{j=1}^{n}f\left(z_j\right)\left(z_{j+1}-z_j\right)R(z_j,T)$$

so

$$\lim_{n\to\infty}F\left(\tilde{f}_n\right)=\frac{1}{2\pi i}\int_{\Gamma}f\left(z\right)R\left(z,T\right)dz, \tag{9}$$

From relations (8) and (9) results that

$$F\left(\tilde{f}\right)=\frac{1}{2\pi i}\int_{\Gamma}f\left(z\right)R\left(z,T\right)dz=F_{T,D}(f)=F_T\left(\tilde{f}\right)$$

which implies that $F_T=F$. ∎

Lemma 3.9 *If $K \subset \mathbb{C}$ is a compact set, then each element of the algebra $\mathcal{A}(K)$ is regular.*

Proof. Let $\tilde{f} \in \mathcal{A}(K)$, $D \supset K$ a relatively compact open set, $f \in \mathcal{O}(D)$ ($f \in \tilde{f}$) and $\omega_0 \notin f(K)$. Then there are two relatively compact open set U and V such that $\omega_0 \in U$, $f(K) \subset V$ and $\overline{U} \cap \overline{V} = \Phi$. For every $\omega \in U$ the function $f_\omega : \overline{f^{-1}(V)} \to \mathbb{C}$ given by relation

$$f_\omega(\lambda) = \frac{1}{\omega - f(\lambda)}, (\forall)\, \lambda \in \overline{f^{-1}(V)}$$

is holomorphic on $f^{-1}(V)$, so $\tilde{f}_\omega \in \mathcal{A}(K)$.

Since for every compact set $A \subset f^{-1}(V)$ we have

$$\sup_{\omega \in \overline{U}} \sup_{\lambda \in A} |f_\omega(\lambda)| < \infty$$

it results that the set $\{ \tilde{f}_\omega | \omega \in \overline{U}\}$ is bounded in $(\mathcal{A}(K), \tau_{ind})$. Moreover,

$$(\omega\tilde{1} - \tilde{f})\tilde{f}_\omega = \widetilde{1}$$

so $\omega \in \sigma_W(\tilde{f})$. Therefore $\sigma_W(\tilde{f}) \subset f(K)$. Since K is compact the set $f(K)$ is compact, hence $\sigma_W(\tilde{f})$ is compact and $\tilde{f}$ is regular. ∎

Lemma 3.10 *If X and Y are unitary locally convex algebra and $F : X \to Y$ is an unitary continuous morphism, then $F(X_r) \subset Y_r$, where X_r and Y_r are the algebras of the regular elements of X, respectively Y.*

Proof. If $x \in X_r$, then there exists $k > 0$ such that $\lambda e - x$ is invertible for every $|\lambda| > k$ and the set $\{R(\lambda, x) \,||\lambda| > k\}$ is bounded in X. Since F is unitary morphism it follows that

$$F(R(\lambda, x)) = R(\lambda, F(x)), (\forall)\, |\lambda| > k,$$

so from continuity of F it results that the set

$$\{F(R(\lambda, x)) \,||\lambda| > k\} = \{R(\lambda, F(x))||\lambda| > k\}$$

is bounded. Hence, $F(x)$ is regular. ∎

Proposition 3.11 *If $\mathcal{P} \in \mathcal{C}_0(X)$ and $T \in (Q_\mathcal{P}(X))_0$, then*

$$F_T(\mathcal{A}(\sigma_W(Q_\mathcal{P}, T))) \subset (Q_\mathcal{P}(X))_0.$$

Proof. From lemmas 3.9 and 3.10 results that $F_T(\tilde{f})$ is a regular element of algebra $Q_\mathcal{P}(X)$, for every $\tilde{f} \in \mathcal{A}(\sigma_W(Q_\mathcal{P}, T))$, so by proposition 2.3 we have that $F_T(\tilde{f}) \in (Q_\mathcal{P}(X))_0$. ∎

Remark 3.12 *If $\mathcal{P} \in \mathcal{C}_0(X)$, $T \in (Q_{\mathcal{P}}(X))_0$ and P is a polynomial, then*

$$F_T(\tilde{P}) = \tilde{P}(T) \text{ and } F_{T,D}(P) = P(T).$$

for each relatively compact open set $D \supset \sigma_W(Q_{\mathcal{P}}, T)$. Hence, for each $T \in (Q_{\mathcal{P}}(X))_0$ we can use the following notation:

$$F_T(\tilde{f}) = \tilde{f}(T) \text{ and } F_{T,D}(f) = f(T).$$

where $\tilde{f} \in \mathcal{A}(K)$, $D \supset K$ open set and $f \in \mathcal{O}(D)$, such that $f \in \tilde{f}$.

The following theorem represents the analogous of the spectral mapping theorem for Banach spaces.

Theorem 3.13 *If $\mathcal{P} \in \mathcal{C}_0(X)$, $T \in (Q_{\mathcal{P}}(X))_0$ and f is a holomorphic function on an open set $D \supset \sigma_W(Q_{\mathcal{P}}, T)$, then*

$$\sigma_W(Q_{\mathcal{P}}, f(T)) = f(\sigma_W(Q_{\mathcal{P}}, T)).$$

Proof. From lemma 3.10 it follows that the operator $\tilde{f}(T)$ is regular element of the algebra $Q_{\mathcal{P}}(X)$, so the spectrum $\sigma_W(Q_{\mathcal{P}}, f(T))$ is compact.

Let $\omega_0 \notin f(\sigma_W(Q_{\mathcal{P}}, T))$. Then there are two relatively compact open set U and V such that $\omega_0 \in U$, $\sigma_W(Q_{\mathcal{P}}, f(T)) \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$. We proved in the proof of lemma 3.9 that if the functions $f_\omega : \overline{f^{-1}(V)} \to \mathbb{C}$, $\omega \in U$, are given by

$$f_\omega(\lambda) = \frac{1}{\omega - f(\lambda)}, (\forall)\, \lambda \in \overline{f^{-1}(V)}$$

then the set $\{\tilde{f}_\omega | \omega \in U\}$ is bounded in $(\mathcal{A}(\sigma_W(Q_{\mathcal{P}}, f(T))), \tau_{ind})$. The morphism F_T is unitary, so

$$F_T(\tilde{f}_\omega)(\omega I - F_T(\tilde{f})) = F_T(\tilde{1}) = I.$$

Now from the continuity of F_T results that the set

$$\{F_T(\tilde{f}_\omega) \mid \omega \in \overline{U}\} = \{R(\omega, F_T(\tilde{f})) \mid \omega \in \overline{U}\} = \{R(\omega, \tilde{f}(T)) \mid \omega \in \overline{U}\}$$

is bounded in $Q_{\mathcal{P}}(X)$. Therefore, $\omega_0 \notin \sigma_W(Q_{\mathcal{P}}, f(T))$ and

$$\sigma_W(Q_{\mathcal{P}}, f(T)) \subset f(\sigma_W(Q_{\mathcal{P}}, T)).$$

If $\omega_0 \in \sigma_W(Q_{\mathcal{P}}, T)$ and $g_{\omega_0} : D \to \mathbb{C}$ is defined by

$$g_{\omega_0}(\lambda) = \begin{cases} \frac{f(\lambda) - f(\omega_0)}{\lambda - \omega_0}, & \text{for } \lambda \neq \omega_0, \\ f'(\omega_0), & \text{for } \lambda = \omega_0, \end{cases}$$

then $g_{\omega_0} \in \mathcal{O}(D)$ and

$$f(\omega_0) - f(\lambda) = (\omega_0 - \lambda) g_{\omega_0}(\lambda), (\forall)\, \lambda \in D.$$

Therefore

$$f(\omega_0) I - f(T) = (\omega_0 I - T) g_{\omega_0}(T).$$

and since $\omega_0 I - T$ is not invertible, it results that $f(\omega_0) \in \sigma_W(Q_{\mathcal{P}}, f(T))$, so

$$f(\sigma_W(Q_{\mathcal{P}},T)) \subset \sigma_W(Q_{\mathcal{P}}, f(T)).$$

∎

Theorem 3.14 *Let $\mathcal{P} \in \mathcal{C}_0(X)$ and $T \in (Q_{\mathcal{P}}(X))_0$. If f is holomorphic function on the open set $D \supset \sigma_W(Q_{\mathcal{P}},T)$ and $g \in \mathcal{O}(D_g)$, such that $D_g \supset f(D)$, then $(g \circ f)(T) = g(f(T))$.*

Proof. Let G be a Cauchy domain for the pair $(\sigma_W(Q_{\mathcal{P}},T), D)$ and Γ the boundary of G. For each $\omega \notin \overline{f(G)}$, the function $f_\omega : \overline{G} \to \mathbb{C}$,

$$f_\omega(\lambda) = \frac{1}{\omega - f(\lambda)}, (\forall)\, \lambda \in \overline{G},$$

is holomorphic, hence we can define $f_\omega(T)$, where

$$f_\omega(T) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\omega - f(\lambda)} R(\lambda,T)d\lambda = R(\omega, f(T)). \tag{10}$$

If we chose a Cauchy domain G' for the pair $(\sigma_W(Q_{\mathcal{P}}, f(T)), D_g)$ with boundary Γ' such that $\overline{f(G)} \subset G'$, then $f(\Gamma) \cap \Gamma' = \varnothing$, so we can define the function given by (10) for all $\lambda \in \Gamma$ and $\omega \in \Gamma'$. Thus, from relation (10) and Cauchy formula it results

$$g(f(T)) = \frac{1}{2\pi i}\int_{\Gamma'} g(\omega)R(\omega, f(T))d\omega =$$

$$= \frac{1}{2\pi i}\int_{\Gamma'} g(\omega)\left(\frac{1}{2\pi i}\int_\Gamma \frac{1}{\omega - f(\lambda)} R(\lambda,T)d\lambda\right)d\omega =$$

$$= \frac{1}{2\pi i}\int_\Gamma R(\lambda,T)\left(\frac{1}{2\pi i}\int_{\Gamma'} \frac{g(\omega)}{\omega - f(\lambda)} d\omega\right)d\lambda = \frac{1}{2\pi i}\int_\Gamma g(f(\lambda))R(\lambda,T)d\lambda =$$

$$= \frac{1}{2\pi i}\int_\Gamma (g \circ f)(\lambda)R(\lambda,T)d\lambda = (g \circ f)(T)$$

∎

Next, we develop the properties of the exponential function of a quotient bounded operator.

Lemma 3.15 *Assume that $\mathcal{P} \in C_0(X)$ and $T \in (Q_{\mathcal{P}}(X))_0$. If f is a holomorphic function on the open set $D \supset \sigma_W(Q_{\mathcal{P}},T)$ and $f(\lambda) = \sum_{k=0}^{\infty} \alpha_k \lambda^k$ on D, then $f(T) = \sum_{k=0}^{\infty} \alpha_k T^k$.*

Proof. For $\epsilon > 0$, sufficiently small, the power series $\sum_{k=0}^{\infty} \alpha_k \lambda^k$ converges uniformly on the boundary Γ of the disc $D = \{\ \lambda \mid |\lambda| = |\sigma_W(Q_{\mathcal{P}},T)| + \epsilon\}$.

From corollary 3.8 it results that

$$f(T) = \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)\, R(\lambda, T)\, d\lambda = \frac{1}{2\pi i}\int_{\Gamma}\left(\sum_{k=0}^{\infty}\alpha_k\lambda^k\right) R(\lambda, T)\, d\lambda =$$

$$= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\alpha_k\int_{\Gamma}\lambda^k R(\lambda, T)\ d\lambda$$

For every $|\lambda| > |\sigma_W(Q_{\mathcal{P}}, T)|$ we have $R(\lambda, T) = \sum\limits_{k=0}^{\infty}\frac{T^k}{\lambda^{k+1}}$, so from Cauchy formula it follows

$$f(T) = \tfrac{1}{2\pi i}\sum_{k=0}^{\infty}\alpha_k\int_{\Gamma}\lambda^k\left(\sum_{n=0}^{\infty}\tfrac{T^n}{\lambda^{n+1}}\right)\ d\lambda = \sum_{k=0}^{\infty}\alpha_k T^k.$$

∎

Corollary 3.16 *If $\mathcal{P} \in \mathcal{C}_0(X)$ and $T \in (Q_{\mathcal{P}}(X))_0$, then* $\exp T = \sum\limits_{k=0}^{\infty}\frac{T^k}{k!}$.

Definition 3.17 *If $\mathcal{P} \in \mathcal{C}_0(X)$ and $T \in (Q_{\mathcal{P}}(X))_0$, then a subset of $\sigma_W(Q_{\mathcal{P}}, T)$ which is both open and closed in $\sigma_W(Q_{\mathcal{P}}, T)$ is called a spectral set of T.*

Denote by δ_T the class of spectral sets of T.

Proposition 3.18 *If $\mathcal{P} \in \mathcal{C}_0(X)$ and $T \in (Q_{\mathcal{P}}(X))_0$, then for each spectral set $H \in \delta_T$ there exists a unique idempotent $T_H \in Q_{\mathcal{P}}(X)$ with the following properties:*

1. *$T_H S = S T_H$, whenever $S \in Q_{\mathcal{P}}(X)$ and $ST = TS$;*
2. *$T_\varnothing$ is the null element of $Q_{\mathcal{P}}(X)$;*
3. *$T_{H\cap K} = T_H T_K, (\forall)\, H, K \in \delta_T$;*
4. *$T_{H\cup K} = T_H + T_K$, for each $H, K \in \delta_T$ with the property $H \cap K = \emptyset$.*

Proof. First we make the observation that for each set $H \in \delta_T$ there exists an unique germ $\tilde{f}_H \in \mathcal{A}(\sigma_W(Q_{\mathcal{P}}, T))$ with the property (H), where

$$(H)\begin{cases}\text{for every pair } (D, D') \text{ of relatively compact open sets of the complex} \\ \text{plane which satisfies the conditions} \\ \qquad H \subset D,\ \sigma_W(Q_{\mathcal{P}}, T) \subset D' \text{ and } D \cap D' = \emptyset \\ \text{then there exists } f_H \in \tilde{f}_H \text{ such that } f_H/_D = 1 \text{ and } f_H\ /_{D'} = 0.\end{cases}$$

If Γ is the Cauchy boundary for the pair $(\overline{H}, D)$ (the closure of H is taken in the topology of $\mathbb{C}$) then we define

$$T_H = F_T(\tilde{f}_H) = \frac{1}{2\pi i}\int_{\Gamma} R(\lambda, T) d\lambda.$$

1) If $ST = TS$, then $SR(\lambda, T) = R(\lambda, T)S$, so $T_H S = ST_H$.
2) Results from the definition of T_H.
3) Let $H, K \in \delta_T$ and $\tilde{f}_H, \tilde{f}_K, \tilde{f}_{H\cap K} \in \mathcal{A}(\sigma_W(Q_\mathcal{P}, T))$ which verifies the properties (H), (K), respectively $(H \cap K)$.

Assume that the pairs (D, D') and (G, G') are like in (H) and (K) properties. Then there exists $f \in \tilde{f}_H$, such that $f/_D = 1$ and $f/_{D'} = 0$, and $g \in \tilde{f}_K$, such that $g/_G = 1$ and $g/_{G'} = 0$. It is obvious that $fg/_{D\cap G} = 1$ and $fg/_{D'\cap G'} = 0$, so $fg \in \tilde{f}_{H\cap K}$ and

$$T_{H\cap M} = F_T(\tilde{f}_{H\cap K}) = F_{T,D\cap G}(fg) = F_{T,D\cap G}(f)F_{T,D\cap G}(g) =$$

$$= F_{T,D}(f)F_{T,G}(g) = F(\tilde{f}_H)F(\tilde{f}_K) = T_H T_M.$$

4) We consider the above notations and with the supplementary conditions $D \cap G = \varnothing$ and $D\,' \cap G\,' = \varnothing$ (since $H \cap M = \varnothing$). Then

$$f(\lambda) + g(\lambda) = \begin{cases} 1, & \text{if } \lambda \in D \cup G, \\ 0, & \text{for } \lambda \in D\,' \cup G\,', \end{cases}$$

Therefore, if $\tilde{f}_{H\cup K} \in \mathcal{A}(\sigma_W(Q_\mathcal{P}, T))$ has the property $(H \cup K)$, then $f + g \in \tilde{f}_{H\cup K}$, so

$$T_{H\cup M} = F_T(\tilde{f}_{H\cup K}) = F_{T,D\cup G}(f + g) = F_{T,D\cup G}(f) + F_{T,D\cup G}(g) =$$

$$= F_{T,D}(f) + F_{T,G}(g) = F_T(\tilde{f}_H) + F_T(\tilde{f}_K) = T_H + T_M.$$

∎

Corollary 3.19 *If $\mathcal{P} \in \mathcal{C}_0(X)$ and $T \in (Q_\mathcal{P}(X))_0$, then for every pair of spectral sets spectral set $H, K \in \delta_T$, which have the properties $H \cap K = \varnothing$ and $H \cup K = \sigma_W(Q_\mathcal{P}, T)$, we have*

$$T_H + T_K = I \text{ and } T_H T_K = O.$$

Remark 3.20 *From proposition 3.11 results that $T_H \in (Q_\mathcal{P}(X))_0$, for each $H \in \delta_T$*

Lemma 3.21 *Assume that $\mathcal{P} \in \mathcal{C}_0(X)$ and $T \in (Q_\mathcal{P}(X))_0$. If $F \subset \mathbb{C}$ has the property $dist(\sigma_W(Q_\mathcal{P}, T), F) > \varepsilon_0 > 0$, then for each $p \in \mathcal{P}$ there exists $c_p > 0$ such that*

$$\hat{p}(R(\lambda, T)^n) \leq \frac{c_p}{\varepsilon_0^n}, \ \ (\forall)\, \lambda \in F, (\forall)\, n \in \mathbb{N}$$

Proof. Let be $\varepsilon \in (0, \varepsilon_0)$, be arbitrarily fixed. If $D = \mathbb{C}\backslash\overline{F}$, then for the pair $(\sigma_W(Q_\mathcal{P}, T)), D)$ there exists a Cauchy domain G such that

$$|\lambda - \omega| > \varepsilon_0 - \varepsilon, \ \ (\forall)\, \lambda \in F, (\forall)\, \omega \in \overline{G}.$$

If Γ is boundary of G, then

$$\hat{p}(R(\lambda, T)^n) = \hat{p}\left(\frac{1}{2\pi i}\int_\Gamma \frac{R(\omega, T)}{(\omega - \lambda)^n} d\omega\right) \leq \frac{L(\Gamma)}{2\pi} \sup_{\omega\in\Gamma} \frac{\hat{p}(R(\omega, T)^n)}{|\omega - \lambda|^n} <$$

$$< \frac{\frac{L(\Gamma)}{2\pi} \sup_{\omega\in\Gamma} \hat{p}(R(\omega, T)^n)}{(\varepsilon_0 - \varepsilon)^n}.$$

Since ε is arbitrary, the lemma is proved if we chose

$$c_p = \frac{L(\Gamma)}{2\pi} \sup_{\omega\in\Gamma} \hat{p}(R(\omega, T)^n).$$

■

The next theorem gives an extension for Taylor's theorem to functions of an operator.

Theorem 3.22 *Let $\mathcal{P} \in \mathcal{C}_0(X)$ and $T \in (Q_{\mathcal{P}}(X))_0$. If D is an relatively compact open set which contains the set $\sigma_W(Q_{\mathcal{P}}, T)$, $f \in \mathcal{O}(D)$ and $S \in (Q\ _{\mathcal{P}}(X))_0$, such that $r_{\mathcal{P}}(S) < dist(\sigma_W(Q_{\mathcal{P}}, T), \mathbb{C}\backslash D)$ and $TS = ST$, then the following statements are true:*

1. $\sigma_W(Q_{\mathcal{P}}, T + S) \subset D$;
2. $f(T + S) = \sum\limits_{n\geq 0} \frac{f^{(n)}(T)}{n!} S^n$.

Proof. Let $d, d_1 > 0$ such that

$$r_{\mathcal{P}}(S) < d_1 < d < dist\,(\sigma_W(Q_{\mathcal{P}}, T), \mathbb{C}\backslash D)\,.$$

If $\Gamma_1 = \{\lambda \in \mathbb{C}|\ \ |\lambda| = d_1\}$, then for each $p \in \mathcal{P}$ and every $n \in \mathbb{N}$ we have

$$\hat{p}(S^n) = \hat{p}\left(\frac{1}{2\pi i}\int_\Gamma \lambda^n R(\lambda, S) d\lambda\right) \leq \frac{L(\Gamma_1)}{2\pi} \sup_{\omega\in\Gamma_1} (|\lambda^n|\, \hat{p}(R(\lambda, S)) \leq$$

$$\leq \frac{L(\Gamma_1)}{2\pi} \sup_{\omega\in\Gamma_1} \hat{p}(R(\lambda, S) \sup_{\omega\in\Gamma_1} |\lambda|^n \leq k_p d_1^n \tag{11}$$

where $k_p = \frac{L(\Gamma_1)}{2\pi} \sup_{\omega\in\Gamma_1} \hat{p}(R(\lambda, T)$.

Moreover, the previous lemma implies that for each $p \in \mathcal{P}$ there is $c_p > 0$ such that

$$\hat{p}(R(\lambda, T)^{n+1}) \leq \frac{c_p}{d^{n+1}},\ \ (\forall)\,\lambda \in \mathbb{C}\backslash D, (\forall)\, n \in \mathbb{N} \tag{12}$$

so from relations (11) and (12) it follows that

$$\hat{p}(R(\lambda, T)^{n+1} S^n) = \hat{p}(R(\lambda, T)^{n+1})\hat{p}(S^n) \leq \frac{k_p c_p}{d_1}\left(\frac{d_1}{d}\right)^{n+1} \tag{13}$$

for every $p \in \mathcal{P}$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}\backslash D$. Since $\frac{d_1}{d} < 1$ the relation (13) proves that the series $\sum_{n=1}^\infty R(\lambda, T)^{n+1} S^n$ converge uniformly on $\mathbb{C}\backslash D$.

From the equalities

$$(\lambda I - T - S)\sum_{n=1}^\infty R(\lambda, T)^{n+1} S^n = \sum_{n=1}^\infty R(\lambda, T)^{n+1} S^n (\lambda I - T - S) =$$

$$= \sum_{n=1}^{\infty} R(\lambda, T)^n S^n - \sum_{n=1}^{\infty} R(\lambda, T)^{n+1} S^{n+1} = I$$

it follows that $\lambda I - T - S$ is invertible in $Q_{\mathcal{P}}(X)$, for all $\lambda \in \mathbb{C} \backslash D$, and

$$R(\lambda, T + S) = \sum_{n=1}^{\infty} R(\lambda, T)^{n+1} S^n. \tag{14}$$

Therefore the relation (13) implies that the set $\{R(\lambda, T+S) | \lambda \in \mathbb{C} \backslash D\}$ is bounded in $Q_{\mathcal{P}}(X)$, so $\sigma_W(Q_{\mathcal{P}}, T+S) \subset D$.

If Γ is a Cauchy boundary for the pair $(\sigma_W(Q_{\mathcal{P}}, T+S), D)$, then from (14) and lemma 2.2 it results

$$f(\lambda, T+S) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R(\lambda, T+S) d\lambda =$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R(\lambda, T)^{n+1} d\lambda \right) S^n =$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \frac{(-1)^n}{n!} \int_{\Gamma} f(\lambda) \frac{d^n}{d\lambda^n} R(\lambda, T) d\lambda \right) S^n =$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \frac{1}{n!} \int_{\Gamma} f^{(n)}(\lambda) R(\lambda, T) d\lambda \right) S^n = \sum_{n=1}^{\infty} \frac{f^{(n)}(T)}{n!} S^n$$

■

Corollary 3.23 *Let $\mathcal{P} \in \mathcal{C}_0(X)$ and $T \in (Q_{\mathcal{P}}(X))_0$. If $S \in Q_{\mathcal{P}}(X)$ is $\mathcal{P}$-quasinilpotent, such that $TS = ST$, then*

$$\tilde{f}(T+S) = \sum_{n \geq 0} \frac{\tilde{f}^{(n)}(T)}{n!} S^n, \ (\forall) \tilde{f} \in \mathcal{A}(\sigma_W(Q_{\mathcal{P}}, T))$$

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