# Capitolul III

# FUNCȚII SPECIALE

**1. Să se calculeze integrala:** $I = \int_0^\infty \frac{x^2 dx}{(1+x^6)^3}$ **.**

*Soluție:*

Notăm $\frac{1}{1+x^6} = t$. Dacă $x = 0$, $t = 1$ şi dacă $x = \infty, t = 0$. Din substituția făcută observăm că: $x = \left(\frac{1-t}{t}\right)^{\frac{1}{6}}$ de unde $dx = \frac{1}{6}\left(\frac{1-t}{t}\right)^{-\frac{5}{6}}\left(-\frac{1}{t^2}\right)dt.$

Integrala devine: $I = \frac{1}{6}\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{3}} t^3 \left(\frac{1-t}{t}\right)^{-\frac{5}{6}} \frac{1}{t^2} dt = \frac{1}{6}\int_0^1 t^{\frac{3}{2}}(1-t)^{-\frac{1}{2}} dt$.

Din $p - 1 = \frac{3}{2}$ şi $q - 1 = -\frac{1}{2}$ obținem $p = \frac{5}{2}$ şi $q = \frac{1}{2}$.

Deci $I = \frac{1}{6}B\left(\frac{5}{2}, \frac{1}{2}\right) = \frac{1}{6}\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)} = \frac{1}{6}\frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2} = \frac{\frac{3}{2}\frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{12}$, de unde ( $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ ) obținem:

$$I = \frac{\pi}{16}.$$

**2. Să se calculeze integrala:** $I = \int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x dx$**.**

*Soluție:*

Notăm $\cos^2 x = t; \sin^6 x = (1-t)^3$; obținem $x = \arccos\sqrt{t}, dx = -\frac{1}{2\sqrt{t}\sqrt{1-t}} dt$.

Integrala devine:

$$I=-\frac{1}{2}\int_1^0 (1-t)^3 t^2 \frac{1}{\sqrt{t}\sqrt{1-t}}dt=\frac{1}{2}\int_0^1 t^{\frac{3}{2}}(1-t)^{\frac{5}{2}}dt=\frac{1}{2}B\left(\frac{5}{2},\frac{7}{2}\right)$$

(din $p-1=\frac{3}{2}$ şi $q-1=\frac{5}{2}$ obţinem $p=\frac{5}{2}$ şi $q=\frac{7}{2}$).

sau
$$I=\frac{1}{2}\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma(6)}=\frac{1}{2}\frac{\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{5!}\underset{\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}}{\Rightarrow} I=\frac{3\pi}{512}.$$

**3. Să se calculeze integrala:** $I=\int_0^\infty \frac{dx}{1+x^8}$.

*Soluţie:*

Notăm $\frac{1}{1+x^8}=t$. Observăm că, dacă $x$ = 0, $t$ = 1 şi dacă $x=\infty$, $t=0$, din substituţia făcută, $x=\left(\frac{1-t}{t}\right)^{\frac{1}{8}}$ de unde $dx=\frac{1}{8}\left(\frac{1-t}{t}\right)^{-\frac{7}{8}}\left(-\frac{1}{t^2}\right)dt$.

Integrala devine:

$$I=-\frac{1}{8}\int_1^0 t(1-t)^{-\frac{7}{8}}t^{\frac{7}{8}}t^{-2}dt=\frac{1}{8}\int_0^1 t^{-\frac{1}{8}}(1-t)^{-\frac{7}{8}}dt.$$

Observăm că $p-1=-\frac{1}{8}$ şi $q-1=-\frac{7}{8}$, de unde $p=\frac{7}{8}$ şi $q=\frac{1}{8}$.

Deci

$$I=\frac{1}{8}B\left(\frac{7}{8},\frac{1}{8}\right)=\frac{1}{8}\frac{\Gamma\left(\frac{7}{8}\right)\Gamma\left(\frac{1}{8}\right)}{\Gamma\left(\frac{7}{8}+\frac{1}{8}\right)}\underset{z=\frac{1}{8}}{=}\frac{1}{8}\frac{\pi}{\sin\frac{\pi}{8}}.$$ ($\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$ (ecuaţia complementelor), $z=\frac{1}{8}$).

sau

$$I=\frac{\pi}{8\sin\frac{\pi}{8}}.$$