Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843 - 7265 (print)
Volume **3** (2008), 123 – 165

50 YEARS SETS WITH POSITIVE REACH - A SURVEY -

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Abstract. The purpose of this paper is to summarize results on various aspects of sets with positive reach, which are up to now not available in such a compact form. After recalling briefly the results before 1959, sets with positive reach and their associated curvature measures are introduced. We develop an integral and current representation of these curvature measures and show how the current representation helps to prove integralgeometric formulas, such as the principal kinematic formula. Also random sets with positive reach and random mosaics (or the more general random cell-complexes) with general cell shape are considered.

1 Introduction

This paper is a collection of various aspects of sets with positive reach, which were introduced by Federer in 1959 [4]. Thus, the paper is also a celebration of their 50-th birthday in 2009.

After the developments of integral geometry for convex sets as well as for smooth manifolds in differential geometry, the situation around 1950 was the following: There were two tube formulas (Steiner's formula and Weyl's formula), which say that the volume of a sufficiently small r-parallel neighborhood of a convex set or a C^2-smooth submanifold X in $\mathbb{R}^d$ is a polynomial in r of degree d, the coefficients of which are (up to some constant) geometric invariants of the underlying set. Unfortunately the the assumptions of both results are quite different, such that each case does not contain or imply the other one. This problem was solved by Federer in this famous paper [4], where he introduced sets with positive reach and their associated curvatures and curvature measures. He was also able tho show a certain tube formula for this class of sets. A comparison with the former cases from convex and differential geometry shows that in this special cases the new invariants coincide with the known ones. Thus, sets with positive reach generalize the notion of convex sets on the one

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2000 Mathematics Subject Classification:49Q15; 28A75; 53C65; 60G55; 60D05; 60G57; 52A39.

Keywords: Sets with Positive Reach; Curvature Measure; Integral Geometry; Kinematic Formula; Random Set; Random Mosaic; Current; Normal Cycle; Random Cell Complex.

This work was supported by the Schweizerischer Nationalfonds grant SNF PP002-114715/1.

hand side and the notion of a smooth submanifold on the other. It was also Federer, who proved the fundamental integralgeometric relationship for sets with positive reach, the principal kinematic formula.

With the development of geometric measure theory and especially the calculus of currents, the idea of the so-called normal cycle of a set with positive reach came into play in the early 1980th. This idea paved the wary for explicit representations of Federer's curvature measures as well as for a simple approach to integral geometry, because many of these problems could be reduced to an application of the famous Coarea Formula. Also extensions to other classes of sets are possible by following this way.

After having developed a solid theory for deterministic sets with positive reach, several well known models from stochastic geometry were lifted up to the case of random sets with positive reach. This includes the theory of random processes of sets with positive reach and their associated union sets. This in particular allows to treat random cell complexes and random mosaics with general cell shape. The main integralgeometric relationships were extended to this random setting, which leads to stochastic versions of the principal kinematic formula and Crofton's formula.

In this paper we like to sketch these developments from the last 50 years. Of course, the material is a selection, which relies more or less on the authors taste. We also do not qualify for completeness. Since proofs are sketched mostly, we try to give detailed references trough the existing literature. We like to point out that there is up to now a lack of a comprehensive monograph on this very interesting and beautiful topic. We remark that we will restrict in this paper ourself to the case of curvature measures defined on $\mathbb{R}^d$, even if there is also a theory dealing with directional curvature measures on $\mathbb{R}^d \times S^{d-1}$.

The paper is organized as follows: Section 2 recalls the situation before 1959. In 2.1 important notions and notations from convex geometry are introduced. Section 3 deals with basic properties of sets with positive reach (Section 3.1) and the most important tools, associated curvature measures and unit normal cycles (Section 3.2). In Section 3.3 the notion of the normal cycle and the curvature measures are extended to the case of locally finite unions of sets with positive reach. Characterization theorems of these curvature measures using tools from geometric measure theory are explained in 3.4. The topic of Section 4 is integral geometry. First we prove a translative integral formula for sets of positive reach (Section 4.1), which leads to the principal kinematic formula in Section 4.2. Here the power of the concept of the unit normal cycle is demonstrated in interplay with the Coarea Formula. In Section 4.3 we extend the theory again to locally finite unions of sets with positive reach. The results are applied in Section 5, where integralgeometric formulas from Section 4 are extended to certain stochastic variants. This will be done in the context of random processes of sets with positive reach in Section . The results will be applied to random cell complexes and the more special random mosaics with a very general cell shape in Section 5.2 at the end of this paper.

2 Results before 1959

Before 1959 there were two main branches in mathematics dealing with curvature and curvature measures. This are convex geometry and differential geometry. The most important results in these fields will be summarized below. This background provides a solid basis for the understanding and motivation for Federer's sets with positive reach.

2.1 Convex Geometry

We fix a convex set $X \subseteq \mathbb{R}^d$. For $r > 0$ its r-parallel set or neighborhood X_r is the set of all points $x \in \mathbb{R}^d$ with distance to X at most r, i.e. $X_r := \{x \in \mathbb{R}^d : \operatorname{dist}(X, x) \leq r\}$. If we denote by $A \oplus B = \{a + b : a \in A, b \in B\}$ the Minkowski sum of two sets A and B, the set X_r can be interpreted as $X_r = X \oplus B(r)$, where $B(r)$ is a ball with radius r. A fundamental result in convex geometry is Steiner's formula:

Theorem 1. *For a convex body $X \subset \mathbb{R}^d$ (this is a compact convex set with non-empty interior) and $r > 0$, the volume $vol(X_r) = \mathcal{H}^d(X_r)$ is a polynomial in r, i.e.*

$$vol(X_r) = \mathcal{H}^d(X_r) = \sum_{i=0}^{d} \omega_i V_{d-i}(X) r^i,$$

where $V_j(X)$ are coefficients with only depend on X, ω_j is the volume of the j-dimensional unit ball and $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure (see [5, 2.10.2]).

The proof of this formula is quite easy if one knows that any convex body X can be approximated by a sequence (P_n) of polyhedra. Now one observes that the formula is true for polyhedra and transfers the result via the above approximation to arbitrary convex bodies. For more details see for example the monograph [25]. The numbers $V_0(X), \ldots, V_d(X)$ are usually called intrinsic volumes of X. In particular we have for any convex body $X \subset \mathbb{R}^d$

1. $V_0(X) = 1$,
2. $V_1(X) = \frac{d\omega_d}{2\omega_{d-1}} b(X)$, where $b(K)$ is the mean breadth of X (cf. [25]),
3. $V_{d-1}(X) = \frac{1}{2}\mathcal{H}^{d-1}(\partial X)$, where ∂X is the boundary of the set X,
4. $V_d(X) = vol(X) = \mathcal{H}^d(X)$.

In the literature there is also another normalization used. We call

$$W_i(X) := \frac{\omega_i}{\binom{n}{i}} V_{d-i}(X)$$

the i-th Quermassintegral of X. The name comes from the following projection formula, which is often used as a definition: Let X be a convex body in $\mathbb{R}^d$ and for $i \in \{1, \dots, d-1\}$, $\mathcal{L}(d, i)$ the family of i-dimensional linear subspaces of $\mathbb{R}^d$ equipped with the unique probability measure dL_i. We denote for $L \in \mathcal{L}(d, i)$ by $\pi_L(X)$ the orthogonal projection of X onto L, which is again a convex set. Then we have

$$V_i(X) = \frac{\binom{d}{i}\omega_d}{\omega_i \omega_{d-i}} \int_{\mathcal{L}(d,j)} V_j(\pi_L(K)) dL_i(L).$$

Here, the integrand $V_j(\pi_L(X))$ is the volume of the projection of X onto L. Hence, we can call it Quermass of X in direction $L^\perp$.

The functionals $V_i : \mathcal{K} \to \mathbb{R}$, where $\mathcal{K}$ is the family of convex bodies, have the following important properties: They are

(i) motion invariant, i.e. $V_i(gX) = V_i(X)$ for any euclidean motion g,

(ii) additive, i.e. $V_i(X \cup Y) = V_i(X) + V_i(Y) - V_i(K \cap Y)$ for all $X, Y, X \cup Y \in \mathcal{K}$,

(iii) continuous, i.e. if $X_n \to X$ in Hausdorff metric then $V_i(X_n) \to V_i(X)$,

(iv) homogeneous, i.e. $V_i(\lambda X) = \lambda^i V_i(X)$ for all $\lambda > 0$,

(v) monotone, i.e. $X \subseteq Y$ implies $V_i(X) \leq V_i(Y)$,

(vi) non-negative, i.e. $V_i(X) \geq 0$ for all $X \in \mathcal{K}$.

We will see now that properties (i)-(iii) are sufficient to characterize the intrinsic volumes. This is the content of Hadwiger's Theorem:

Theorem 2. *Let $\Psi : \mathcal{K} \to \mathbb{R}$ a functional which is motion invariant, additive and continuous. Then Ψ can be written as a linear combination of the intrinsic volumes, i.e. there are real constants $c_0, \dots, c_d$, such that*

$$\Psi = \sum_{i=0}^{d} c_i V_i.$$

The proof of this theorem uses deep methods of discrete geometry, see [10]. A short proof was given by Klain [11]. The so-called principal kinematic formula is now an easy consequence of Hadwiger's Theorem:

Corollary 3. *Let $X, Y \in \mathcal{K}$ and $i \in \{0, \dots, d\}$. Then*

$$\int_{SO(d)\ltimes\mathbb{R}^d} V_i(X \cap gY) dg = \sum_{m+n=d+i} \gamma(m, n, d) V_m(X) V_n(Y),$$

where $SO(d) \ltimes \mathbb{R}^d$ is the group of euclidean motions with Haar measure dg and $\gamma(m, n, d) = \Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\left(\Gamma\left(\frac{m+n-d+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)\right)^{-1}$.

Remark 4. *The measure dg is the product measure with factors $d\mathcal{H}^d$ and $d\vartheta$, where $d\vartheta$ is a Haar measure on the group $SO(d)$. Here and for the rest of this paper we will use the following normalization of $d\vartheta$:*

$$\vartheta\{g \in SO(d) : g\mathcal{O} \in M\} = \mathcal{H}^d(M),$$

where $\mathcal{O}$ is the origin and M some subset of $\mathbb{R}^d$. With this normalization in mind it is clear that $d\vartheta$ is not a probability measure on $SO(d)$.

For the proof one has to observe that for fixed X the left hand side is a functional in the sense of Theorem 2. Now, fixing Y instead of X we have the same situation and can apply Hadwiger's Theorem once again. It remains to shown that the constant equals $\gamma(m, n, d)$. This can be done, by plugging balls with varying radii into the formula.
An obvious consequence is the so-called Crofton formula:

Corollary 5. *For $X \in \mathcal{K}$, $k \in \{0, \ldots, d\}$ and $i \in \{0, \ldots, k\}$ we have*

$$\int_{\mathcal{E}(d,k)} V_i(X \cap E)dE = \gamma(i, k, d)V_{d+i-k}(X),$$

where $\mathcal{E}(d, k)$ is the space of k-dimensional affine subspaces of $\mathbb{R}^d$ with Haar measure dE.

We remark here that it is possible to localize all these formulas in the language of curvature measures. We omit the details in the convex case, since curvature measures will be considered in detail below for sets with positive reach, which includes the case of convex sets. For more details we refer to [25]. We also like to remark that it is possible to extend the intrinsic volumes as well as the curvature measures to the so-called convex ring $\mathcal{R}$. This is the family of locally finite unions of convex sets. For details we also refer to [25], because we will work out in Section 3.3 in detail such an extension in the case of locally finite unions of sets with positive reach and the convex ring $\mathcal{R}$ is included in these considerations.
We like to finish this section with an introduction to translative integral geometry for convex sets (see for example [9] for more details). As a main tool we introduce the so-called mixed volumes:

Theorem 6. *Let $X_1, \ldots, X_m \in \mathcal{K}$, $m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \geq 0$. Then there exists a representation of the volume of the linear combination $\lambda_1 X_1 \oplus \ldots \oplus \lambda_m X_m$ of the following form:*

$$vol(\lambda_1 X_1 \oplus \ldots \oplus \lambda_m X_m)) \sum_{k_1,\ldots,k_d=1}^{m} \lambda_{k_1} \cdots \lambda_{k_d} V_{k_1 \ldots k_d},$$

where the coefficient $V_{k_1 \ldots k_d}$ only depends on the sets $X_{k_1}, \ldots, X_{k_d}$.

We write $V_{1\ldots d} = V(X_1,\ldots,X_d)$ and called it mixed volume of $X_1,\ldots,X_d$. The mixed volumes have the following important properties:

1. $V(X_1,\ldots,X_d)$ is symmetric, i.e.
$$V(X_1,\ldots,X_m,\ldots,X_n,\ldots,X_d) = V(X_1,\ldots,X_n,\ldots,X_m,\ldots,X_d)$$
for all $1 \leq n < m \leq d$,
2. $V(X_1,\ldots,X_d) \geq 0$ and V is monotone in each component,
3. V is translation invariant in each component and
$$V(\vartheta X_1,\ldots,\vartheta X_d) = V(X_1,\ldots,X_d)$$
for all $\vartheta \in SO(d)$,
4. V is continuous on $\mathcal{K}^d$ wrt. the natural product topology,
5. we have for $x \in \mathcal{K}$ and $r > 0$
$$vol(X_r) = \sum_{i=0}^{d} \binom{d}{d-i} r^i V(\underbrace{X,\ldots,X}_{d-j},\underbrace{B(1),\ldots B(1)}_{j}).$$

A comparison of the last point and Steiner's formula especially shows

$$V_{d-i}(X) = \frac{\binom{d}{d-i}}{\omega_i} V(\underbrace{X,\ldots,X}_{d-i},\underbrace{B(1),\ldots,B(1)}_{i}),\ \ i = 0,\ldots,d.$$

Tis concept can also be localized, which leads to mixed curvature measures. They will be introduced below for sets with positive reach.

Translative integral geometry for convex sets deals with integrands of the form $V_i(X \cap \tau_z(Y))$, where $\tau_z(A)$, $z \in \mathbb{R}^d$, denotes the translation of a set A by a vector z. As a main result we state the following principal translative integral formula for convex bodies, where $i = 0$:

Theorem 7. *For two convex bodies X and X we have*

$$\int_{\mathbb{R}^d} V_0(X \cap \tau_z(Y))dz = \sum_{m+n=d+k} \binom{d}{m} V(\underbrace{X,\ldots,X}_{r},\underbrace{-Y,\ldots,-Y}_{s}).$$

A similar formula holds also true for $V_i(X \cap \tau_z(Y))$ and $i > 0$. But in this case the summands do not have in general a simple explicit interpretation. In section 4.1 we will introduce so-called mixed curvature measures in a more general setting.

2.2 Differential Geometry

We consider a d-dimensional submanifold M_d in $\mathbb{R}^d$ with C^2-smooth boundary ∂M_d and denote by $\nu(x)$ the unique unit outer normal vector of M_d at $x \in \partial M_d$. The map $\nu : \partial M_d \to S^{d-1}$ is called Gauss map. Since ∂M_d is C^2-smooth we know that the differential

$$D\nu(x) : T_x\partial M_d \to T_{\nu(x)}S^{d-1} \equiv T_x\partial M_d$$

exists in all points $x \in \partial M_d$. We assume that in a neighborhood of $x \in \partial M_d$ the surface is parameterized by $F : U \to \mathbb{R}^d$. Then

$$L = -D\nu \circ (DF)^{-1}$$

is a well defined symmetric endomorphism on $T_x\partial M_d$. Hence, there exist eigenvalues $k_1(x), ..., k_{d-1}(x)$ (called principal curvatures) and eigenvectors $a_1(x), ..., a_{d-1}(x)$ (usually called principal directions).

Definition 8. *The elementary symmetric functions σ_k of order $k = 0, ..., d-1$ are defined as*

$$\sigma_k(k_1(x), ..., k_{d-1}(x)) = \sum_{1 \le i_1 < ... < i_k \le d-1} k_{i_1}(x) \cdots k_{i_k}(x).$$

They are used in the following

Definition 9. *The k-th integral of mean curvature (also called Lipschitz-Killing curvature) of M_d is defined as*

$$C_k(M_d) := \mathcal{O}^{-1}_{d-1-k} \int_{\partial M_d} \sigma_{d-1-k}(k_1(x), ..., k_{d-1}(x)) d\mathcal{H}^{d-1}(x), \ k = 0, \dots, d-1$$

where $\mathcal{O}_m$ is the surface area of the m-dimensional unit ball. Define further $C_d(M_d) := \mathcal{H}^d(M_d)$.

In particular we have in the case, where X is compact

1. $C_0(M_d) = \chi(M_d)$ (Gauss-Bonnet Theorem),
2. $C_{d-1}(M_d) = \frac{1}{2}\mathcal{H}^{d-1}(\partial M_d)$,
3. $C_d(M_d) = \mathcal{H}^d(M_d) = vol(M_d)$.

One of the fundamental theorems in differential geometry is Wely's Tube Formula, which has its origin in a statistical problem [28]:

Theorem 10.

$$\mathcal{H}^d((M_d)_\varepsilon) = \sum_{k=0}^{d} \omega_k C_{d-k}(M_d)\varepsilon^k,$$

where the ε-parallel set $(M_d)_\varepsilon$ of M_d is defined as $(M_d)_\varepsilon := \{x \in \mathbb{R}^d : dist(x, M_d) \le \varepsilon\}$ for sufficiently small $\varepsilon > 0$.

Integral geometry for smooth hypersurfaces and m-surfaces ($m < d-1$) in $\mathbb{R}^d$ was developed in [26, Ch. V] and we refer to this monograph for further details. We only like to mention here, that similar formulas as in Corollary 3 and Corollary 5 are true in this case. But up to a constant, the k-th intrinsic volume is replaced by the k-th integral of mean curvature, $k = 0, \dots, d$. This is the reason, why we omit to state them here explicitly. Furthermore, the integralgeometric results below will cover the case of C^2-smooth submanifolds.

3 Sets with Positive Reach

We introduce in this section the class of sets with positive reach and their geometric properties. The focus of our considerations lies on curvature measures for this class of sets. Therefore, the so-called unit normal cycle is used as a fundamental toll in singular curvature theory. It also helps to extend the curvature measures to the class of locally finite unions of sets with positive reach.

3.1 Definition and Basic Properties

Sets with positive reach are characterized by their unique foot point property in a positive r-parallel set. This property ensures that suitable small parallel neighborhoods have no self-intersections and this allows to compute their volume. This will lead to a Steiner-type formula and a definition of curvature measures for sets with positive reach, which extends the cases treated in Section 2.

Definition 11. *The reach of a set $X \subseteq \mathbb{R}^d$ is defined as*

$$\textit{reach } X := \sup\{r \geq 0 : \forall y \in X_r \ \exists!!x \in X \ \textit{nearest to } y\}.$$

We say that a set X has positive reach, if reach $X > 0$ and denote by PR the family of sets with positive reach.

We can also formulate this property as follows: A set X has positive reach, if one can roll up a ball of radius at most reach $X > 0$ on the boundary ∂X. Note that sets with positive reach are necessarily closed subsets of $\mathbb{R}^d$. This will be useful when we deal with random sets with positive reach in Section 5, since there exists a well developed theory of random closed sets in $\mathbb{R}^d$, see for example [12].

Remark 12. *It is also possible to define the class of sets with positive reach on smooth and connected Riemannian manifolds. By a Theorem of Bangert [1, p. 57] this property does not depend on the Riemannian structure of the underlying manifold. Hence, the theory of sets with positive reach in $\mathbb{R}^d$ (and also their additive extension) can be lifted up to the case of smooth, connected Riemannian manifolds. But we will not follow this direction further in this paper.*

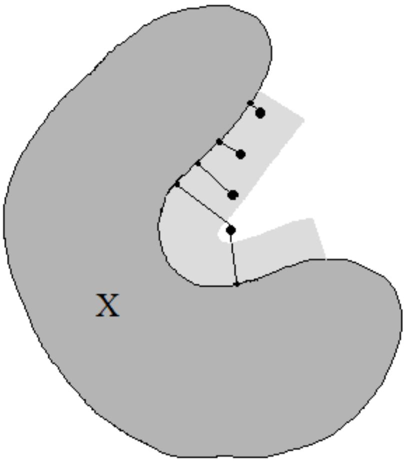

Figure 1: A non-convex set X with positive reach: 4 points in $\mathbb{R}^2$ and its foot points on X, the lower one is not unique

We will now show that the classes of sets introduced in Section 2 are included in our discussion:

Proposition 13. *A set X is convex if and only if reach $X = +\infty$.*

One direction is clear, the other corresponds to [25, Thm. 1.2.4]. It is a well known fact in differential geometry that the exponential map of a closed C^2-submanifold is a bijection in a suitable small neighborhood of the submanifold. But this leads immediately to

Proposition 14. *Compact C^2-smooth submanifolds X of $\mathbb{R}^d$ have positive reach.*

The closed convex cone of tangent vectors of $X \in PR$ at x will be denoted by $\operatorname{Tan}(X, x)$. Here, a vector $u \in S^{d-1}$ belongs to $\operatorname{Tan}(X, x)$ if there exists a sequence $(x_n) \subset X \setminus \{x\}$, such that $\frac{x_n - x}{|x_n - x|}$ converges to u. The normal cone of X at x

$$\operatorname{Nor}(X, x) = \{u \in S^{d-1} : \langle v, u\rangle \leq 0,\ v \in \operatorname{Tan}(X, x)\}$$

is the dual cone of $\operatorname{Tan}(X, x)$. For an illustration of these concepts see Figure 2. The set

$$\operatorname{nor} X := \{(x, u) \in \mathbb{R}^d \times S^{d-1} : x \in X,\ u \in \operatorname{nor}(X, x)\}$$

is said to be the (unit) normal bundle of X. Remark that this is a set in the $(2d-1)$-dimensional manifold $\mathbb{R}^d \times S^{d-1}$, whereas X itself is a set in $\mathbb{R}^d$, which is d-dimensional.

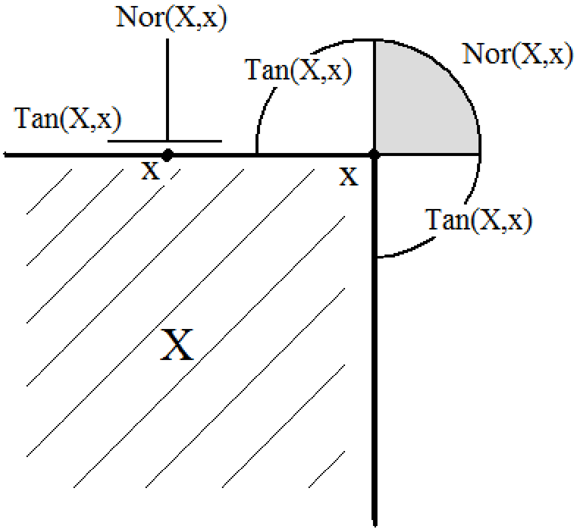

Figure 2: A set X and two points $x \in X$ with associated tangent and normal cone

Remark 15. *The unit normal bundle is a* $(d-1)$*-dimensional rectifiable set in* $\mathbb{R}^d \times S^{d-1} \subseteq \mathbb{R}^{2d}$ *in the sense of Federer [5, 3.2.14]. This means that nor* X *is* $\mathcal{H}^{d-1}$*-measurable and there exist Lipschitz functions* $f_1, f_2, \ldots : \mathbb{R}^{d-1} \to \mathbb{R}^{2d}$ *and bounded sets* $E_1, E_2, \ldots \subset \mathbb{R}^{d-1}$ *such that*

$$\mathcal{H}^{d-1}\left(nor\ X \setminus \bigcup_{i=1}^{\infty} f_i(E_i)\right) = 0.$$

We recall here the following result, which was proved by Federer [4]. It relates the boundary of a set $X \in PR$ with its unit normal bundle:

Proposition 16. *Assume* $0 < r \leq \varepsilon < R = reach\ X$*. Then*

(1) $\varphi : \partial X_r \to nor\ X : y \mapsto \left(\Pi_X(y), \frac{y - \Pi_X(y)}{r}\right)$ *is bijective and bi-Lipschitz.*

(2) $f : nor\ X \times (0, \varepsilon] \to (X_\varepsilon \setminus X) : (x, u, r) \mapsto x + ru$ *is bijective and bi-Lipschitz.*

Here $\Pi_X : \mathbb{R}^d \to X$ *is the metric projection onto* X*, i.e.* $\Pi_X(x)$ *is the set of nearest points of* X *to* $x \in \mathbb{R}^d$*.*

This proposition is (together with the Area Formula) the key to obtain a Steiner-type formula and a definition of curvature measures for sets with positive reach.

Sets with positive reach are closely connected with Lipschitz functions and semiconcave functions. Federer has shown in [4, 4.20] that a Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}^n$ has Lipschitz derivative if and only if the graph of f has positive reach. This illustrates very well of what it means for a submanifold to have positive reach. For a function $f : \mathbb{R}^m \to \mathbb{R}$, $U \subset \mathbb{R}^m$ open, we define its epigraph and its catograph (see Figure 3) as

$$\begin{aligned} \text{epi } f &:= \{(x,y) : x \in U, y \geq f(x)\}, \\ \text{cato } f &:= \{(x,y) : x \in U, y < f(x)\}. \end{aligned}$$

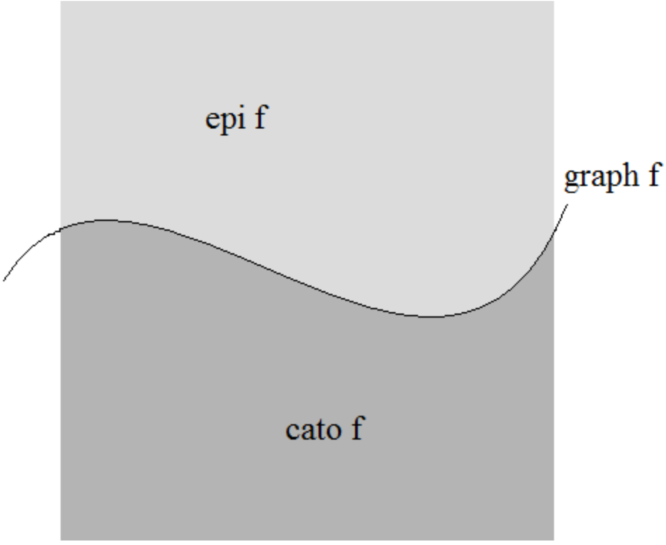

Figure 3: The graph of a function $f(x)$ with its epi- and catograph

We say that f is semiconcave, if for each bounded open set $V \subset U$ with closure$(V) \subseteq U$ there exists a constant $C < \infty$, such that the restriction of $g(x) := C\frac{\|x\|^2}{2} - f(x)$ to the set V is a convex function. We define $sc(f,V)$ to be the smallest such constant C, $sc(f,U) := \sup_V sc(f,V)$ and $sc_0(f,U) := \max\{sc(f,U), 0\}$. Then Fu [6, Th. 2.3] has proved

Proposition 17. *For a locally Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$ we have*

$$sc_0(f, \mathbb{R}^m) \geq reach(cato\ f)^{-1}.$$

For the opposite direction of the inequality we have [6, Cor. 2.8]

Proposition 18. *Let $U \subset \mathbb{R}^m$ be open and convex, $f : U \to \mathbb{R}$ Lipschitz with Lipschitz constant L. Suppose there exists an $r > 0$ such that for all $u \in U$ there* *exists a point* $p \in \mathbb{R}^{m+1}$ *for which* $B(p,r) \cap \mathit{cato}\ f = \{a, f(a)\}$*, where* $B(p,r)$ *is the closed ball around* p *with radius* r*. Then*

$$\mathit{reach}(\mathit{cato}\ f)^{-1} \geq (1+L^2)^{-3/2} sc(f,U).$$

Summarizing these results we get under the conditions of Proposition 18

$$sc_0(f,U)^{-1} \leq \mathrm{reach}(\mathrm{cato}\ f) \leq (1+L^2)^{3/2} sc(f,U)^{-1}.$$

From a version of the implicit function theorem for Lipschitz functions, which says that if $p \in U \subset \mathbb{R}^m$ is a regular value (this is 0 does not belong to the subgradient at p) and $f : U \to \mathbb{R}$ is semiconcave then there exists $V \subset U$, $p \in V$, a rotation $\vartheta \in SO(m)$, an open set $W \subset \mathbb{R}^{m-1}$ and a semiconcave function $g : W \to \mathbb{R}$, such that $\vartheta(f^{-1}(f(p) \cap V)) = \text{graph } g$ and $f^{-1}([f(p), \infty))$ is locally the catograph of g, we obtain [6, Cor. 3.4], which is also a special case of a result in [1]:

Theorem 19. *Suppose that* $f : \mathbb{R}^m \to \mathbb{R}$ *is semiconcave and proper (this is that the pre-image of a compact set is compact). Let* t *be a regular value of* f*. Then*

$$\mathit{reach}(f^{-1}([f(t), \infty))) > 0.$$

An immediate consequence of the last Theorem is

Corollary 20. *Let* $S \subset \mathbb{R}^d$ *be a compact set. Denote by* $\mathit{dist}_S(x) := \inf\{\|x-s\| : s \in S\}$ *the distance function of* S*, by* $\mathit{crit}(\mathit{dist}_S)$ *the set of critical points of* dist_S *(a point is critical if it is not regular) and by* $C := \mathit{dist}_S(\mathit{crit}(\mathit{dist}_S))$ *the set of critical values. Then for* $r \in (0,\infty) \setminus C$*, the set* $\mathit{closure}(\mathbb{R}^d \setminus S_r)$ *has positive reach.*

Moreover one can show that $\mathcal{H}^{(d-1)/2}(C) = 0$. This in particular implies that for $d = 2$, $\mathrm{closure}(\mathbb{R}^d \setminus S_r)$ has positive reach for *all* $r > 0$. For $d = 3$ this is only true for almost all r.

Corollary 20 has various applications. For example one can show that $\mathrm{closure}(\mathbb{R}^d \setminus X_r)$ has positive reach if $X \in \mathcal{R}$ or $X \in U_{PR}$ (for a definition see Section 3.3). This property is also fulfilled for certain Lipschitz manifolds (cf. [22]) or if X is semialgebraic set X (cf. [6, Section 5.3]). One can use this property to approximate or to construct for example curvature measures or normal cycles for more complicated classes of sets. An example for this approach can be found in [22]. We think that this construction can also be applied in other situations.

3.2 Curvature Measures and Normal Cycles

We will need the following important result - called Area Theorem - with is the key to prove a Steiner-type formula for sets with positive reach [5, 3.2.3]:

Theorem 21. *Let $f:\mathbb{R}^m\to\mathbb{R}^n$ $(m\le n)$ be Lipschitz, $A\subseteq\mathbb{R}^m$ $\mathcal{L}^m$-measurable and $g:\mathbb{R}^m\to\mathbb{R}$ $\mathcal{L}^m$-integrable. Then*

$$\int_A g(x)J_m f(x)d\mathcal{L}^m(x)=\int_{\mathbb{R}^n}\sum_{x\in f^{-1}(y)\cap A} g(x)d\mathcal{H}^n(y).$$

Remark 22. *The k-Jacobian $J_k f(x)$ of f at x can be introduced as*

$$J_k f(x)=\left\|\bigwedge_k Df(x)\right\|=\sup\{\mathcal{H}^k(Df(x)(C)) : C \textit{ is a } k\textit{-dimensional unit cube}\}.$$

In the special case $k=n=m$ we have $J_k f(x)=|\det Df(x)|$, which is the same as in linear algebra.

We apply now the Area Formula of Theorem 21 to the function f of Proposition 16. This yields

$$\int_{\text{nor } X\times(0,\varepsilon]} g(f(x,u,r))|\det Df(x,u,r)|d\mathcal{H}^d(x,u,r)=\int_{X_\varepsilon\setminus X} g(y)d\mathcal{H}^d(y).$$

Choose now $g(y):=\mathbf{1}_{\Pi_X^{-1}(B)}(y)=\mathbf{1}_B(x)$ for $y=x+ru$ and B a bounded Borel set in $\mathbb{R}^d$ (We change nothing if we choose the Borel set B to be contained in the boundary ∂X of $X\in PR$. The advantage of our approach is that we get a measure on $\mathbb{R}^d$ instead of a measure defined on ∂X.). Then the right hand side of the last equation equals

$$RHS=\int_{X_\varepsilon\setminus X}\mathbf{1}_{\Pi_X^{-1}(B)}(y)d\mathcal{H}^d(y)=\mathcal{H}^d((X_\varepsilon\setminus X)\cap\Pi_X^{-1}(B)).$$

For the left hand side we get by Fubini

$$\begin{aligned}LHS&=\int_{\text{nor } X\times(0,\varepsilon]}\mathbf{1}_{\Pi_X^{-1}(B)}(x+ru)|\det Df(x,u,r)|d\mathcal{H}^d(x,u,r)\\&=\int_{\text{nor } X}\mathbf{1}_B(x)\int_0^\varepsilon|\det Df(x,u,r)|drd\mathcal{H}^{d-1}(x,u).\end{aligned}$$

We calculate now the determinant with the help of multilinear algebra (cf. [5, Chap. 1]). We first define the coordinate projections π_0 and π_1 by

$$\pi_0(x,u)=x \text{ and } \pi_1(x,u)=u.$$

Since nor X is a $(d-1)$-dimensional rectifiable set in $\mathbb{R}^{2d}$ we know that $\operatorname{Tan}(\text{nor } X,(x,u))$ is for almost all (x,u) a linear subspace (cf. [5, 3.2.16]). Hence, there exists for almost all $(x,u)\in$ nor X a basis $a_1(x,u),\ldots,a_{d-1}(x,u)$ with positive orientation, i.e.

$$\operatorname{sgn}\langle(\pi_0+r\pi_1)a_1(x,u)\wedge\ldots\wedge(\pi_0+r\pi_1)a_{d-1}(x,u)\wedge n,\Omega_d\rangle=1,$$

where $\Omega_d = dx_1 \wedge \ldots, dx_d$ is the volume form in $\mathbb{R}^d$ and the property that $|a_1(x,u) \wedge \ldots \wedge a_{d-1}(x,u)| = 1$. By definition of the determinant we have

$$\begin{aligned} |\det Df(x,u,r)| &= \langle (\pi_0 + r\pi_1)a_1(x,u) \wedge \ldots \wedge (\pi_0 + r\pi_1)a_{d-1}(x,u) \wedge n, \Omega_d\rangle \\ &= \sum_{k=0}^{d-1} r^k \sum_{\substack{\varepsilon_i=0,1 \\ \varepsilon_1+\ldots+\varepsilon_{d-1}=k}} \left\langle \pi_{\varepsilon_1} a_1(x,u) \wedge \ldots \wedge \pi_{\varepsilon_{d-1}} a_{d-1}(x,u) \wedge u, \Omega_d \right\rangle \\ &= \sum_{k=0}^{d-1} r^k \omega_k \langle a_1(x,u), \wedge \ldots \wedge a_{d-1}(x,u), \varphi_{d-1-k}(x,u)\rangle . \end{aligned}$$

Definition 23. *The k-th Lipschitz-Killing $(d-1)$-form $\varphi_k(x,u) = \varphi_k(u)$ is defined via the relation*

$$\begin{aligned} &\langle \xi_1(x,u) \wedge \ldots \wedge \xi_{d-1}(x,u), \varphi_k(u)\rangle \\ &= \mathcal{O}_{d-k}^{-1} \sum_{\substack{\varepsilon_i=0,1 \\ \varepsilon_1+\ldots+\varepsilon_{d-1}=d-1-k}} \left\langle \pi_{\varepsilon_1}\xi_1(x,u) \wedge \ldots \wedge \pi_{\varepsilon_{d-1}}\xi_{d-1}(x,u) \wedge u, \Omega_d \right\rangle . \end{aligned}$$

Remark 24. *The Lipschitz-Killing forms are universal differential forms. We will see in Theorem 30 below that they can be used to define the curvature measures of a set X. The forms are universal in the sense that they do not depend on the set X. This is the reason, why it is possible to define the Lipschitz-Killing curvature measures for other classes of sets with the help of these forms. This will be shown for example in Section 3.3.*

This means (by LHS=RHS) that

$$\begin{aligned} &\mathcal{H}^d((X_\varepsilon \setminus X) \cap \Pi_X^{-1}(B)) \\ &= \sum_{k=0}^{d-1} \omega_k r^k \int_{\mathrm{nor}\ X} \mathbf{1}_B(x) \langle a_X(x,u), \varphi_{d-k}(x,u)\rangle \, d\mathcal{H}^{d-1}(x,u), \end{aligned}$$

where $a_X(x,u) = a_1(x,u) \wedge \ldots \wedge a_{d-1}(x,u)$ is a unit simple orienting vector field of X.

Definition 25. *The k-the Lipschitz-Killing curvature measure of X is defined as*

$$C_k(X,B) := \int_{nor\ X} \mathbf{1}_B(x) \langle a_X, \varphi_k\rangle \, d\mathcal{H}^{d-1}$$

if $0 \le k < d$ and $C_d(X,B) := \mathcal{H}^d(X \cap B)$.

Thus, we obtain the following tube formula, which originally is due to Federer [4] (but he gave a quite different proof using approximations of sets with positive reach by smooth manifolds) and unifies the formulas of Steiner and Wely:

Theorem 26. *For all $X \in PR$, $r <$ reach X and Borel sets $B \subseteq \mathbb{R}^d$ we have*

$$\mathcal{H}^d((X_\varepsilon \setminus X) \cap \Pi_X^{-1}(B)) = \sum_{k=0}^{d-1} \omega_k C_{d-k}(X, B) r^k.$$

A comparison of Theorem 26 with the formulas of Steiner and Wely shows

Proposition 27.

1. *If X is a convex set in $\mathbb{R}^d$ then $V_k(X) = C_k(X, \mathbb{R}^d)$, $k = 0, \dots, d$.*
2. *If X is a compact C^2-submanifold of $\mathbb{R}^d$ then $M_k(X) = C_k(X, \mathbb{R}^d)$, $k = 0, \dots, d$.*

We like to summarize some other important properties of the curvature measures $C_k(X, \cdot)$ here:

1. $C_k(X, \cdot)$ is a signed Radon measure on the Borel σ-algebra of $\mathbb{R}^d$,
2. $C_k(X, \cdot)$ is motion invariant, i.e. $C_k(gX, g\cdot) = C_k(X, \cdot)$ for all euclidean motions g,
3. $C_k(X, \cdot)$ is additive, i.e. $C_k(X \cup Y, \cdot) = C_k(X, \cdot) + C_k(Y, \cdot) - C_k(X \cap Y, \cdot)$, whenever $X, Y, X \cup Y, X \cap Y \in PR$,
4. $C_k(X, \cdot)$ is homogeneous, i.e. $C_k(\lambda X, \lambda \cdot) = \lambda^k C_k(X, \cdot)$ for $\lambda > 0$,
5. C_k is continuous, i.e. if $X_n \to X$ in Hausdorff metric, then $C_k(X_n, \cdot) \to C_k(X, \cdot)$ in the sense of weak convergence of measures.

It is now our goal to give explicit representations of these curvature measures. We start by introducing a fundamental tool in singular curvature theory, the unit normal cycle N_X of a set X. If we denote by $\mathcal{D}^k(M)$ the set of k-forms with compact support on a manifold M, the space $\mathcal{D}_k(M)$ of k-currents can be introduced as the dual space $\mathcal{D}_k(M) = (\mathcal{D}^k(M))^*$. The normal cycle will be a $(d-1)$-current on the manifold $M = \mathbb{R}^d \times S^{d-1}$, whose support is the unit normal bundle nor X of $X \in PR$.

Definition 28. *The functional or $(d-1)$-current*

$$N_X(\omega) := \int_{nor\ X} \langle a_X(x, u), \omega(x, u) \rangle \, d\mathcal{H}^{d-1}(x, u),$$

where $\omega \in \mathcal{D}_{d-1}(\mathbb{R}^d \times S^{d-1})$ is a $(d-1)$-form, is called the (unit) normal cycle of X.

The idea to use this functional goes back to the ideas of Wintgen [29] and Zähle [32] in the early 80th. It is nowadays one of the fundamental tolls in singular curvature theory and integral geometry, because the proofs of many integral geometric formulas can be reduced to an application of Federer's Coarea Formula (Theorem 44 below). This will be shown in Section 4.

We summarize now the properties of the normal cycle N_X of a set $X \in PR$:

Proposition 29.

1. *N_X is a cycle, i.e. $\partial N_X(\omega') = N_X(d\omega') = 0$, where ω' is a $(d-2)$-form.*
2. *N_X is Legendrian, i.e. $N_X \llcorner \alpha = 0$ for $\alpha = \sum_{i=1}^{d} n_i dx_i$, i.e. the normal vectors are orthogonal to the associated tangent vectors.*
3. *N_X is a locally $(\mathcal{H}^{d-1}, d-1)$-rectifiable current in $\mathbb{R}^d \times S^{d-1}$.*
4. *N_X is additive, i.e. $N_{X\cup Y} = N_X + N_Y - N_{X\cap Y}$, if $X, Y, X \cup Y, X \cap Y \in PR$.*

For the prove of 1. we use the fact of [4], that ∂X_r, $X \in PR$, $r <$ reach X, is a $C^{1,1}$-hypersurface (this is a C^1-hypersurface with Lipschitz unit outer normal) without boundary. Thus, 1. follows by Stokes Theorem. 2. is clear by the construction and 3. follows from the fact that the support nor X is a $(d-1)$-dimensional rectifiable set in $\mathbb{R}^d \times S^{d-1}$. The additivity uses Theorem 33 below and can be shown as in [8, Thm. 4.2].

The normal cycle leads immediately to the following explicit representation of the curvatures measures established by Zähle [32]:

Theorem 30.

$$C_k(X, B) = (N_X \llcorner \mathbf{1}_{B\times S^{d-1}})(\varphi_k),\ 0 \le k < d.$$

We know from above that the boundary ∂X_ε is a $C^{1,1}$-hypersurface. Thus, there exists $d-1$ principal curvatures $k_i^\varepsilon(x+\varepsilon u)$ for almost all $x+\varepsilon u$. The limits

$$k_i(x,u) := \lim_{\varepsilon\to 0} k_i^\varepsilon(x+\varepsilon u)$$

are well defined for almost all $(x,u) \in$ nor X. An appropriate choice of an orthonormal basis of Tan(nor $X,(x,u)$), i.e.

$$a_i(x,u) = \left(\frac{1}{\sqrt{1+k_i^2(x,u)}} b_i(x,u), \frac{k_i(x,u)}{\sqrt{1+k_i^2(x,u)}} b_i(x,u)\right)_{i=1}^{d-1}$$

(here we use the following convention: if $k_i = \infty$ then $\frac{1}{\sqrt{1+\infty^2}} = 0$ and $\frac{\infty}{\sqrt{1+\infty^2}} = 1$) and $\{b_1(x,u), \dots, b_{d-1}(x,u)\}$ is a basis of $\mathrm{Tan}(X_r, x+ru)$, leads to the following integral representation of the curvature measures also due to Zähle [32]:

Theorem 31.

$$C_k(X,B) = \int_{nor\ X} \mathbf{1}_B(x) \prod_{i=1}^{d-1} \frac{\sigma_{d-1-k}(k_1(x,u),\dots,k_{d-1}(x,u))}{\sqrt{1+k_i^2(x,u)}} d\mathcal{H}^{d-1}(x,u).$$

This is the positive reach analogue to the definition of the k-th integral of mean curvature of a C^2-submanifold with C^2-smooth boundary, see Definition 9.

We return again to the normal cycle: Joseph Fu has worked out in [7] the following characteristic properties of normal cycles and introduced the family of so-called geometric sets:

Definition 32. *A compact set $X \subset \mathbb{R}^d$ is called geometric if it admits a normal cycle, i.e. a current $N_X \in \mathcal{D}_{d-1}(\mathbb{R}^d \times S^{d-1})$ in $\mathbb{R}^d \times S^{d-1}$ with the following properties:*

(1) N_X is a compact supported locally $(d-1)$-rectifiable current,

(2) N_X is a cycle, i.e. $\partial N_X = 0$,

(3) N_X is Legendrian, i.e. $N_X \llcorner \alpha = 0$, where $\alpha = \sum_{i=1}^{d} dx_i$ is the contact 1-form, i.e. the normal vectors are orthogonal to the associated tangent vectors,

(4) N_X satisfies

$$N_X(g\varphi_0) = \mathcal{O}_{d-1}^{-1} \int_{S^{d-1}} \sum_{x \in \mathbb{R}^d} g(x,u) j_X(x,u) d\mathcal{H}^{d-1}(x,u),$$

where $g : \mathbb{R}^d \times S^{d-1} \to \mathbb{R}$ is an arbitrary differentiable function,

$$j_X(x,u) := \mathbf{1}_X(x) \left(1 - \lim_{\varepsilon \to 0} \lim_{\delta \to 0} (X \cap B(x,\varepsilon) \cap H_{u,\delta}(x))\right)$$

and $H_{u,\delta}(x)$ is the hyperplane with unit normal u, which contains the point $x + \delta u$ (compare with Figure 4).

We remark that in [23] it was shown that the last condition (4) is equivalent to the following explicit representation of the normal cycle N_X:

$$N_X(\phi) = \int_{\mathbb{R}^d \times S^{d-1}} \langle j_X(x,u) a_X(x,u), \phi \rangle d\mathcal{H}^{d-1}(x,u) = (\mathcal{H}^{d-1} \llcorner \mathrm{nor}\ X) \wedge j_X a_X.$$

In the case of sets with positive reach X we have $j_X(x,u) = 1$ for almost all $(x,u) \in$ nor X and we deduce that PR-sets are geometric. We will see in Section 3.3 that also locally finite unions of sets with positive reach admit a normal cycle, i.e. are geometric sets in the sense of Definition 32.
We also mention the following uniqueness theorem due fu Fu [8]:

Theorem 33. *For any compact set $X \subset \mathbb{R}^d$ there is at most one current N_X satisfying the properties $(1)-(4)$ of Definition 32.*

The proof of this theorem is very involved and uses deep methods from geometric measure theory. We therefore omit even to sketch the idea of the proof.
It is clear that not every compact set $X \subset \mathbb{R}^d$ admits a normal cycle. The set X has at least to be locally rectifiable in the sense of Federer [5]. For example the so-called Koch curve (see [3]) is a non-rectifiable set in the euclidean plane and therefore not geometric in the sense of Definition 32. It is still an open problem to give another, more explicit and more geometric characterization of the class of geometric sets.

3.3 Additive Extension and U_{PR}-Sets

Curvatures and curvature measures for convex sets admit an additive extension to the so-called convex ring $\mathcal{R}$ (cf. [25]). This is the family of subsets of $\mathbb{R}^d$, which are locally representable as finite union of convex sets. It is clear that not every set $X \in \mathcal{R}$ has positive reach. Therefore it would be desirable to have a family of subsets of $\mathbb{R}^d$, which contains both, the classes PR and $\mathcal{R}$ and extends the notion of curvature in this sense. We introduce to this end the class U_{PR} of locally finite unions of sets with positive reach, whose arbitrary finite intersections have also positive reach (the last condition is of course not necessary for the definition of $\mathcal{R}$, because intersections of convex sets are always convex). It is our goal to extend now the Lipschitz-Killing curvatures and curvature measures to the class U_{PR}. Here we follow [33] and [20].
We start by introducing the following index function for a closed set $X \subseteq \mathbb{R}^d$, $x \in \mathbb{R}^d$ and $u \in S^{d-1}$:

$$i_X(x,u) := \mathbf{1}_X(x)\left(1 - \lim_{\varepsilon\to 0}\lim_{\varepsilon\to 0}\chi(X \cap B(x+(\varepsilon+\delta)u,\varepsilon))\right),$$

where χ is the Euler characteristic in the sense of singular homology and $B(y,r)$ is the closed ball around y with radius $r \geq 0$, see Figure 4.

We remark here that $i_X(x,u) = (-1)^{\lambda(x,u)} j_X(x,u)$ for almost all $(x,u) \in \mathbb{R}^d \times S^{d-1}$, where $\lambda(x,u)$ is the number of negative principal curvatures $k_1(x,u),\ldots,k_{d-1}(x,u)$. Since χ is additive on U_{PR}, i.e. $\chi(X\cup Y) = \chi(X)+\chi(Y)-\chi(X\cap Y)$ for $X,Y \in U_{PR}$ we have additivity of the index function:

$$i_{X\cup Y} = i_X + i_Y - i_{X\cap Y}$$

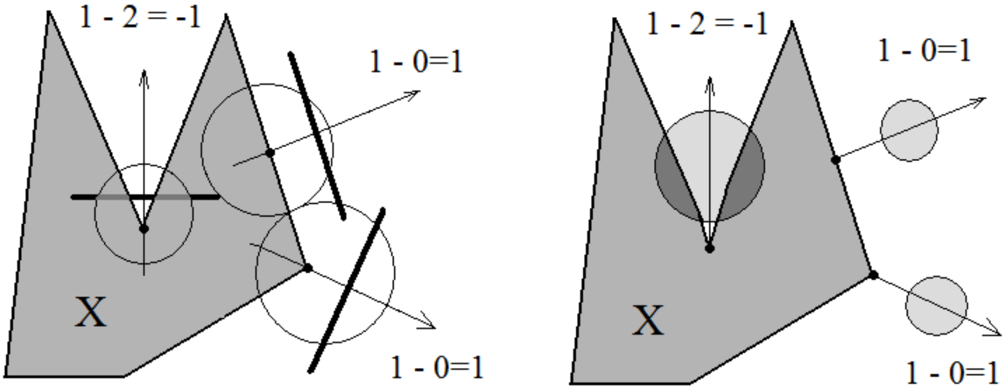

Figure 4: A set X with its associated index functions j_X (left picture) and i_X (right picture)

for such $X, Y \in U_{PR}$ with $X \cap Y \in U_{PR}$. The generalized unit normal bundle of a set $X \in U_{PR}$ is now defined as

$$\operatorname{nor} X := \{(x,u) \in \mathbb{R}^d \times S^{d-1} : i_X(x,u) \neq 0\}.$$

This is a locally $(\mathcal{H}^{d-1}, d-1)$-rectifiable subset in $\mathbb{R}^d \times S^{d-1}$ (cf. [5, 3.2.14]). This implies again that for almost all $(x,u) \in \operatorname{nor} X$ the approximate tangent space $\operatorname{Tan}^{d-1}(\operatorname{nor} X,(x,u))$ is a $(d-1)$-dimensional linear subspace of $\mathbb{R}^{2d}$. Therefore there exist vectors $b_1(x,u), \dots, b_{d-1}(x,u)$ (principal directions) in $\mathbb{R}^d$ perpendicular to u and real numbers $k_1(x,u), \dots, k_{d-1}(x,u)$ (principal curvatures), such that the vectors

$$a_i(x,u) = \left(\frac{1}{\sqrt{1+k_i^2(x,u)}} b_i(x,u), \frac{k_i(x,u)}{\sqrt{1+k_i^2(x,u)}} b_i(x,u) \right), \ i = 1, \dots, d-1$$

form an orthonormal basis of $\operatorname{Tan}^{d-1}(\operatorname{nor} X,(x,u))$. If $k_i = \infty$ then we put again $\frac{1}{\sqrt{1+\infty^2}} = 0$ and $\frac{\infty}{\sqrt{1+\infty^2}} = 1$. For any $X \in U_{PR}$ we now define its unit normal current as

$$N_X := (\mathcal{H}^{d-1} \llcorner \operatorname{nor} X) \wedge i_X a_X,$$

where $a_X(x,u) = a_1(x,u) \wedge \ldots \wedge a_{d-1}(x,u)$ is a unit simple orienting vector field of $\operatorname{nor} X$. From the additivity of the index function i on easily deduces [20, Thm. 2.2]

Theorem 34. *If $X, Y, X \cap Y \in U_{PR}$ then*

$$N_{X \cup Y} = N_X + N_Y - N_{X \cap Y}.$$

The following properties are an immediate consequence of the additivity and the corresponding validity in the case of PR-sets:

Proposition 35. *For $X \in U_{PR}$ we have*

1. *$\partial N_X = 0$, which means that the $(d-1)$-current N_X is a cycle.*
2. *$N_X \llcorner \alpha = 0$, where $\alpha = \sum_{i=1}^{d} dx_i$ is the contact 1-form, i.e. the normal vectors are orthogonal to the associated tangent vectors.*

Hence, by Theorem 33 the current N_X is the unique normal cycle of the U_{PR}-set $X \subset \mathbb{R}^d$ (1. and 4. are clear).

The curvature measures for an U_{PR}-set X can now be introduced as

$$C_k(X,B) := (N_X \llcorner \mathbf{1}_{B\times S^{d-1}})(\varphi_k),\ k=1,\dots,d-1,\ B \subseteq \mathbb{R}^d \text{ Borel.}$$

This are signed Radon measures on $\mathbb{R}^d$, whose support is given by the projection of the generalized unit normal bundle nor X onto the first component. Using the additivity from Theorem 34, the following properties carry over from the PR-case [20, Prop. 4.1]:

Proposition 36. *For $X, Y, X\cap Y \in U_{PR}$, $k=0,\dots,d-1$ and $B \in \mathcal{B}(\mathbb{R}^d)$ bounded we have*

1. *Motion invariance, i.e. $C_k(gX, gB) = C_k(X,B)$ for any euclidean motion $g \in SO(d) \ltimes \mathbb{R}^d$,*
2. *Additivity, i.e. $C_k(X\cup Y,\cdot) = C_k(X,\cdot) + C_k(Y,\cdot) - C_k(X\cap Y,\cdot)$,*
3. *Homogeneity: $C_k(\lambda X, \lambda B) = \lambda^k C_k(X,B)$, $\lambda \geq 0$,*
4. *Continuity: $\boldsymbol{F} - \lim_{n\to\infty} N_{X_n} = N_X$ implies $w - \lim_{n\to\infty} C_k(X_n,\cdot) = C_k(X,\cdot)$, $X_n \in U_{PR}$ (compare with Section 3.4).*

Using the description of the approximate tangent space $\mathrm{Tan}^{d-1}(\mathrm{nor}\ X,(x,u))$ (and the experience from the PR-case) one obtains the following integral representation for the curvature measures [33, Thm. 4.5.1], [20, Thm. 4.1]:

Theorem 37. *Let $X \in U_{PR}$, $B \in \mathcal{B}(\mathbb{R}^d)$ and $k \in \{0,\dots,d-1\}$ Then*

$$C_k(X,B) = \mathcal{O}_{d-1-k}^{-1} \int_{nor\ X} \mathbf{1}_B(x) i_X(x,u) \frac{\sigma_{d-1-k}(k_1(x,u),\dots,k_{d-1}(x,u))}{\prod_{i=1}^{d-1}\sqrt{1+k_i^2(x,u)}} d\mathcal{H}^{d-1}(x,u).$$

The integral and current representation of the curvature measures will be used in Section 4.3 to develop an integral geometry for U_{PR}-sets.
We close this section with the following version of the famous Gauss-Bonnet Theorem for U_{PR}-sets:

Theorem 38. *Let* $X \subset \mathbb{R}^d$ *a compact* U_{PR}*-set. Then*

$$\chi(X) = N_X(\varphi_0) = \sum_{x \in \partial X} j_X(x,u)$$

for almost all $n \in S^{d-1}$*.*

The first equality is proved in [22, Thm. 3.2] and the second one corresponds to [23, Thm. 4.4 (ii)]. We further remark that the sum in Theorem 38 is finite, i.e. there are only finitely many $x \in \partial X$ with $j_X(x,u) \neq 0$ for almost all $n \in S^{d-1}$.
After interpreting $N_X(\varphi_0)$ as the Euler-Characteristic of X, we now give an interpretation of the $(d-1)$-st curvature measure $C_{d-1}(X,\cdot)$:

Theorem 39. *For a set* $X \in U_{PR}$*,* $B \subseteq \mathbb{R}^d$ *Borel with the property that for all* $x \in \partial X \cap B$*,* $u \in Nor(X,x)$ *implies* $u \notin Nor(X,x)$*, we have*

$$(N_X \llcorner \mathbf{1}_{B \times S^{d-1}})(\varphi_{d-1}) = C_{d-1}(X,B) = \mathcal{H}^{d-1}(\partial X \cap B).$$

This was recently shown in [18, Cor. 2.2]. We mention that a similar result is also true for general Borel sets B. In this case the points $x \in \partial X$, where $\pm u \in \mathrm{Nor}(X,x)$ have to be weighted by a factor 2.

3.4 Characterization of Curvature Measures

We start by recalling some basic notions and notations from geometric measure theory [5]. The set of k-forms on some manifold M will be denoted by $\mathcal{D}^k(M)$. Its dual $\mathcal{D}_k(M) = (\mathcal{D}^k(M))^*$ is the space of k-currents. For $S \in \mathcal{D}_k(M)$ and a compact set $K \subset M$ we define the flat seminorm of S as

$$\mathbf{F}_K(S) = \sup\left\{ S(\varphi) : \varphi \in \mathcal{D}^k(M), \sup_{x \in K} \|\varphi(x)\| \leq 1, \sup_{x \in K} \|d\varphi(x)\| \leq 1 \right\},$$

where $\|\varphi\|$ is the comass of the k-form φ. We will write

$$S = \mathbf{F} - \lim_{n \to \infty} S_n, \;\; S_n \in \mathcal{D}_k(M)$$

if $\lim\limits_{n \to \infty} \mathbf{F}_K((S_n - S) \llcorner K) = 0$ for any compact set $K \subset M$.
We now put $M := \mathbb{R}^d \times S^{d-1}$ and fix a set $X \subseteq \mathbb{R}^d$ with positive reach, i.e. $X \in PR$. The normal cycle of X will be denoted by N_X.
We start now by the characterization of Lipschitz-Killing curvatures [36, Thm. 5.3]. Let therefore $\mathcal{C}$ be one of the classes PR or U_{PR}.

Theorem 40. *Let* $\psi : \mathcal{C} \to \mathbb{R}$ *be a functional such that*

(1) Ψ *is motion invariant, i.e.* $\Psi(gX) = \Psi(X)$ *for all euclidean motions,*

(2) Ψ *is additive, i.e.* $\Psi(X \cup Y) = \Psi(X) + \Psi(Y) - \Psi(X \cap Y)$ *whenever* $X, Y, X \cup Y, X \cap Y \in \mathcal{C}$*,*

(3) Ψ *is continuous, i.e.* $\lim_{n\to\infty} \Psi(X_n) = \Psi(X)$ *if* $\boldsymbol{F} - \lim_{n\to\infty} N_{X_n} = N_X$*,* $X, X_n \in \mathcal{C}$*,*

(4) $\Psi(X) \geq 0$ *for any compact convex polyhedron* X*.*

Then there exist certain constants $c_0, \dots, c_d$ *such that*

$$\Psi(X) = \sum_{k=0}^{d-1} c_k N_X(\varphi_k) + c_d \mathcal{H}^d(X), \ \ X \in \mathcal{C}$$

where φ_k *is the* k*-th Lipschitz-Killing curvature form.*

We next turn to the characterization of Lipschitz-Killing curvature measures [36, Th. 5.5]:

Theorem 41. *Let* $\Psi : \mathcal{C} \times \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}$ *a functional such that*

(1) for any $X \in \mathcal{C}$*,* $\Psi(X, \cdot)$ *is a signed Radon measure,*

(2) Ψ *is motion invariant, i.e.* $\Psi(gX, gB) = \Psi(X, B)$ *for all euclidean motions,*

(3) Ψ *is additive, i.e.* $\Psi(X \cup Y, B) = \Psi(X, B) + \Psi(Y, B) - \Psi(X \cap Y, B)$ *whenever* $X, Y, X \cup Y, X \cap Y \in \mathcal{C}$*,*

(4) Ψ *is continuous, i.e.* $w - \lim_{n\to\infty} \Psi(X_n, B) = \Psi(X, B)$ *(the weak limit of measures) if* $\boldsymbol{F} - \lim_{n\to\infty} N_{X_n} = N_X$*,* $X, X_n \in \mathcal{C}$*,*

(5) Ψ *is locally determined, i.e.* $\Psi(X, B) = \Psi(Y, B)$ *if* $N_{X\llcorner}(B \times S^{d-1}) = N_{Y\llcorner}(B \times S^{d-1})$*,*

(5) $\Psi(X, \cdot) \geq 0$ *if* X *is a compact convex polyhedron.*

Then there exist certain constants $c_0, \dots, c_{d-1}$ *such that*

$$\Psi(X, \cdot) = \sum_{k=0}^{d-1} c_k N_X(\varphi_k), \ \ X \in \mathcal{C}$$

and φ_k *is the* k*-th Lipschitz-Killing curvature form.*

The proof of these results is based on the following two approximation theorems [36, Thm. 3.1] and [36, Thm. 4.2]:

Theorem 42. *For any set with positive reach $X \in PR$ there exists a sequence (P_n) of simplicial polyhedra such that*

$$\boldsymbol{F} - \lim_{n\to\infty} N_{P_n} = N_X,$$

where N_{P_n} is the normal cycle associated with P_n.

By a simplicial polyhedron in $\mathbb{R}^d$ we mean a euclidean polyhedron generated by a locally finite number of euclidean d-simplices.

Theorem 43. *Let $X, X_n \in \mathcal{C}$ (here $\mathcal{C}$ is again one of the classes PR or U_{PR}) such that $\boldsymbol{F} - \lim_{n\to\infty} N_{X_n} = N_X$. Then*

$$w - \lim_{n\to\infty} C_k(X_n, B) = C_k(X, B), \ k = 0, \ldots, d-1, \ B \in \mathcal{B}(\mathbb{R}^d).$$

The last statement is clear, since flat convergence implies weak convergence of currents and the curvature measures are introduced by means of currents. Clearly Theorem 42 and Theorem 43 imply Theorem 40 and Theorem 41, because all statements may be reduced to the case of polytopes and in this case the situation is clear (cf. [25]).

Theorem 42 is proved in several steps. The first is to approximate the set X by its parallel set X_r, $0 < r <$ reach X. The boundary of these parallel sets are $(d-1)$-dimensional C^1-submanifolds with Lipschitz unit outer normal field, which may be triangulated. The edges of the triangulations generate now the boundary of a simplicial polyhedron. In a next step one shows that these polyhedra behave 'good', which means that their associated normal cycles (they are well defined by the results of [2]) converge in flat seminorm to the normal cycle of X.

4 Integral Geometry for Sets with Positive Reach and Extensions

It is the aim of this section to show how an integral geometry for sets with positive reach can be developed by using the normal cycle. This approach can be extended to U_{PR}-sets using the index function introduced in Section 3.3.

4.1 A Translative Integral Formula

The most important integralgeometric formula, the principal kinematic formula, deals with the integral

$$\int_{SO(d)\ltimes\mathbb{R}^d} C_k(X \cap gY, A \cap gB)dg,$$

where $X, Y \in PR$ and $A, B \subseteq \mathbb{R}^d$ are Borel sets. Using the product structure of the group of euclidean motions, we can write the last integral also as

$$\int_{SO(d)} \int_{\mathbb{R}^d} C_k(X \cap \vartheta(\tau_z Y)) dz d\vartheta.$$

It is the goal of this section to obtain an expression for the inner integral, i.e. for fixed $\vartheta \in SO(d)$. Such a formula is called translative integral formula.
Before starting, we will recall the following fundamental result from geometric measure theory, the so-called Coarea Formula [5, 3.2.22]:

Theorem 44. *Consider a Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}^n$ with $m > n$. If A is $\mathcal{L}^m$-measurable and $g : \mathbb{R}^m \to \mathbb{R}$ $\mathcal{L}^m$-integrable. Then*

$$\int_A g(x) J_n f(x) d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} \int_{f^{-1}(y)} g(x) d\mathcal{H}^n(y) d\mathcal{H}^{m-n} d\mathcal{H}^n(y).$$

Let us now fix two sets $X, Y \in PR$ such that also $X \cap Y \in PR$. Denote by U the set of pairs $(u, v) \in \mathbb{R}^d \times \mathbb{R}^d$ such that the closed segment with endpoints u and v does not contain the origin (this is the shorter geodesic arc on S^{d-1} connecting u and v), $R := \{(x, u, y, v) \in \mathbb{R}^{4d} : (u, v) \in U\}$ and consider the map

$$n : U \times [0, 1] \to \mathbb{R}^d : (u, v, t) \mapsto \frac{\sin t\alpha}{\sin \alpha} u + \frac{\sin(1-t)\alpha}{\sin \alpha} v,$$

where $\cos \alpha = \langle u, v \rangle$. Consider further the differentiable mapping

$$f : R \times [0, 1] \to \mathbb{R}^{2d} \times S^{d-1} : (x, u, y, v, t) \mapsto (x, y, n(u, v, t)),$$

which is locally Lipschitz and not necessarily proper. The joint unit normal bundle of X and Y is defined as

$$\text{nor}(X, Y) := f_{\#}(((\text{nor } X \times \text{nor } Y) \cap R) \times [0, 1]),$$

the joint normal cycle as

$$N_{X,Y} := f_{\#}(((N_X \times N_Y) \llcorner \mathbf{1}_R) \times [0, 1]).$$

We further introduce the following two mappings

$$G : \mathbb{R}^{3d} \to \mathbb{R} : (x, y, u) \mapsto x - y,$$

$$\pi : \mathbb{R}^{3d} \to \mathbb{R}^{2d} : (x, y, u) \mapsto (x, u).$$

From a remark in [22, p.112] we infer that the slices $\langle N_{X,\vartheta Y}, G, z \rangle$ are well defined for almost all rotations $\vartheta \in SO(d)$ and almost all $z \in \mathbb{R}^d$, where the slice $\langle T, h, z \rangle$ is defined as (compare with [5, 4.3.1])

$$\langle T, h, z \rangle := \lim_{r \downarrow 0} \frac{T \llcorner h^{\#}(\mathbf{1}_{B(z,r)} \Omega_d)}{\mathcal{H}^d(B(0, r))}.$$

For a Borel set $A \subseteq \mathbb{R}^{2d}$ we define for $1 \le i, j \le d-1$ the mixed curvature measures by

$$C_{i,j}(X, Y; A) :=$$

$$\int_{\operatorname{nor}(X,Y)} \mathbf{1}_A(x, y) \, \langle i_X(x, u) i_Y(y, u) \eta(x, y, u), \psi_{i,j}(x, y, u) \rangle \, d\mathcal{H}^{2d-1}(x, y, u),$$

$$C_{i,d}(X, Y; B \times C) := C_i(X, B) \cdot C_d(Y, C),$$

$$C_{d,j}(X, Y; B \times C) := C_d(X, B) \cdot C_j(Y, C)$$

where the $\psi_{i,j}(x, y, u) = \psi_{i,j}(u)$'s are the mixed Lipschitz-Killing curvature forms defined in [19, Section 2]. This are again universal differential forms like the Lipschitz-Killing curvature forms. In the special case they correspond to the mixed volumes of Section 2.1. Here $\eta(x, y, u)$ is the unit simple orienting vector field of the joint normal bundle of X and Y, such that

$$\lim_{\varepsilon \downarrow 0} \operatorname{sgn} \left\langle \eta(x, y, u), \sum_{\substack{1 \le i,j \le d-1 \\ i+j \ge d}} \varepsilon^{2d-1-i-j} \psi_{i,j}(x, y, u) \right\rangle = 1.$$

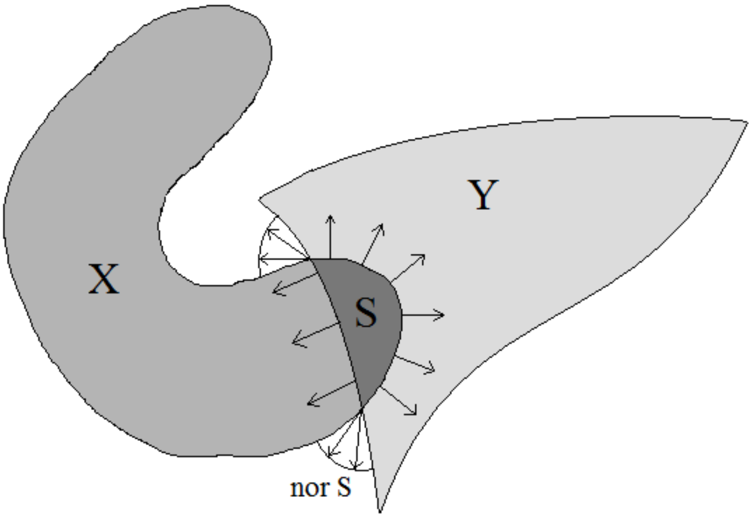

Figure 5: Two sets X and Y with positive reach and their intersection $S = X \cap Y$ with associated normal cycle nor S

Observe that the normal cycle of $X \cap \tau_z Y$ can be written as $N_{X \cap \tau_z Y} = N_1 + N_2 + N_3$, where $N_1 = N_X \llcorner (\operatorname{int} \tau_z Y \times S^{d-1})$, $N_2 = N_{\tau_z Y} \llcorner (\operatorname{int} X \times S^{d-1})$ and $N_3 = (\mathcal{H}^{d-1} \llcorner \operatorname{nor}(\partial X \cap \partial(\tau_z Y))) \wedge a_{X \cap \tau_z Y} i_{X \cap \tau_z Y}$, see Figure 5. Here $a_{X \cap \tau_z Y}$ is the unit simple orienting vector field of $X \cap \tau_z Y$ and $i_{X\cap\tau_z Y}(x,n) = i_X(x,n)\cdot i_{\tau_z Y}(x,n)$. We use now the current version [5, 4.3.8] of the Coarea Formula 44 to conclude that $N_3 = \pi_{\#}\left\langle N_{X,Y}, G, z\right\rangle$, whenever the slice is well defined.

Theorem 45. *Let $X, Y \subseteq \mathbb{R}^d$ be two sets of positive reach. Let further $h : \mathbb{R}^{3d} \to \mathbb{R}^d$ be a bounded Borel measurable function with compact support supp $h \subset \mathbb{R}^{3d}$. Assume further that $C_{i,j}(X, Y; K)$ is well defined for any compact set $K \subseteq \mathbb{R}^{2d}$. Then for $0 \le k \le d-1$ we have*

$$\int\int h(z,x,u)C_k(X\cap\tau_z Y, d(x,u))dz$$

$$= \sum_{i+j=k+d}\int h(x-y,x,u)C_{i,j}(X,Y;d(x,y,u)).$$

Proof. We have

$$C_k(X\cap\tau_z Y,\cdot) = N_{X\cap\tau_z Y}(\varphi_k) = N_1(\varphi_k) + N_2(\varphi_k) + N_3(\varphi_k)$$

by the definition of the curvature measures and the additivity of normal cycles for all $z \in \mathbb{R}^d$ for which the intersection $X \cap \tau_z Y$ has positive reach. Hence, we can write the left hand side as

$$\int\int h(x-y,x,u)C_k(X,d(x,u))C_d(Y,dy)$$

$$+\int\int h(x-y,x,u)C_d(X,dx)C_k(Y,d(y,u))$$

$$+\int_{\mathbb{R}^d}\pi_{\#}\left\langle N_{X,Y}\llcorner h, G, z\right\rangle(\varphi_k)d\mathcal{L}^d(z) = (*)$$

by using [19, Theorem 1] and the assumption of the theorem. Applying the Coarea Formula 44 we get for the last integral

$$\int_{\mathbb{R}^d}\pi_{\#}\left\langle N_{X,Y}\llcorner h, G, z\right\rangle(\varphi_k)d\mathcal{L}^d(z)$$

$$= ((N_{X,Y}\llcorner h)\llcorner G^{\#}\Omega_d)(\pi^{\#}\varphi_k) = (N_{X,Y}\llcorner h)(G^{\#}\Omega_d\wedge\pi^{\#}\varphi_k).$$

Thus, by using [19, Eq. (7)] we get

$$(*) = \int\int h(x-y,x,u)C_k(X,d(x,u))C_d(Y,dy)$$

$$+\int\int h(x-y,x,u)C_d(X,dx)C_k(Y,d(y,u)) + \sum_{\substack{i+j=k+d\\1\le i,j\le d-1}}(N_{X,Y}\llcorner h)(\psi_{i,j})$$

$$= \sum_{i+j=k+d}\int h(x-y,x,u)C_{i,j}(X,Y;d(x,y,u)),$$

which gives the result. □

For an iterated version of the translative integral formula for sets with positive reach see [17]. In the original version of this formula, the non-osculating condition

$$\mathcal{H}^d(\{z \in \mathbb{R}^d : \exists (x,u) \in \text{nor } X, (x-z,-u) \in \text{nor } Y\}) = 0$$

was assumed additionally. However, it was shown in [37] that this condition is not necessary to prove that $\text{reach}(X \cap \tau_z Y) > 0$ for almost all $z \in \mathbb{R}^d$. It can therefore by omitted. We further remark that Rataj [16, Thm. 1] gave an example of two $(d-1)$-dimensional C^{d-2}-submanifolds, $d \geq 3$, which violate the condition $\mathcal{H}^d(\{z \in \mathbb{R}^d : \exists (x,u) \in \text{nor } X, (x-z,-u) \in \text{nor } Y\}) = 0$.
The assumption that $C_{i,j}(X,Y;K)$ is well defined for any compact set $K \subseteq \mathbb{R}^{2d}$ can unfortunately not be omitted. Rataj and Zähle gave an example of a compact set $X \subset \mathbb{R}^4$ with positive reach and $u \in S^{d-1}$, such that

$$\mathcal{H}^1(\{\langle x,u\rangle : (x,u) \in \text{nor } X \text{ or } (x,-u) \in \text{nor } X\}) = 0$$

and the positive part of the mixed curvature measure $C_{1,3}(X,u^\perp,\cdot)$ is infinite on a compact set. They also gave sufficient conditions for the assumption to hold. One of them is the following: If for any compact subset $K \subset \mathbb{R}^{4d}$

$$\int_{K \cap (\text{nor } X \times \text{nor } Y) \cap R} (\sin \angle(u,v))^{3-d} d\mathcal{H}^{2d-2}(x,u,y,v) < +\infty$$

then all mixed curvature measures $C_{i,j}(X,Y;\cdot)$ are well defined (this is especially the case for $d \leq 3$). Moreover, the $C_{i,j}(X,\vartheta Y;\cdot)$'s are well defined for almost all rotations $\vartheta \in SO(d)$. For details and another condition involving absolute curvature measures and tangential projections we refer to [21].

4.2 The Principal Kinematic Formula

The principal kinematic formula follows now from an integration of the translative integral formula of Theorem 45 over the rotation group $SO(d)$. Therefore we will need the following integral representation of the mixed curvature measures [19, Thm. 3.2]:

Proposition 46. *For two sets of positive reach X and Y in $\mathbb{R}^d$ let $a_X = a_1 \wedge \ldots \wedge a_{d-1}$ and $b_Y = b_1 \wedge \ldots \wedge b_{d-1}$ be unit simple orienting vector field of nor X and nor Y respectively, both having positive orientation determined by $sgn\langle \xi(x,n) \wedge n, \Omega_d\rangle = 1$, where ξ is one of the vector fields a_X or b_X. Let further $1 \leq i,j \leq d-1$, $i+j \geq d$ and A be a bounded Borel set of $\mathbb{R}^{2d}$. Then*

$$C_{i,j}(X,Y;A) = \int_{(nor\ X \times nor\ Y) \cap R} \frac{\mathbf{1}_A}{\sigma_{2d-1-i-j}} F(i,j,\alpha)$$

$$\times \frac{\sum_{|I|=i}\sum_{|J|=j}\prod_{r\in I^c}\kappa_r\prod_{s\in J^c}\lambda_s\left[\bigwedge_{r\in I}a_r,\bigwedge_{s\in J}b_s\right]^2}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2}\prod_{j=1}^{d-1}\sqrt{1+\lambda_j^2}}d\mathcal{H}^{2d-2},$$

whenever the integral exists. Here

$$F(i,j,\alpha) := \frac{\alpha}{\sin\alpha}\int_0^1\left(\frac{\sin t\alpha}{\sin\alpha}\right)^{d-1-i}\left(\frac{\sin(1-t)\alpha}{\sin\alpha}\right)^{d-1-j}dt,$$

$\left[\bigwedge_{i\in I}a_i,\bigwedge_{j\in J}b_j\right]$ *is the Jacobian of the orthogonal projection of the linear subspace spanned by* $\{a_i : i\in I\}$ *onto the orthogonal complement of the subspace spanned by* $\{b_j : j\in J\}$*,* $I,J\subseteq\{1,\dots,d-1\}$ *and* κ_i *and* λ_j *are the generalized principal curvatures of* X *and* Y*, respectively. (*α *was defined at the beginning of Section 4.1.)*

By the help of this integral representation, we are now able to show the principal kinematic formula:

Theorem 47. *Suppose* X *and* Y *are subsets with positive reach and* A *and* B *are bounded Borel sets of* $\mathbb{R}^d$*. Then*

$$\int_{SO(d)\ltimes\mathbb{R}^d}C_k(X\cap gY,A\cap gB)dg = \sum_{i+j=k+d}\gamma(i,j,d)C_i(X,A)C_j(Y,B),$$

$$\textit{where } \gamma(i,j,d) = \frac{\Gamma(\frac{i+1}{2})\Gamma(\frac{j+1}{2})}{\Gamma(\frac{i+j-d+1}{2})\Gamma(\frac{d+1}{2})}.$$

Proof. We distinguish the two cases 1. $k=d$ and 2. $k<d$. For the first one we have

$$\int_{SO(d)\ltimes\mathbb{R}^d}\mathcal{H}^d(X\cap A\cap(gY\cap gB))dg = \int_{SO(d)\ltimes\mathbb{R}^d}\int_{X\cap gY}\mathbf{1}_{A\cap gB}(x)d\mathcal{H}^d(x)dg$$

$$= \int_{SO(d)\ltimes\mathbb{R}^d}\int_{X\cap gY}\mathbf{1}_A(x)dg\cdot\int_{SO(d)\ltimes\mathbb{R}^d}\int_{X\cap gY}\mathbf{1}_{gB}(x)dg$$

$$= C_d(X,A)C_d(Y,B)$$

and $\gamma(d,d,d)$=1. We now treat the case $k\le d-1$. Choose for the function h of Theorem 45 the following: $h(x,y,u)=\mathbf{1}_A(y)\mathbf{1}_{\vartheta B}(y-x)$, for a rotation $\vartheta\in SO(d)$. We now integrate both sides of the translative integral formula and obtain for the left hand side

$$\int_{SO(d)\ltimes\mathbb{R}^d}C_k(X\cap gY,A\cap gB)dg.$$

For the right hand side we get

$$\int_{SO(d)}\sum_{i+j=k+d}(N_{X,\vartheta Y}\llcorner\mathbf{1}_{A\times\vartheta B})(\psi_{i,j})d\vartheta$$

$$= \sum_{i+j=k+d} \int_{SO(d)} C_{i,j}(X, \vartheta Y; A \times \vartheta B) d\vartheta.$$

Note, that $C_{i,j}(X, \vartheta Y; \cdot)$ is well defined for almost all rotations $\vartheta \in SO(d)$ by the remark at the end of Section 4.1, see also [22, p. 125]. We can therefore make use of the integral representation provided by Proposition 46 (the intersection with R can be omitted after applying ϑ, see [19, Corollary 1]) and conclude that

$$= \sum_{i+j=k+d} \int_{\mathrm{nor}\ X \times \mathrm{nor}\ Y} \mathbf{1}_{A\times B} i_X i_Y \sum_{|I|=i} \sum_{|J|=j} \frac{\prod_{r\in I^c} \kappa_r \prod_{s\in J^c} \lambda_s}{\prod_{i=1}^{d-1} \sqrt{1+\kappa_i^2} \prod_{j=1}^{d-1} \sqrt{1+\lambda_j^2}}$$

$$\times \int_{SO(d)} \frac{F(i,j,\alpha(n, \vartheta m))}{\sigma_{2d-1-i-j}} \left[\bigwedge_I a_i, \bigwedge_J \vartheta b_j \right]^2 d\vartheta d\mathcal{H}^{2d-2}.$$

The inner integral is a constant $c(i,j,d)$ and the outer one can be written as $C_i(X,A) \cdot C_j(Y,B)$, by using the integral representation of the generalized curvature measures in Theorem 31:

$$= \sum_{i+j=k+d} c(i,j,d) \int_{\mathrm{nor}\ X\cap A} i_X \frac{\sum_{|I|=i} \prod_{r\in I^c} \kappa_r}{\prod_{r=1}^{d-1} \sqrt{1+\kappa_r^2}} d\mathcal{H}^{d-1}$$

$$\times \int_{\mathrm{nor}\ Y\cap B} i_Y \frac{\sum_{|J|=j} \prod_{s\in J^c} \lambda_s}{\prod_{s=1}^{d-1} \sqrt{1+\lambda_s^2}} d\mathcal{H}^{d-1}$$

$$= \sum_{i+j=k+d} c'(i,j,d) C_i(X,A) \cdot C_j(Y,B).$$

Hence, we have

$$\int_{SO(d)\ltimes\mathbb{R}^d} C_k(X \cap gY, A \cap gB) dg = \sum_{i+j=k+d} c'(i,j,d) C_i(X,A) C_j(Y,B).$$

The exact value of $c'(i,j,d)$ may be determined by letting X and Y balls with varying radii. This leads to $c'(i,j,d) = \gamma(i,j,d)$. ☐

We can also give the following short alternative proof of the principal kinematic formula:

Proof. For fixed X and variable Y or variable X and fixed Y it is easy to see that

$$\int_{SO(d)\ltimes\mathbb{R}^d} C_k(X \cap gY, A \cap gB) dg$$

is a functional as in Theorem 40. Applying this result twice, we get

$$\int_{SO(d)\ltimes\mathbb{R}^d} C_k(X\cap gY, A\cap gB)dg = \sum_{i+j=k+d} d(i,j,d)C_i(X,A)C_j(Y,B)$$

for some constants $d(i,j,d)$. The exact values may again be determined by using balls with different radii. □

4.3 Integral Geometry for U_{PR}-Sets

Using the notions and notations from the last section, we will sketch now, how an integral geometry can be developed for U_{PR}-sets. The joint unit normal bundle $\mathrm{nor}(X,Y)$ of two sets $X,Y\in U_{PR}$ is introduced in analogy to the PR-case:

$$\mathrm{nor}(X,Y) = f_{\#}(((\mathrm{nor}(X)\times\mathrm{nor}(Y))\cap R)\times[0,1]).$$

If it exists, the joint unit normal cycle is given by

$$N_{X,Y} = f_{\#}(((N_X\times N_Y)\llcorner\mathbf{1}_R)\times[0,1]).$$

Once again it is guarantied $N_{X,\vartheta Y}$ is well defined for almost all rotations $\vartheta\in SO(d)$ (cf. [20]). In this case the mixed curvature measures can be introduced: $C_{r,s}(X,Y,A) = (N_{X,Y}\llcorner\mathbf{1}_{A\times S^{d-1}})(\psi_{r,s})$, $A\subseteq\mathbb{R}^{2d}$ Borel. For these measures we have the following integral representation:

$$C_{i,j}(X,Y;A) = \int_{(\mathrm{nor}\ X\times\mathrm{nor}\ Y)\cap R} \mathbf{1}_A(x,y)\cdot\frac{i_X(x,u)i_Y(y,v)}{\sigma_{2d-1-i-j}}F(i,j,\alpha)$$

$$\times\frac{\sum_{|I|=i}\sum_{|J|=j}\prod_{r\in I^c}\kappa_r(x,u)\prod_{s\in J^c}\lambda_s(y,v)\left[\bigwedge_{r\in I}a_r(x,u),\bigwedge_{s\in J}b_s(y,v)\right]^2}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2(x,u)}\prod_{j=1}^{d-1}\sqrt{1+\lambda_j^2(y,v)}}$$

$$\times d\mathcal{H}^{2d-2}(x,u,y,v),$$

whenever the integral exists (cf. [20]). This is for example the case, if X and Y belong to the convex ring $\mathcal{R}$ [20, Prop. 4.5]. We also have that $C_{r,s}(X,\vartheta Y,\cdot)$ is well defined for almost all rotations $\vartheta\in SO(d)$ [20, Prop. 4.6]. Moreover, the translative integral formula as well as the principal kinematic formula hold true and can be proved in the same way as demonstrated in the last section:

Theorem 48. *Let $X=\bigcup_i X_i$, $Y=\bigcup_j Y_j$ be two locally finite unions of sets with positive reach in $\mathbb{R}^d$. Let further $h:\mathbb{R}^{3d}\to\mathbb{R}^d$ be a bounded Borel measurable function with compact support supp $h\subset\mathbb{R}^{3d}$. Assume further that $C_{i,j}(X,Y;K)$ is well defined for any compact set $K\subseteq\mathbb{R}^{2d}$ and that for all index subsets $I,J\subset\mathbb{N}$ with non-empty intersection sets $\bigcap_{i\in I}X_i$, $\bigcap_{j\in J}Y_j$ the sets $\bigcap_{i\in I}X_i$, $\bigcap_{j\in J}\tau_zY_j$ are* *non-osculating for $\mathcal{H}^d$-almost all $z \in \mathbb{R}^d$. Then $X \cap \tau_z Y \in U_{PR}$ for almost all $z \in \mathbb{R}^d$ and for $0 \le k \le d-1$ we have*

$$\int\int h(z,x,u)C_k(X \cap \tau_z Y, d(x,u))dz$$

$$= \sum_{i+j=k+d} \int h(x-y,x,u)C_{i,j}(X,Y;d(x,y,u)).$$

By integration over $SO(d)$ we get the principal kinematic formula for U_{PR}-sets:

Theorem 49. *Suppose $X, Y \in U_{PR}$ and A and B are bounded Borel sets of $\mathbb{R}^d$. Then*

$$\int_{SO(d)\ltimes\mathbb{R}^d} C_k(X \cap gY, A \cap gB)dg = \sum_{i+j=k+d} \gamma(i,j,d)C_i(X,A)C_j(Y,B),$$

$$\textit{where } \gamma(i,j,d) = \frac{\Gamma(\frac{i+1}{2})\Gamma(\frac{j+1}{2})}{\Gamma(\frac{i+j-d+1}{2})\Gamma(\frac{d+1}{2})}.$$

Remark 50. *Again, using Theorem 40 one can give another short proof of this formula as in the PR-case.*

The principal kinematic formula will be useful in the context of random processes of sets with positive reach and their associated union sets in Section 5.1. There, a stochastic version Theorem 49 will be derived. We also remark that the principal kinematic formula implies a Crofton-type formula for sets with positive reach as well as for locally finite unions from U_{PR}.

5 Random Sets with Positive Reach

As in the case of convex sets, a theory of random sets with positive reach or a theory of random processes of sets with positive reach can be developed. This general approach and concrete models will be shown within this section.

5.1 Definition and Integralgeometric Formulas

Following [31] we can construct random processes of sets with positive reach. Denote therefore by $\mathcal{G}, \mathcal{F}, \mathcal{K}$ the spaces of open, closed and compact sets in $\mathbb{R}^d$, respectively. As usual, a subbasis of the topology of $\mathcal{F}$ is generated by

$$\{\mathcal{F}_G : G \in \mathcal{G}\} \cup \{\mathcal{F}^K : K \in \mathcal{K}\},$$

where $\mathcal{F}_G = \{F \in \mathcal{F} : F \cap G \neq \emptyset\}$ and $\mathcal{F}^K = \{F \in \mathcal{F} : F \cap K \neq \emptyset\}$ (see for example [12] or [14]). The σ-algebra $\mathfrak{F}$ on $\mathcal{F}$ is generated by $\{\mathcal{F}_G : G \in \mathcal{G}\}$ and

$\{\mathcal{F}^K : K \in \mathcal{K}\}$. Denote here by PR the family of all compact sets with positive reach of $\mathbb{R}^d$. The trace of $\mathfrak{F}$ on PR will be denoted by $\mathfrak{PR}$. It was shown in [32, Prop. 1.1.1] that PR is a measurable subset of $\mathcal{F}$ (here we used the fact that sets with positive reach are closed).

We can now introduce random processes of sets with positive reach: Let $\mathcal{N}$ be the space of nonnegative, integer-valued, locally finite measures φ on $(PR, \mathfrak{PR})$. Any such measure may be represented as

$$\varphi(\cdot) = \sum_{X \in PR : \varphi(\{X\}) > 0} \varphi(\{X\}) \delta_X(\cdot),$$

where δ_X is the Dirac measure concentrated on X. Let further $\mathfrak{N}$ be the σ-algebra on $\mathcal{N}$, which is generated by the mappings $\varphi \mapsto \varphi(X)$ for all $X \in PR$. A random point process on $(PR, \mathfrak{PR})$ with sample space $(\mathcal{N}, \mathfrak{N})$ is now called a random process of sets with positive reach. Since $PR \in \mathfrak{F}$ and $\mathcal{F}$ is a compact separable Hausdorff space we have that $(\mathcal{F}, \mathfrak{F})$ and $(PR, \mathfrak{PR})$ are full (in the sense of [13]). Hence, by [13, Thm. 4], random processes of sets with positive reach can be constructed by finite dimensional distributions. We can for example construct Poissonian random processes of sets with positive reach with some given intensity measure. This will be demonstrated in Example 60.

For any $\varphi \in \mathcal{N}$ exists an associated union set φ_u, which is defined as

$$\varphi_u := \bigcup_{X : \varphi(\{X\}) > 0} X. \tag{1}$$

As in [31, Prop. 1.3.1] we have that the mapping $U : \mathcal{N} \to \mathcal{F} : \varphi \mapsto \varphi_u$ is measurable. Hence, φ_u is a random closed set (in the sense of [12] or [14]) for any random PR-process φ. To ensure that φ_u is a U_{PR}-set, for which integralgeometric formulas are valid, we have to restrict the class of processes to a subclass satisfying some regularity conditions. We require therefor the components of the union set φ_u to be in a general relative position. This ensures later that we can investigate second order properties of the union set. It is clear that for any U_{PR}-set $Z \in U_{PR}$ there exists at least one $\varphi \in \mathcal{N}$ such that $\varphi_u = Z$. We now restrict our attention to the opposite direction, i.e. those point measures $\varphi \in \mathcal{N}$, for which $\varphi_u \in U_{PR}$ and introduce the space

$$PR_r^n := \{(X_1, \dots, X_n) \in PR^n : \forall I \subseteq \{1, \dots, n\} \text{ we have } \bigcap_{i \in I} X_i \in PR\},$$

where $PR^n = \bigtimes_{i=1}^{n} PR$. Denote further by $\mathfrak{PR}^n$ the product σ-algebra $\bigotimes_{i=1}^{n} \mathfrak{PR}$ (analogously the n-fold product σ-algebra of $\mathfrak{F}$ by $\mathfrak{F}^n$). We have that PR_r^n is measurable in $\mathfrak{PR}^n$. The n-fold product of $\varphi \in \mathcal{N}$ with itself will be denoted by φ^n. Since the families PR_r^n are measurable, we deduce that for each $n \geq 1$

$$\{\varphi \in \mathcal{N} : \varphi^n(PR^n \setminus PR_r^n) = 0\} \in \mathfrak{N}.$$

The space of regular processes of sets with positive reach can now be defined as

$$\mathcal{N}_r = \bigcap_{n=2}^{\infty} \{\varphi \in \mathcal{N} : \varphi^n(PR^n \setminus PR_r^n) = 0\}.$$

Definition 51. *A random PR-process* Φ *will be called regular, if* $\mathbb{P}(\Phi \in \mathcal{N}_r) = 1$.

The following result is now obvious:

Proposition 52. *We have* $\mathcal{N}_r \in \mathfrak{N}$ *and* $\Phi \in \mathcal{N}$ *is a regular iff* $\mathbb{P}(\Phi^n(PR^n \setminus PR_r^n) = 0) = 1$ *for any* $n \geq 2$.

For a regular PR-process $\Phi \in \mathcal{N}_r$ it is now clear that its associated union set Φ_u defined by (1) is a locally finite union of sets with positive reach, for which the integralgeometric tools of Section 4.3 are available. This will be essential for the study of second order properties in the next section.

We denote by $G_d = SO(d) \ltimes T_d$ the group of euclidean motions, where $SO(d)$ is the special orthogonal group and T_d the group of translations of $\mathbb{R}^d$. G_d acts naturally on space of sets with positive reach, namely by rotations, translations and their compositions. This action induces a natural counterpart on the space $\mathcal{N}$ of point measures by

$$g\varphi(X) := \varphi(gX),$$

where $g \in G_d$ and $\varphi \in \mathcal{N}$. Using standard arguments, one easily shows that these actions are measurable [31, Prop. 1.7.1]

Definition 53. *We say that a random* PR*-process* Φ *with distribution* $P_\Phi = \mathbb{P} \circ \Phi^{-1}$ *is stationary, if* P_Φ *is invariant under all translations of* $\mathbb{R}^d$ *and isotropic, if* P_Φ *is invariant under the action of* $\vartheta \in SO(d)$ *on* $\mathbb{R}^d$*. The process* Φ *will be called motion invariant, if it is stationary and isotropic, i.e. invariant under all euclidean motions* $g \in G_d$.

Curvature measures of U_{PR}-sets were considered in Section 3.3. We fix now a regular random PR-process Φ, which ensures that the curvature measures of its associated union set Φ_u are well defined.

Definition 54. $C_k(\Phi_u, \cdot)$ *is said to be the* k*-th (random and signed) curvature measure of the measure* $\Phi \in \mathcal{N}_r$ *(or better its associated union set).*

Mean values of curvature measures will play an important roll in the considerations of Section 5.2. Corresponding results and definitions are well known in the convex case.

Definition 55. *Let $\Phi \in \mathcal{N}_r$ a regular PR-process such that*

$$\mathbb{E}|C_k|(\Phi_u, B) < \infty \text{ and } \mathbb{E}|C_k|(\Phi_t, B) < \infty$$

for any bounded Borel set $B \subseteq \mathbb{R}^d$, where $|C_k|$ denotes the total variation of the measure C_k. Then the measures $\overline{C}_k(\cdot) := \mathbb{E}C_k(\Phi_u, \cdot)$ exist and are called the curvature intensity measures.

From the general result [31, Thm. 6.3.1] for signed random measures, one obtains that if Φ is stationary and $\overline{C}_k$ is determined, it is a multiple of the d-dimensional Lebesgue measure. The proportionality factors, determined by $\overline{C}_k = c_k\mathcal{L}^d$, $k = 0, \ldots, d$, are called curvature intensities of Φ, respectively.

We study now the intersection (and union) of processes of sets with positive reach [31, Thm. 3.1.1, Thm. 3.1.3]:

Proposition 56. *Let Φ and Ψ two independent regular PR-processes and further Φ motion invariant and Φ or Ψ concentrated on compact sets. Then*

$$(\Psi \cap \Phi)(\cdot) := \int\int \delta_{X\cap Y}(\cdot) d\Phi(X) d\Psi(Y)$$

is a regular PR-process a.s. Moreover, we have $\Phi_u \cup \Psi_u \in U_{PR}$ and $\Phi_u \cap \Psi_u \in U_{PR}$ a.s. for their associated unions sets Φ_i and Ψ_u.

The union and the intersection of Φ_u and Ψ_u can be defined as

$$\begin{aligned} \Phi_u \cup \Psi_u &:= \bigcup_{X:\Phi(\{X\})>0} X \cup \bigcup_{Y:\Psi(\{Y\})>0} Y, \\ \Phi_u \cap \Psi_u &:= \bigcup_{X:\Phi(\{X\})>0} \bigcup_{Y:\Psi(\{Y\})>0} (X \cap Y). \end{aligned}$$

Suppose that Φ and Ψ are two independent regular random PR-processes, such that for their associated union sets we have $\Phi_u \cup \Psi_u \in U_{PR}$ and $\Phi_u \cap \Psi_u \in U_{PR}$ a.s. - we say that Φ_u and Ψ_u are compatible a.s. Then the measures

$$\begin{aligned} C_k(\Phi_u \cup \Psi_u, \cdot) &= C_k(\Phi_u, \cdot) + C_k(\Psi_u, \cdot) - C_k(\Phi_u \cap \Psi_u), \\ \overline{C}_k(\Phi_u \cup \Psi_u, \cdot) &= \overline{C}_k(\Phi_u, \cdot) + \overline{C}_k(\Psi_u, \cdot) - \overline{C}_k(\Phi_u \cap \Psi_u) \end{aligned}$$

are well defined, provided the right hand side exists. We use now the principal kinematic formula of Theorem 49 do derive the following result [31, Thm. 4.2], which is a stochastic version of the principal kinematic formula:

Theorem 57. *Let Φ and Ψ two independent regular random PR-processes with the property that Φ_u and Ψ_u are compatible. Assume further that Φ is motion invariant and that for any bounded Borel set $B \subset \mathbb{R}^d$*

$$\begin{aligned}\mathbb{E}|C_k|(\Phi_u, B) &< \infty, k = 0, \ldots, d,\\ \mathbb{E}|C_k|(\Psi_u, B) &< \infty, k = 0, \ldots, d,\\ \mathbb{E}|C_k|(\Phi_u \cap \Psi_u, B) &< \infty.\end{aligned}$$

Then we have for bounded Borel sets $B \subseteq \mathbb{R}^d$

$$\overline{C}_k(\Phi_u \cap \Psi_u, B) = \sum_{i+j=k+d} \gamma(i,j,d) c_i^{\Phi} \overline{C}_j^{\Psi}(B),$$

where $\gamma(i,j,d)$ is the same constant as in Theorem 49.

Proof. We take expectation on both sides of the principal kinematic formula for U_{PR}-sets. The independence of Φ and Ψ yields together with Fubini's theorem for the right hand side

$$\begin{aligned}\mathbb{E} \sum_{i+j=k+d} \gamma(i,j,d) C_i^{\Phi}(A) C_j^{\Psi}(B) &= \sum_{i+j=k+d} \gamma(i,j,d) \overline{C}_i^{\Phi}(A) \overline{C}_j^{\Psi}(B)\\ &= \sum_{i+j=k+d} \gamma(i,j,d) c_i^{\Phi} \overline{C}_j^{\Psi}(B),\end{aligned}$$

where A was chosen in such a way that $\mathcal{L}^d(A) = 1$. Using the motion invariance of Φ and $\int_{G_d} \mathbf{1}_A(gx) d\mathcal{L}^d(x) = 1$ we infer for the left hand side

$$\begin{aligned} &= \mathbb{E} \int_{G_d} C_k(\Phi_u \cap g\Psi_u, A \cap gB)\\ &= \int_{G_d} \mathbb{E} \int_{\mathbb{R}^d} \mathbf{1}_A(x) \mathbf{1}_B(g^{-1}x) C_k(\Phi_u \cap g\Psi_u, dx) dg\\ &= \int_{G_d} \mathbb{E} \int_{\mathbb{R}^d} \mathbf{1}_A(gx) \mathbf{1}_B(x) C_k(g^{-1}\Phi_u \cap \Psi_u, dx) dg\\ &= \int_{G_d} \mathbb{E} \int_{\mathbb{R}^d} \mathbf{1}_A(gx) \mathbf{1}_B(x) C_k(\Phi_u \cap \Psi_u, dx) dg\\ &= \mathbb{E} \int_{\mathbb{R}^d} \mathbf{1}_B(x) \int_{G_d} \mathbf{1}_A(gx) dg C_k(\Phi_u \cap \Psi_u, dx)\\ &= \mathbb{E} C_k(\Phi_u \cap \Psi_u, B) = \overline{C}_k(\Phi_u \cap \Psi_u, B).\end{aligned}$$

□

We also mention the following two important corollaries [31, Cor. 4.4 and Cor. 4.5], which are a stochastic variant of Crofton's formula and a steriological formula for the curvature intensities:

Corollary 58. *Let* Φ *and* Ψ *as in Theorem 57 and assume additionally that* Ψ *is stationary. Then*

$$c_k^{\Phi\cap\Psi} = \sum_{i+j=k+d} \gamma(i,j,d) c_i^{\Phi} c_j^{\Psi}.$$

Corollary 59. *Let* E *be a generic* p*-dimensional plane,* $p = 0, \dots, d-1$*. Then*

$$c_k^{\Phi\cap E} = \gamma(d+k-p, p, d) c_{d+k-p}^{\Phi}.$$

Example 60. *As pointed out at the beginning of this section, random* PR*-processes may be constructed via their finite dimensional distributions. We consider in this example, Poissonian* PR*-processes (cf. [31, Sec. 1.6]). Let therefore* μ *be a non-negative, locally finite measure on the space* $(PR, \mathfrak{PR})$ *and* $\Phi \in \mathcal{N}$ *such that*

$$\mathbb{P}(\Phi(B_1) = k_1, \dots, \Phi(B_n) = k_n) = \prod_{j=1}^{n} \frac{(\mu(B_i))^{k_j}}{k_j!} e^{-\mu(B_j)},$$

where $B_1, \dots, B_n$ *are disjoint bounded sets with positive reach and* $k_1, \dots, k_n \geq 0$*. Such a* Φ *is called Poissonian* PR*-process. It can now be shown [31, Sec. 5] that if* Φ *is a motion invariant Poissonian* PR*-process then* Φ *is regular, i.e.* $\Phi \in \mathcal{N}_r$*.*

This theory will now be applied to random mosaics or random cell complexes whose cells (also the lower dimensional) are random sets with positive reach.

5.2 Random Cell Complexes and Random Curved Mosaics

In this section we apply the theory of deterministic and random sets with positive reach to random cell complexes and random mosaics in $\mathbb{R}^d$. We will follow here the lines of [35] and [27]. Let therefore $\mathcal{M}_i$, $i = 0, \dots, d$, be the space of connected compact i-dimensional submanifolds m_i with boundary and positive reach, i.e. $m_i = m_i \cup \partial m_i$ and reach $m_i > 0$. By a k-dimensional cell complex in $\mathbb{R}^d$, $p \leq d$, we mean a $(k+1)$-tuple $M = (M_0, \dots, M_k)$, where for $i \in \{0, \dots, k\}$ the M_i's are locally finite families from $\mathcal{M}_i$ (called i-cells) satisfying the incidence relations:

1. The intersection of two i-cells from M_i is either empty or a j-cell from M_j and $j < i$.
2. Any $(i-1)$-cell from M_{i-1} is contained in the boundary of some i-cell from M_i.
3. The boundary of any i-cell from M_i is the finite union of some $(i-1)$-cells from M_{i-1}.

As usual in algebraic topology, the corresponding union sets $\cup M_i$ are denoted by $|M_i|$ and called i-skeletons of the cell complex M. The cells from M_k are called k-dimensional PR-polyhedra in $\mathbb{R}^d$. We now omit the smoothness conditions and let $\mathcal{U}_i$ be the space of i-dimensional submanifolds with or without boundary, which are representable as PR-polyhedra. Any $(k+1)$-tuple $U = (U_0, \dots, U_k)$ of locally finite families of U_i from $\mathcal{U}_i$ satisfying the incidence relations 1.-3. is called a k-dimensional U_{PR}-cell complex. By $|U_i|$ we denote its i-skeleton and by $|U_k|$ the U_{PR}-polyhedron associated with U.

For a stochastic model we use again the language of point processes. Let $\mathcal{N}_i$ be the space of locally finite, non-negative and integer-valued measures on $(\mathcal{U}_i, \mathfrak{U}_i)$, where $\mathfrak{U}_i$ is the the trace of the σ-algebra $\mathfrak{U}_{\mathfrak{P}\mathfrak{R}}$, which is the smallest σ-algebra for which the mappings $f : U_{PR} \to \mathbb{R}^d \times S^{d-1} : X \mapsto \text{closure}(\text{nor } X)$ are measurable. The set of atoms $A(\varphi_i)$, $\varphi_i \in \mathcal{N}_i$ will correspond to the family U_i of i-cells. We will identify φ_i with $A(\varphi_i)$ and write $|\varphi_i|$ instead of $|A(\varphi_i)|$. The usual σ-algebra on $\mathcal{N}_i$ will be denoted by $\mathfrak{N}_i$. The space of k-dimensional random U_{PR}-complexes can now be introduced as

$$\mathcal{N}^k := \{\eta = (\eta_0, \dots, \eta_k) : \eta_i \in \mathcal{N}_i, A(\eta) = (A(\eta_0), \dots, A(\eta_k)) \text{ is a } U_{PR} - compex\}.$$

A random k-dimensional U_{PR} complex is defined as a random variable ξ with values in $(\mathcal{N}^k, \mathfrak{N}^k)$ (here $\mathfrak{N}^k$ is given by $(\mathfrak{N}_0 \otimes \ldots \otimes \mathfrak{N}_k) \cap \mathcal{N})$. We also write $\xi = (\xi_0, \dots, \xi_k)$ as a random vector and $|\xi_i|$ for the associated random i-skeleton. We call $|\xi_k|$ also the random k-polyhedron.

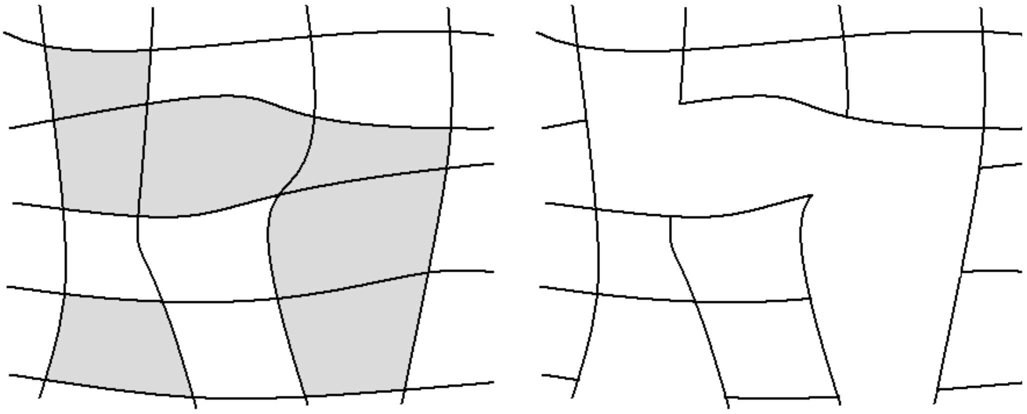

Figure 6: A random mosaic and a random cell complex whose cells are sets with positive reach. The random cell complex is obtained by a so-called p-thinning of the underlying random mosaic. Here the gray marked cells are deleted

For a k-dimensional U_{PR}-complex $U = (U_0, \dots, U_k)$ the curvature measures $C_n(|U_i|, \cdot)$ are well defined. The following result relates now the curvature measures of i-cells with its underlying complex (see [34, Thm. 4.2]):

Theorem 61. *Let* $U = (U_0, \dots, U_k)$ *be a* k*-dimensional* U_{PR}*-complex. Then for* $i = 0, \dots, k$ *we have*

$$C_n(|U_i|, B) = \sum_{j=n}^{i} (-1)^{j-n} \sum_{u_j \in U_j} C_n(u_j, B)$$

for all bounded Borel sets $B \subset \mathbb{R}^d$*.*

It follows the following Euler-type relation:

Corollary 62. *Under the conditions of Theorem 61 we have*

$$C_0(|u_k|, \mathbb{R}^d) = \sum_{i=0}^{k} (-1)^i a_i,$$

where a_i *is the number of* i*-cells of* u_i*.*

If we denote by $N_i(u_j)$ the number of i-cells adjacent to the j-cell u_j then Theorem 61 implies for $i = 0, \dots, n-1$

$$\sum_{j=n}^{i-1} (-1)^{j-n} \sum_{u_j \in U_j} N_i(u_j) C_n(u_j, B) = (1 - (-1)^{i-n}) \sum_{u_i \in U_i} C_n(u_i, B).$$

We now want to apply this theory to random cell complexes. There for we need the fact [35, Thm. 3.1.1] that for any random k-dimensional U_{PR}-complex $\xi = (\xi_0, \dots, \xi_k)$ the curvature measures $C_n(|\xi_i|, \cdot)$ are random signed Radon measures on $\mathbb{R}^d$. In analogy to Definition 53 we call a random U_{PR}-complex ξ (defined on an abstract underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$) stationary if its distribution $\mathbb{P} \circ \xi$ is invariant under translations of $\mathbb{R}^d$. We will restrict from now on out attention to stationary random U_{PR}-complexes which are integrable, i.e. for which

$$\mathbb{E} \sum_{u_i \in \xi_i} \int_{\text{closure(nor } u_i)} \mathbf{1}_B(x) |i_{u_i}(x,n)| d\mathcal{H}^{d-1}(x,n) < +\infty$$

for any bounded Borel set $B \subset \mathbb{R}^d$ and $i = 0, \dots, k$. Again, for such a stationary random cell-complex its associated mean curvature measures $\mathbb{E} C_n(|\xi_i|, \cdot)$ are multiples of the d-dimensional Lebesgue measures. The multiplicities (i.e. the intensities of the mean curvature measures) c_n^i are called curvature intensities. For integrable, stationary random U_{PR}-complexes $\xi = (\xi_0, \dots, \xi_k)$ the mean number N^i of i-cells per unit volume and the shape distribution P^i of the typical cell from ξ_i are well defined (cf. [35]). We denote by $C_n^i := \int C_n(u_i, \mathbb{R}^d) dP^i(u_i)$ the mean value of the n-th curvature of the typical i-cell. In particular

1. C_0^i is the mean Euler characteristic,
2. $2C_{i-1}^i$ is the mean $(i-1)$-volume of the boundary and
3. C_i^i is the mean i-volume of the typical i-cell of the random complex ξ.

The main result for random U_{PR}-complexes is [35, Thm. 3.3.6]:

Theorem 63. *For an integrable, stationary random U_{PR}-complex $\xi = (\xi_0, \dots, \xi_k)$ in $\mathbb{R}^d$ we have*

$$c_n^i = \sum_{j=n}^{i} (-1)^{j-n} N^j C_n^j.$$

This means that the curvature intensities may be computed by the curvature properties and the mean number of the typical j-cells, $j = 0, \dots, i$, $i = 0, \dots, k$. We can also conclude the following inversion formula:

$$C_n^i = (-1)^{i-n} (N^i)^{-1} (c_n^i - c_n^{i-1}).$$

If all cells are simply connected we also have [33, Cor. 3.3.7]

$$N^i = (-1)^i (c_0^i - c_0^{i-1})$$

As a special case we study now random stationary mosaic of $\mathbb{R}^d$. This are d-dimensional stationary random cell complexes $\xi = (\xi_0, \dots, \xi_d)$ (in the above sense) with the property that $|\xi_d| = \mathbb{R}^d$, a similar concept was studied in [27]. We remark that this model is quite more general than the one usually used in the literature on stochastic geometry (see for example [15]).
The above formulas may be completed in the mosaic case by the relation $c_d^d = N^d C_d^d = 1$. Moreover the relations from above yield

$$\sum_{j=n}^{d} (-1)^{j-n} N^j C_n^j = 0, \ \ n < d.$$

If moreover the cells are simply connected, the following Euler-type relation holds true (cf. [32, Eqn. (18)]):

$$\sum_{j=0}^{d} (-1)^j N^j = 0.$$

Example 64. *We assume that ξ is a d-dimensional integrable, stationary random mosaic in the above sense. In this case we use the following special notations: c_k^i is the k-th curvature intensity of the typical i-face, $\overline{C_k^i}$ is the mean total k-th curvature of the typical i-face, N^i is the mean number of i-faces per unit volume, $N^{i,j}$ is the man number of j-faces adjacent to the typical i-cell, $V^{i,i} = \overline{C_i^i}$ the mean i-volume of* *the typical i-face and $V^i = c_i^i$ the mean total i-dimensional volume of all i-faces per unit volume. Then we have for $d = 2$:*

$$c_0^0 = N^0,\ c_0^1 = N^0 - N^1,\ c_0^2 = 0 = N^0 - N‘1 + N^2,$$

$$c_1^1 = V^1 = N^1V^{1,1},\ c_1^2 = 0 = N^1V^{1,1} - \frac{1}{2}N^2V^{2,1},\ c_2^2 = 1 = N^2V^{2,2}.$$

For $d = 3$ we have

$$c_0^0 = N^0,\ c_0^1 = N^0 - N^1,\ c_0^2 = N^0 - N^1 + N^2,\ c_0^3 = 0 = N^0 - N^1 + N^2 - N^3,$$

$$c_1^1 = V^1 = N^1V^{1,1},\ c_1^2 = N^1V^{1,1} - \frac{1}{2}N^2V^{2,1},\ c_1^3 = 0 = N^1V^{1,1} - \frac{1}{2}N^2V^{2,1} + N^2V^{3,1},$$

$$c_2^2 = V^2 = N^2V^{2,2},\ c_2^3 = 0 = N^2V^{2,2} - \frac{1}{2}N^3V^{3,2},\ c_3^3 = 1 = N^3V^{3,3}.$$

Furthermore we conclude

$$N^0N^{0,1} = 2N^1,\ N^0N^{0,2} = N^1N^{1,2},$$

$$N^3N^{3,2} = 2N^2,\ N^0N^{0,3} - N^1N^{1,3} = 2N^3 - 2N^2.$$

We can also apply the stochastic Crofton formula of Corollary 59 from Section 5.1 to our situation. Let therefore E be p-dimensional plane. The intersection of ξ with E is a random stationary mosaic in E. The curvature intensities of $\xi \cap E$ can be calculated as follows:

$$c_{k,\xi\cap E}^i = \gamma(d+k-p,p,d)c_{d+k-p,\xi}^{d+i-p},\ i = 0,\dots,p;\ k = 0,\dots,i$$

and $\gamma(i,j,d)$ is the same constant as in Theorem 49.

Example 65. *We have in particular for $d = 3$ and $p = 2$*

$$c_{0,\xi\cap E}^0 = \frac{1}{2}c_{1,\xi}^1,\ c_{0,\xi\cap E}^1 = \frac{1}{2}c_{1,\xi}^2,\ c_{1,\xi\cap E}^1 = \frac{\pi}{4}c_{2,\xi}^1.$$

A similar holds also true for general random U_{PR}-complexes. This follows also immediately from Corollary 59.

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