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LOG-CONCAVITY PROPERTY FOR SOME WELL-KNOWN DISTRIBUTIONS

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Abstract. Interesting properties and propositions, in many branches of science such as economics have been obtained according to the property of cumulative distribution function of a random variable as a concave function. Caplin and Nalebuff (1988 [10],1989 [11]), Bagnoli and Khanna (1989 [7]) and Bagnoli and Bergstrom (1989 [4], 1989 [5], 2005 [6]) have discussed the log-concavity property of probability distributions and their applications, especially in economics.

Log-concavity concerns twice differentiable real-valued function g whose domain is an interval on extended real line. g as a function is said to be log-concave on the interval (a, b) if the function $ln(g)$ is a concave function on (a, b). Log-concavity of g on (a, b) is equivalent to $g^{'}/g$ being monotone decreasing on (a, b) or $(ln(g))^{''} < 0$. Bagnoli and Bergstrom (2005 [6]) have obtained log-concavity for distributions such as normal, logistic, extreme-value, exponential, Laplace, Weibull, power function, uniform, gamma, beta, Pareto, log-normal, Student's t, Cauchy and F distributions. We have discussed and introduced the continuous versions of the Pearson family, also found the log-concavity for this family in general cases, and then obtained the log-concavity property for each distribution that is a member of Pearson family. For the Burr family these cases have been calculated, even for each distribution that belongs to Burr family. Also, log-concavity results for distributions such as generalized gamma distributions, Feller-Pareto distributions, generalized Inverse Gaussian distributions and generalized Log-normal distributions have been obtained.

1 Introduction

The log-concavity and log-convexity property have an important role in economics, social sciences, information theory and optimization. Most of the time logarithm of cumulative function of a random variable is concave. In papers such as Laffont and Tirole (1988 [17]), Lewis and Sappington (1988 [18]), Baron and Myerson (1982 [8]), Riordan and Sappington (1989 [28]), Myerson and Satterthwaite (1983 [22]), Maskin and Riley (1984 [19]), Caplin and Nalebuff (1988 [10],1989 [11]) and Matthews (1987 [20]), many results due to concavity, log-concavity and their applications in many

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Log-concavity and log-convexity of survival functions are important in reliability theory that is equivalent to the failure rate being increasing and decreasing respectively. An (1994) studied classes of log-concave distributions that arise in economics of uncertainly and information. Bagnoli and Khanna (1989) obtained a model where there is a distribution reservation demand for houses via log-concavity of reliability function and similar research by Jegadeesh and Chowdhry (1989 [14]) for studying the log-concavity of reliability function in finance literature. Let F be the distribution function, Flinn and Heckman (1983 [12]) stated that if the function

$$H(x) = \int_x^{\infty} (1 - F(t))dt$$

is log-concave, then with optimal search strategies, an increase in the rate of arrivals of jobs offers will increase the exit rate from unemployment. Also, Bagnoli and Bergstorm (1989a) used by the log-concavity of

$$G(x) = \int_{-\infty}^{x} F(t)dt$$

developed a marring market model.
Fortunately, it happens that sufficient condition for *cdf* to be log-concave is that the density function be log-concave. A sufficient condition for the integral of the *cdf* being log-concave is that the *cdf* be log-concave. These results are proved by Prekopa (1972 [25]) in Hungarian Mathematics Journal. Flinn and Heckman (1983 [12]) introduced these results to the economics literature and were applied by Caplin and Nalebuff (1988 [10]).
In this paper, based on the theorems and properties mentioned in Bagnoli and Bergstrom (2005), we have obtained results due to log-concavity for Pearson type, Burr type distributions and some generalized version of distributions such as generalized gamma, Feller-Pareto and generalized inverse Gaussian distributions. Also, we have discussed log-concavity for each members of these families. On noting that, if the density function f is log-concave on (a, b), then properties of reliability measures connected to concavity and convexity are discussed.

2 Preliminaries

The concept of log-concavity was revolutionized by introducing log-concave probability measures due to Prekopa(1971 [24],1973 [26]) and An (1994 [1], 1995 [2], 1997 [3]) completed and reproduced several results on log-concavity from the previous literature and obtained some new results.

The following definitions and theorems that are discussed here (for more details see Bagnoli and Bergstrom 2005 [6] that has an important role in our results).

Definition 1. *A function g is said to be log-concave on interval (a, b) if the function $ln(g)$ is a concave function on (a, b).*

Definition 2. *Log-concavity of g on (a, b) is equivalent to each of the following two conditions:*
(i) g'/g is monotone decreasing on (a, b).
(ii) $(lng)'' < 0$.

Lemma 3. *Let g be strictly monotonic(increasing or decreasing) defined on the interval (a, b), it must be that $g(x)$ is also a log-concave function on (a, b).*

Let X be continuous random variable with density function $f(x)$ and *cdf* $F(x)$ whose support Ω is an open interval such as $(a, b) \subset \Re$. Define in the interval (a, b),
$S(x) \equiv 1 - F(x)$ as its survivor function,
$h(x) \equiv f(x)/S(x)$ as its hazard function,
$G(x) \equiv \int_a^x F(u)du$ as its left side integral,
$H(x) \equiv \int_x^b S(u)du$ as its right side integral.

Thus, the following notes are needed (See Bagnoli and Bergstrom 2005 [6] and An 1995 [2]), so:

- If the density function f, is monotone decreasing(increasing), then its cdf.,F(S), and its left side integral, G, are both log-concave.
- (i). If the density function f, is log-concave on (l, h), then the survivor (reliability) function S, is also log-concave on (l, h).

  (ii). If the reliability function S, is log-concave on (l, h), then the right hand integral H, is log-concave function on (l, h).
- If the density function f, be log-concave on (a, b), then the failure rate is monotone increasing on (a, b). If the failure rate is monotone increasing on (a, b), then H'/H is monotone decreasing.
- If the density function f is monotone increasing, then the reliability function, S, is log-concave.
- Among the properties of log-concave distributions, the most surprising result is that the class of log-concave densities coincides with the class of strongly unimodal densities.
- Let $g : \Re \to \Re_+$ be a measurable function. Suppose $\{x : g(x) > 0\} = (a, b)$. If $g(x)$ is log-concave on (a, b), then $G_l(x) \equiv \int_b^x g(y)dy$ is log-concave on (a, b).

For the functions defined above, the following logical implications hold :
$f(x)$ is log-concave $\Rightarrow h(x)$ is non-decreasing in x,

$f(x)$ is log-concave $\Leftrightarrow S(x)$ is log-concave,
$f(x)$ is log-concave $\Rightarrow H(x)$ is log-concave,
$f(x)$ is log-concave $\Rightarrow F(x)$ is log-concave,
$f(x)$ is log-concave $\Rightarrow G(x)$ is log-concave.

Theorem 4. *Let X be a random variable whose density function, $f(x)$, is log-concave. Then for any $a \neq 0$ the random variable $Y = \alpha X + \beta$ is log-concave.*

Proof. See An (1995 [2]). □

Theorem 5. *Let X be a random variable with monotonically decreasing density function $f(x)$. Then,*
1. If X is Log-concave(log-convex) then, for any $\alpha \neq 0$, the random variable $\alpha X + \beta$ is log-concave(log-convex). In particular, the mirror image, $Y = -X$, is log-concave(log-convex).
2. If X is log-concave and positive valued then $log(X)$ is log-concave.
3. If X is log-convex, then, e^X is log-convex.

Proof. See An (1997 [3]) . □

- Let X be a random variable whose density function, $f(x)$, is log-concave and monotonic decreasing. Consider a function $l(.)$ satisfying:

  (i) $x = l(y)$ is strictly increasing, differentiable and convex,

  (ii) $l''(y)$ is log-concave.
  Then the random variable $Y = l^{-1}(X)$ is log-concave.

- The distributions such as uniform, normal, logistic, gamma $(G(\alpha, \beta), \alpha \geq 1)$, beta $(B(a, b), a \geq 1, b \geq 1)$ and Weibull $(W(\gamma, \alpha), \alpha \geq 1)$ are log-concave and Pareto, gamma $(G(\alpha, \beta), \alpha < 1)$, beta $(B(a, b), a < 1, b < 1)$, Weibull $(W(\gamma, \alpha), \alpha < 1)$ and F-distribution $(F(m_1, m_2), m_1 \leq 2)$ are log-convex.

- There are distributions which are both log-concave and log-convex. For example, the negative exponential distributions are such cases. In fact, since linear functions are the only functions which are both concave and convex, the only distributions which are both log-concave and log-convex are exponential or truncated exponential.

- There are distributions which are neither log-concave nor log-convex over the entire support. Examples include the log-normal distribution, the Beta distribution with $a > 1$ and $b < 1$ and the F-distribution with the first degree of freedom $m_1 > 2$.

3 Log-concavity of the Pearson and Burr families of distributions

In this section, via the arguments in Bagnoli and Bergstrom (2005 [6]), the easiest distributions to deal with that, are the one's with log-concave or log-convex density functions where distribution function and failure rate function of them are listed in Table 1.

The Pearson and Burr families will be introduced before we discuss about log-concavity of the Pearson and Burr families of distributions.

3.1 Pearson family

Pearson (1895 [23]) used as a solution of differential equation

$$\frac{f^{'}(x)}{f(x)} = \frac{x-a}{b_0 + b_1 x + b_2 x^2}, \tag{3.1}$$

where f is the density of the random variable X and it's derivative exists as the densities of Pearson family. Also, discrete version of their family is obtained that we can not use in this discussion. For various values of a, b_0, b_1 and b_2, we have some members of this family that is shown in Table 2.

Theorem 6. *For the Pearson family with the form (3.1):*
I. If $b_2 > 0$, $a^2 b_2 + b_0 + b_1 a > 0$ for $x > a + \frac{\sqrt{a^2 {b_2}^2 + b_2(b_0+b_1 a)}}{b_2}$ or $x < a - \frac{\sqrt{a^2 {b_2}^2 + b_2(b_0+b_1 a)}}{b_2}$, the Pearson family is log-concave.
II. If $b_2 > 0$, $a^2 b_2 + b_0 + b_1 a < 0$, then the Pearson family is log-concave.
III. If $b_2 < 0$, $a^2 b_2 + b_0 + b_1 a < 0$ for $a - \frac{\sqrt{a^2 {b_2}^2 + b_2(b_0+b_1 a)}}{b_2} < x < a + \frac{\sqrt{a^2 {b_2}^2 + b_2(b_0+b_1 a)}}{b_2}$, the Pearson family is log-concave.

Proof. The Pearson family (3.1) is log-concave if $h^{'}(x) = \frac{d}{dx}(\frac{f^{'}(x)}{f(x)}) < 0$.
Thus,

$$h^{'}(x) = \frac{d}{dx}(\frac{x-a}{b_0 + b_1 x + b_2 x^2}) < 0 \Rightarrow \tag{3.2}$$

$$h^{'}(x) = \frac{-b_2 x^2 + 2ab_2 x + b_0 + b_1 a}{(b_0 + b_1 x + b_2 x^2)^2} < 0 \Rightarrow \tag{3.3}$$

$$b_2 x^2 - 2ab_2 x - b_0 - b_1 a > 0. \tag{3.4}$$

So, based on $\Delta = 4a^2 b_2^2 + 4b_2(b_0 + b_1 a)$ and b_2 we have these statements for holding log-concavity property.

Table 1: Log-concavity of some common Distributions

Distribution	Form of density	The derivative of $\ln f$	density	c.d.f.	Reliability
Uniform	1	$\frac{1}{x}$	log-concave	log-concave	log-concave
Normal	$\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$	$-x$	log-concave	log-concave	log-concave
Logistic	$\frac{e^{-x}}{(1+e^{-x})^2}$	$\frac{e^{-x}-1}{e^{-x}+1}$	log-concave	log-concave	log-concave
Extreme Value	$e^{-e^{-x}}$	e^{-x}	log-concave	log-concave	log-concave
Chi-Square	$\frac{x^{\frac{n}{2}-1}2^{\frac{n}{2}}e^{-2x}}{\Gamma(\frac{n}{2})}$	$\frac{n-2-4x}{2x}$	log-concave	log-concave	log-concave
Chi	$\frac{x^{(\frac{n}{2})-1}e^{(-\frac{n}{2})}x^2}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}$	$\frac{n+2}{2x}$	log-concave	log-concave	log-concave
Exponential	$\lambda e^{-\lambda x}$	$-\lambda$	log-concave	log-concave	log-concave
Laplace	$\frac{\lambda}{2}e^{-\lambda\|x\|}$	$\begin{cases} -\lambda & x \geq 0 \\ \lambda & x < 0 \end{cases}$	log-concave	log-concave	log-concave
Weibull($c \geq 1$)	$cx^{c-1}e^{-x^c}$	$-cx^{c-1}$	log-concave	log-concave	log-concave
Gamma($m \geq 1$)	$\frac{x^{m-1}\theta^m e^{-\theta x}}{\Gamma(m)}$	$-\frac{-m+1+\theta x}{x}$	log-concave	log-concave	log-concave
Beta ($a \geq 1, b \geq 1$)	$\frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$	$\frac{x(a+b-2)-a+1}{x(x-1)}$	log-concave	log-concave	log-concave
Log Normal	$\frac{e^{-(lnx)^2}/2}{x\sqrt{2\pi}}$	$-\frac{1+lnx}{x}$	mixed	log-concave	*mixed*⋆
Pareto	$\beta x^{-\beta-1}$	$-\frac{\beta+1}{x}$	log-convex	log-concave	log-convex
Power Function ($\beta < 1$)	$\beta x^{\beta-1}$	$\frac{\beta+1}{x}$	log-convex	log-concave	mixed
Weibull (($c < 1$)	$cx^{c-1}e^{-x^c}$	$-cx^{c-1}$	log-convex	log-concave	log-convex
Gamma ($m < 1$)	$\frac{x^{m-1}\theta^m e^{-\theta x}}{\Gamma(m)}$	$-\frac{-m+1+\theta x}{x}$	log-convex	log-concave	log-convex
Beta ($a = .5, b = .5$)	$\frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$	$\frac{x(-1)+0.5}{x(x-1)}$	log-convex	*mixed*⋆	*mixed*⋆
Beta ($a = 2, b = .5$)	$\frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$	$\frac{x(.5)-1}{x(x-1)}$	*mixed*⋆	*mixed*⋆	log-convex
Student's t	$\frac{(1+\frac{x^2}{n})^{-n+1/2}}{\sqrt(n)B(.5,n/2)}$	$\frac{(1-2n)x}{n+x^2}$	mixed	*mixed*⋆	*mixed*⋆
Cauchy	$\frac{1}{\pi(1+x^2)}$	$\frac{-2x}{1+x^2}$	mixed	*mixed*⋆	*mixed*⋆

*Denotes answers found, not by analytic means, but by numerical simulation for particular parameter values see detailed comments on the particular distribution in Bagnoli and Bergstorm .

Table 2: The Pearson Family Distribution

Type	Density	Support	$\frac{f^{'}(x)}{f(x)}=\frac{x-a}{b_0+b_1x+b_2x^2}$
Normal	$exp(\frac{-x^2}{2})$	$x\in\Re$	$a=b_1=b_2=0, b_0=-1$
I	$(1+x)^{m_1}(1-x)^{m_2}$	$-1\le x\le 1$	$a=-\frac{m_2-m_1}{m_2+m_1}, b_0=-\frac{1}{m_2+m_1}, b_1=0$ $b_2=\frac{1}{m_2+m_1}$
II	$(1-x^2)^m$	$-1\le x\le 1$	$a=0, b_0=-\frac{1}{2m}, b_1=0, b_2=\frac{1}{2m}$
III	$x^m exp(-x)$	$x\ge 0$	$a=m, b_0=0, b_1=-1, b_2=0$
IV	$(1+x^2)^{-m}*exp(-vtan^{-1}(x))$	$x\in\Re$	$a=-\frac{\nu}{2m}, b_0=-\frac{1}{2m}, b_1=0, b_2=-\frac{1}{2m}$
V	$x^{-m}exp(-x^{-1})$	$0\le x<\infty$	$a=\frac{1}{m}, b_0=b_1=0, b_2=-\frac{1}{m}$
VI	$x^{m_2}(1+x)^{-m_1}$	$0\le x<\infty$	$a=-\frac{m_2}{m_2-m_1}, b_0=0, b_1=b_2=\frac{1}{m_2-m_1}$
VII	$(1+x^2)^{-m}$	$x\in\Re$	$a=b_1=0, b_0=b_2=-\frac{1}{2m}$
VIII	$(1+x)^{-m}$	$0\le x\le 1$	$a=m, b_0=b_1=1, b_2=0$
IX	$(1+x)^m$	$0\le x\le 1$	$a=-m, b_0=b_1=1, b_2=0$
X	e^{-x}	$0\le x<\infty$	$a=b_0=1, b_1=b_2=0$
XI	x^{-m}	$1\le x<\infty$	$a=m, b_0=b_2=0, b_1=1$
XII	$(\frac{g+x}{g-x})^h$	$-g\le x\le g$	$a=-2gh, b_0=g^2, b_1=0, b_2=-1$

I. If $b_2>0$, $\Delta>0$ then the answer is $x<x_1=a-\frac{\sqrt{\Delta}}{b_2}$, $x>x_2=a+\frac{\sqrt{\Delta}}{b_2}$ $(x_1<x_2)$.
II. For $b_2>0$, $\Delta<0$ it is obvious.
III. If $b_2<0$, $\Delta>0$, then the answer is $x_1=a-\frac{\sqrt{\Delta}}{b_2}<x<x_2=a+\frac{\sqrt{\Delta}}{b_2}$.
If we simplify the statements above, it will imply the theorem.
On noting that log-concavity of f implies that $h(x)=\frac{f^{'}(x)}{f(x)}$ should be monotone decreasing on it's interval, so $h^{'}(x)<0$.

□

Remark 7. *For Pearson family with the form (3.1) when $b_2=0$, then $b_1a<b_0$ implies log-concavity of f. Also, when $b_1=b_2=0$, then $b_0<0$ implies the log-concavity of f.*

According to Theorem 6 we have discussed log-concavity of the Pearson family in Table 3.
On noting that the normal type of the Pearson family is always log-concave.

Table 3: Log-concavity for Pearson family

Type	Log-concave	Type	Log-concave
I	$\begin{cases} 1.\ m_1 m_2 > 0 \\ 2.\ m_1 m_2 < 0\ ,\ x > \frac{\sqrt{\lvert m_1\rvert}-\sqrt{\lvert m_2\rvert}}{\sqrt{\lvert m_1\rvert}+\sqrt{\lvert m_2\rvert}}. \end{cases}$	VII	$\begin{cases} 1.\ m < 0, -1 < x < 1, \\ 2.\ m > 0, x \in (-\infty, -1) \cup (1, \infty) \end{cases}$
II	$m > 0$	VIII	$m < 0$
III	$m > 0$	IX	$m > 0$
IV	$\begin{cases} 1.\ m > 0\ ,\ x > x_2\ ,\ x < x_1 \\ 2.\ m < 0\ ,\ x_1 < x < x_2 \\ where\quad x_1, x_2 = \frac{-\nu \pm \sqrt{\nu^2+4m^2}}{2m} \end{cases}$	X	Never
V	$x < \frac{2}{m}$	XI	$m < 0$
VI	$\begin{cases} 1.\ m_1 > 0\ ,\ m_2 > 0\ ,\ x < \frac{\sqrt{m_2}}{\sqrt{m_1}-\sqrt{m_2}} \\ 2.\ m_1 < 0\ ,\ m_2 < 0\ ,\ x > \frac{1}{\sqrt{\frac{m_1}{m_2}}-1} \end{cases}$	XII	$hgx < 0$

3.2 Burr family

Burr (1942 [9]) chose to work with cdf $F(x)$ satisfying

$$\frac{dF(x)}{dx} = F(x)(1 - F(x))g(x, F(x)) \tag{3.5}$$

that is the analogue of Pearson system. $g(x, F(x))$ must be positive for $0 < F(x) < 1$ and x in support of x. When $g(x, F(x)) = g(x)$, then $F(x) = \frac{exp\{\int_0^x g(t)dt\}}{1+exp\{\int_0^x g(t)dt\}}$ that implies 12 distributions as Burr family with various values of g. Table 4 shows cdf of the random variable X via various values of g:

Theorem 8. *For Burr family with the form (3.5), for the values that* $(\frac{g^{'}(x)}{g(x)})' < r - \widetilde{r}^{'}$, *then the Burr family is log-concave, where* $r(x) = \frac{f(x)}{\overline{F}(x)}$ *and* $\widetilde{r}(x) = \frac{f(x)}{F(x)}$ *are hazard rate and reversed hazard rate respectively.*

Proof. We know that $f(x) = F(x)\overline{F}(x)g(x)$,
then,

$$\begin{aligned} \frac{f^{'}(x)}{f(x)} &= \frac{f(x)}{F(x)} - \frac{f(x)}{\overline{F}(x)} + \frac{g^{'}(x)}{g(x)} \\ &= \widetilde{r}(x) - r(x) + \frac{g^{'}(x)}{g(x)}, \end{aligned} \tag{3.6}$$

So, $\frac{d}{dx}(\frac{f^{'}(x)}{f(x)}) = (\widetilde{r}(x))' - r^{'}(x) + \frac{d}{dx}(\frac{g^{'}(x)}{g(x)}) < 0$, and this implies the theorem. □

Remark 9. *We can simplify Theorem 8 via special cases of Burr family that is mentioned in Table 4.*

Table 4: The Burr Distributions

Type	$F(x)$	$f(x)$
I	x , $0 < x < 1$	1
II	$(1+e^{-x})^{-k}$, $x \in \Re$	$ke^{-x}(1+e^{-x})^{-k-1}$
III	$(1+x^{-c})^{-k}$, $x > 0$	$(1+x^{-c})^{-k-1}kcx^{-c-1}$
IV	$(1+(\frac{c-x}{x})^{\frac{1}{c}})^{-k}$, $0 < x < c$	$-\frac{k(1+(\frac{c-x}{x})^{\frac{1}{c}})^{-k-1}(\frac{c-x}{x})^{\frac{1}{c}}}{x(x-c)}$
V	$(1+ce^{-tanx})^{-k}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$	$(1+ce^{-tanx})^{-k-1}kc(1+tan^2x)e^{-tanx}$
VI	$(1+ce^{-rsinhx})^{-k}$, $x \in \Re$	$-kce^{-rcoshx}(1+ce^{-rsinhx})^{-k-1}$
VII	$2^{-k}(1+tanhx)^k$, $x \in \Re$	$\frac{k(1-tanhx)(1+tanhx)^k}{2^k}$
VIII	$(\frac{2tan^{-1}e^x}{\pi})^k$, $x \in \Re$	$\frac{ke(\frac{2tan^{-1}e^x}{\pi})^k}{tan^{-1}e^x(1+e^{2x^2})}$
IX	$1-\frac{2}{c((1+e^x)^k-1)+2}$, $x \in \Re$	$\frac{2^k ce(1+e^x)^k}{(c((1+e^x)^k-1)+2)^2(1+e^e)}$
X	$(1+e^{-x^2})^k$, $x > 0$	$k(1+e^{-x^2})^{k-1}e^{-2x}$
XI	$(x-\frac{sin2\pi x}{2\pi})^k$, $0 < x < 1$	$\frac{(x-\frac{sin2\pi x}{2\pi})^k k(1-cos2\pi x)}{x-\frac{sin2\pi x}{2\pi}}$
XII	$1-(1+x^c)^{-k}$, $x > 0$	$(1+x^c)^{-k-1}kcx^{c-1}$

4 Log-concavity for some general version of distributions

In this section, we have discussed log concavity property for some general distributions including: generalized gamma distributions, Feller-Pareto distributions, generalized inverse Gaussian distributions and generalized log-normal distributions.

4.1 Generalized Gamma Distributions

The generalized gamma (GG) distribution offers a flexible family in the varieties of shapes and hazard functions for modeling duration. It was introduced by Stacy (1962 [29]). Difficulties with convergence of algorithms for maximum likelihood estimation (Hager and Bain, 1970, [13]) inhabited applications of the GG model. Prentice (1974 [27]) resolved the covergence problem using a nonlinear transformation of GG model.

Definition 10. *The probability density function of GG distribution, $GG(\alpha, \tau, \lambda)$, is*

$$f_{GG}(y|\alpha,\tau,\lambda) = \frac{\tau}{\lambda^{\alpha\tau}\Gamma(\alpha)} y^{\alpha\tau-1} e^{-(y/\lambda)^{\tau}} \ , \ y \geq 0 \ , \ \alpha > 0 \ , \ \tau > 0 \ , \ \lambda > 0, \qquad (4.1)$$

where $\Gamma(.)$ is the gamma function, α and τ are shape parameters, and λ is the scale parameter.

The GG family is flexible in that it includes several well-known models as sub-families (see, Johnson et al., 1994 [15]). The sub-families of GG considered here are exponential $GG(1,1,\lambda)$, gamma for $GG(\alpha,1,\lambda)$, and Weibull for $GG(1,\tau,\lambda)$. The log-normal distribution is also obtained as a limiting distribution when $\alpha \to \infty$. By letting $GG(\alpha,2,\lambda)$ we obtain a sub-family of GG which is known as the generalized normal distribution, $GN(2\alpha,\lambda)$. The GN itself is a flexible family and includes Half-normal $GG(1/2,\tau,\lambda)$, Rayleigh $GG(1,\tau,\lambda)$, Maxwell-Boltzmann $GG(3/2,\tau,\lambda)$, and Chi $(GG(k/2,\tau,\lambda), k=1,2,...)$ distributions.

Theorem 11. *The generalized Gamma distribution with the form (4.1) is log-concave for*
$y > \lambda \sqrt[\tau]{\frac{1-\alpha\tau}{\tau(\tau-1)}}$.

Proof. f is log-concave if $(\frac{f'(y)}{f(y)})' < 0$ on it's interval, so :

$$\frac{d}{dy}\left(\frac{f'(y)}{f(y)}\right) = -\frac{\alpha\tau - 1 - \tau(\frac{y}{\lambda})^{\tau} + \tau^2(\frac{y}{\lambda})^{\tau}}{y^2} < 0$$

That implies $y > \lambda \sqrt[\tau]{\frac{1-\alpha\tau}{\tau(\tau-1)}}$. □

Remark 12. *For $\tau = 1$, when $\alpha \geq 1$, then the special version of the generalized gamma (gamma distribution) is log-concave.*

So, log-concavity of the GG distribution implies the log-concavity of it's sub-families such as the generalized normal distribution, gamma, exponential, Weibull and log-normal distributions on their interval. On noting that the generalized normal is a flexible distribution and has it's own sub families such as half-normal, Rayleigh, Maxwell-Boltzmann and Chi distributions.

4.2 Feller-Pareto distributions

The Feller-Pareto distributions are denoted by GB2(a,b,p,q) and has the *pdf*

$$f(x) = \frac{ax^{ap-1}}{b^{ap}B(p,q)[1+(x/b)^a]^{p+q}}, x > 0. \tag{4.2}$$

Here all four parameters a,b,p,q are positive, b is a scale and a,p,q are shape parameters. If the distribution of $Y = logX$, with density

$$f(y) = \frac{ae^{ap(y-logb)}}{B(p,q)[1+e^{a(y-logb)}]^{p+q}}, -\infty < y < \infty, \tag{4.3}$$

is considered, a turns out to be a scale parameter, whereas p and q are still shape parameters.

Theorem 13. *The Feller-Pareto distribution that is defined in (4.2), is log-convex if $ap < 1$ and $p + q > 4$.*

Proof. We should prove that

$$\frac{d}{dx}\left(\frac{f'(x)}{f(x)}\right) = \frac{-1 + ap - 2(\frac{x}{b})^a - (\frac{x}{b})^2 a - qa(\frac{x}{b})^a - qa(\frac{x}{b})^2 a + a^2(\frac{x}{b})^a p + a^2 q(\frac{x}{b})^a + ap(\frac{x}{b})^a}{-x^2(1 + (\frac{x}{b})^a)^2} > 0 \tag{4.4}$$

by choosing $(\frac{x}{b})^a = y$, we have

$$(-aq - 1)y^2 + (ap + a^2 q + a^2 p - aq - 2)y + ap - 1 < 0. \tag{4.5}$$

It implies that,

$$(a(p - q) + a^2(p + q) - 2)^2 < 4(ap - 1)(-1 - aq), \tag{4.6}$$

it is necessary that $ap < 1$.
Also, (4.6) leads to $(p + q)((p + q)a^2 + 2(p - q)a + (p + q - 4)) < 0$
that implies

$$\frac{q - p - \sqrt{\Delta}}{p + q} < a < \frac{q - p + \sqrt{\Delta}}{p + q} \quad where \quad \Delta = 4p + 4q - 4pq. \tag{4.7}$$

(4.7) is equivalent to $(a(p + q) + (p - q))^2 < \Delta$, so, a should be positive, which leads to $p + q > 4$.
The special cases of Feller-Pareto size distributions should be log-convex based on the following conditions :

$GB2(a, b, p, 1) \Rightarrow$ Dagum distribution is log-convex if $ap < 1$ and $p > 3$.

$GB2(1, b, p, q) \Rightarrow$ Beta distribution of second kind is log-convex when $p < 1$ and $p + q < 4$.

$GB2(a, b, 1, q) \Rightarrow$ Singh-Maddala distribution is log-convex when $a < 1$ and $q < 3$.

$GB2(1, b, 1, q) \Rightarrow$ Lomax distribution is always log-convex.

$GB2(a, b, 1, 1) \Rightarrow$ Fisk(log-logistic) distribution is log-convex for $a < 1$ and $p > 3$.

$GB2(1, b, p, 1) \Rightarrow$ Inverse Lomax distribution is log-convex for $0 < p < 1$

□

Remark 14. *A distribution introduced by McDonald an Xu (1995 [21]) as the "generalized beta" (GB) distribution. The GB is defined by the pdf*

$$GB(x; a, b, c, p, q) = \frac{|a| x^{ap-1}(1 - (1 - c)(x/b)^a)^{q-1}}{b^{ap} B(p, q)(1 + c(x/b)^a)^{p+q}}, \quad 0 < x^a < b^a/(1 - c), \tag{4.8}$$

and zero otherwise with $0 \le c \le 1$ and b, p and q positive. As in the ordinary beta distribution, the parameters p and q control shape and skewness. Parameters a and b control "peakedness" and scale, respectively. $c = 1$ and $c = 0$ implies the Feller-Pareto and GB1 distributions.
In general case, finding log-concavity or log-convexity of GB is complicated.

4.3 Generalized inverse Gaussian distributions

The generalized inverse Gaussian distribution denoted by GIG(μ, c^2, λ), with parameters (μ, c^2, λ) has the *pdf* given by

$$q(x) = \frac{1}{2K_\lambda(1/c^2)\mu}(\frac{x}{\mu})^{\lambda-1}exp\{-\frac{1}{2c^2}(\frac{x}{\mu}+\frac{\mu}{x})\},$$
$$0 < x < \infty, -\infty < \lambda < \infty, 0 < \mu < \infty, 0 < c < \infty, \quad (4.9)$$

where $K_\lambda(.)$ denotes the modified Bessel function of the third kind and with index λ (Kawamura et al. 2003 [16]). In particular, GIG$(\mu, c^2, -1/2)$ and GIG$(\mu, c^2, 1/2)$ which are the inverse Gaussian and reciprocal inverse Gaussian distributions respectively. Also, for GIG$(\mu, c^2, 0)$ the Halphen distribution which is a prototype of generalized inverse Gaussian distribution can be obtained.

Theorem 15. *The Generalized Inverse Gaussian with the form (4.9) is log-concave for the two conditions below:*
(i) $\lambda < 1$, $x > \frac{\mu}{c^2(1-\lambda)}$
(ii) $\lambda > 1$, $x < \frac{\mu}{c^2(1-\lambda)}$

Proof. For the $GIG(\mu, c^2, \lambda)$, we have $\frac{d}{dx}(\frac{f'(x)}{f(x)}) = -\frac{\lambda xc^2 - xc^2 + \mu}{x^3c^2} < 0$ and after simplifying it, we have the theorem. □

Remark 16. *when $\lambda = 1$ the GIG with the form (4.9) is always log-concave.*

The power inverse Gaussian distribution parameterized by an arbitrarily fixed real number $\lambda \neq 0$ denoted by $PIG_\lambda(\mu, c^2)$ has the *pdf* given by

$$q(x) = \frac{1}{\sqrt{(2\pi)}c\mu}(\frac{x}{\mu})^{-(1+\lambda/2)}exp\{-\frac{1}{2(\lambda c)^2}((\frac{x}{\mu})^{\lambda/2} - (\frac{\mu}{x})^{-(\lambda/2)})^2\},$$
$$0 < x < \infty \ , \ 0 < \mu < \infty \ , \ 0 < c < \infty, \quad (4.10)$$

In particular, $PIG_1(\mu, c^2)$ and $PIG_{-1}(\mu, c^2)$, are the inverse Gaussian and the reciprocal inverse Gaussian distributions respectively. Also when $\lambda \to 0$, the power inverse Gaussian reduces to a log-normal distribution.

Theorem 17. *The Power Inverse Gaussian with the form (4.10) is log-concave for $\lambda > 1$.*

Proof. We have,

$$\frac{d}{dx}(\frac{f'(x)}{f(x)}) = \frac{2\lambda c^2 + \lambda^2 c^2 + (x/\mu)^\lambda - (x/\mu)^{-\lambda} - \lambda(x/\mu)^{-\lambda} - \lambda(x/\mu)^\lambda}{2\lambda c^2 x^2} < 0 \quad (4.11)$$

where on choosing $(x/\mu)^\lambda = A$, we should prove that □

Proof.

$$(1-\lambda)A^2 + (2\lambda c^2 + \lambda^2 c^2)A - (1+\lambda) < 0, \tag{4.12}$$

□

Proof. This inequality is true when $\lambda > 1$. Note that, when $\lambda < 1$, equation (4.12) is not possible.
For an arbitrarily fixed real numbers $\lambda \neq 0$, let a positive random variable X satisfy the relation

$$(1+\lambda\frac{X-\mu}{\sigma})^{1/\lambda} \sim e(1), \tag{4.13}$$

where μ and c are real number with $-\infty < x < \infty$, $0 < \sigma < \infty$ and $e(1)$ denotes the exponential distribution with the mean 1. Also the range of X is assumed to satisfy $1 + \lambda(X-\mu)/\sigma > 0$. We call this distribution of X the generalized Gumbel distribution $GG_\lambda(\mu, \sigma^2)$. The transformation $y = (1+\lambda(x-\mu)/\sigma)^{1/\lambda}$ is one-to-one, and therefore, the *pdf* of generalized Gumbel distribution is presented by □

Proof.

$$q(x) = \frac{1}{\sigma}(1+\lambda\frac{x-\mu}{\sigma})^{(1/\lambda)-1} exp\{-(1+\lambda\frac{x-\mu}{\sigma})^{1/\lambda}\} \tag{4.14}$$

$1 + \lambda\frac{x-\mu}{\sigma} > 0$, $-\infty < \mu < \infty$, $0 < \sigma < \infty$. □

Theorem 18. *The generalized Gumbel distribution is log-concave based on the conditions below:*
(i). When $\lambda \leq 1$ *then, for* $x \leq \frac{\sigma\lambda^\lambda + \lambda\mu - \sigma}{\lambda}$ *generalized Gumbel distribution is log-concave.*
(ii). When $\lambda > 1$ *then, for* $x > \frac{\sigma\lambda^\lambda + \lambda\mu - \sigma}{\lambda}$ *generalized Gumbel distribution is log-concave.*

Proof. For being log-concave $\frac{d}{dx}(\frac{f'(x)}{f(x)}) < 0$. So :

$$\frac{d}{dx}(\frac{f'(x)}{f(x)}) = \frac{(\frac{\sigma+\lambda x-\lambda\mu}{\sigma})^{\frac{1}{\lambda}}(\lambda-1) - \lambda + \lambda^2}{(\sigma+\lambda x-\lambda\mu)^2} < 0, \tag{4.15}$$

$$((\frac{\sigma+\lambda x-\lambda\mu}{\sigma})^{\frac{1}{\lambda}} + \lambda)(\lambda-1) < 0, \tag{4.16}$$

and if we simplify (4.16), we have the theorem. □

Remark 19. *For* $\sigma = \lambda = \mu = 1$, *the generalized Gumbel distribution is log-convex.*

4.4 Generalized log-normal distributions

Vianelli [31, 32, 33] proposed a three-parameter generalized log-normal distribution. It is obtained as the distribution of $X = e^Y$; where Y follows a generalized error distribution, with density

$$f(y) = \frac{1}{2r^{1/r}\sigma_r\Gamma(1+1/r)} exp\{-\frac{1}{r\sigma_r^r}|y-\mu|^r\} \quad , \quad -\infty < y < \infty, \tag{4.17}$$

where $-\infty < \mu < \infty$ is the location parameter, $\sigma_r = [E|Y-\mu|^r]^{1/r}$ is the scale parameter, and $r > 0$ is the shape parameter. For $r = 2$ we arrive at the normal distribution and $r = 1$ yields to the Laplace distribution. The generalized error distribution is thus known as both a generalized normal distribution, in particular in the Italian literature (Vianelli, 1963), as a generalized Laplace distribution. If we start from (4.17), the density of $X = e^Y$ is

$$f(x) = \frac{1}{2xr^{1/r}\sigma_r\Gamma(1+1/r)} exp\{-\frac{1}{r\sigma_r^r}|logx-\mu|^r\} \quad , \quad 0 < x < \infty, \tag{4.18}$$

Here e^μ is a scale parameter and σ_r,r are shape parameters.

Theorem 20. *The generalized Error distribution with the form (4.17) is log-concave for any of these conditions:*
(i) $y \geq \mu$, $r > 1$, $\sigma_r > 0$,
(ii) $y \leq \mu$, $r = 2k(k \in Z)$, $r > 1$,
(iii) $y < \mu$, $r \neq 2k(k \in Z)$, $r < 1$, $\sigma_r > 0$,
(iv) $y \geq \mu$, $r < 1$, $\sigma_r < 0$,
(v) $y < \mu$, $r > 1$, $\sigma_r > 0$.

Proof. For $y \geq \mu$ and $y \leq \mu(r = 2k, k \in Z)$ we have, $\frac{d}{dx}(\frac{f'(x)}{f(x)}) = -\sigma_r^{-r}(y-\mu)^{r-2}(r-1) < 0$,
and for $y \leq \mu(r \neq 2k, k \in Z)$ $\frac{d}{dx}(\frac{f'(x)}{f(x)}) = \sigma_r^{-r}(y-\mu)^{r-2}(r-1) < 0$.
So, if we simplify them, we have the theorem. □

Remark 21. *For* $r = 1$ *or* $\sigma_r = 0$ *the generalized error distribution is log-convex.*

Remark 22. *For generalized log-normal distribution, we can find the log-convexity properties via Theorem 5 on using log-convexity properties of the generalized error distribution.*

Conclusion 23. *In this paper, log-concavity and log-convexity properties for classes of distributions, such as, Pearson, Burr, generalized gamma, Feller-Pareto distributions, generalized inverse Gaussian, power inverse Gaussian, generalized Gumbel, generalized error and generalized log-normal and special cases of them are obtained.*

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