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SOME CRITERIA FOR TWO NEW INTEGRAL OPERATORS

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Abstract. Using the Hadamard product, we define two new integral operators. The main object of the present paper is to discuss some univalence conditions for these operators. Several corollaries of the main results are also considered.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0,$$

$\mathbb{C}$ being the set of complex numbers.
We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathbb{U}$.
For two functions, $f(z) \in \mathcal{A}$ and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.1}$$

their Hadamard product (or convolution) is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.2}$$

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For a function $g \in \mathcal{A}$ defined by (1.1), where $b_n \geq 0$, $(n \geq 2)$, we define the family $\mathcal{S}(g,p)$ so that it consists of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| \frac{z^2 (f*g)'(z)}{[(f*g)(z)]^2} - 1 \right| \leq p\,|z|^2 \qquad (z \in \mathbb{U}; 0 < p \leq 2)\,, \tag{1.3}$$

provided that $(f*g)(z) \neq 0$.
Note that $\mathcal{S}(\frac{z}{1-z}, p) = \mathcal{S}(p)$, where the class $\mathcal{S}(p)$, $0 < p \leq 2$ of analytic and univalent functions was defined by D. Yang and J. Liu [4].
Also, if $f \in \mathcal{S}(p)$ then the following property is true

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| \leq p\,|z|^2 \qquad (z \in \mathbb{U}) \tag{1.4}$$

relation proved in [3].
Using the Hadamard product defined by (1.2), we define two families of integral operators:

$$F_{n,\alpha}\,(f,g)(z) = \left((1 + n(\alpha - 1)) \int_0^z \prod_{i=1}^{n} ((f_i * e^{g_i})(t))^{\alpha-1}\, dt \right)^{\frac{1}{1+n(\alpha-1)}} \tag{1.5}$$

$$\alpha_i \in \mathbb{C};\, f_i, g_i \in \mathcal{A} \text{ for all } i \in \{1,2,...,n\}.$$

$$G_{n,\beta}\,(f,g)(z) = \left(\beta \int_0^z t^{\beta - n\alpha - 1} \prod_{i=1}^{n} ((f_i * g_i)(t))^{\alpha}\, dt \right)^{\frac{1}{\beta}} \tag{1.6}$$

$$\alpha_i \in \mathbb{C};\, f_i, g_i \in \mathcal{A};\ M \geq 1 \text{ for all } i \in \{1,2,...,n\}.$$

In the present paper, we study the univalence conditions involving the families of integral operators defined by (1.5) and (1.6).
In the proof of our main results (Theorem 3 and Theorem 6) we need the following univalence criterion. The univalence criterion, asserted by Theorem 1 below, was proven by Pescar [2].

Theorem 1. [2] *Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}\alpha > 0$, $c \in \mathbb{C}$ with $|c| \leq 1$, $c \neq -1$. If $f \in \mathcal{A}$ satisfies*

$$\left| c\,|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{z f''(z)}{\alpha f'(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$ then the integral operator

$$F_\alpha(z) = \left(\alpha \int_0^z t^{\alpha-1} f'(t) dt \right)^{\frac{1}{\alpha}}$$

is in the class $\mathcal{S}$.

Finally, in our present investigation, we shall also need the familiar Schwarz Lemma (see, for details, [1]).

Lemma 2. (**General Schwarz Lemma**) [1] *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \qquad (z \in \mathbb{U}_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

2 Main Results

Theorem 3. *Let $f_i \in \mathcal{A}$ for all $i \in \{1, 2, ..., n\}$, $\alpha \in \mathbb{C}$ and $M_i \geq 1$ with*

$$|c| \leq 1 - \left|\frac{\alpha - 1}{\alpha}\right| \sum_{i=1}^{n} [(p_i + 1)M_i + 1], \qquad c \in \mathbb{C}, c \neq -1. \tag{2.1}$$

If for all $i \in \{1, 2, ..., n\}$, $f_i \in \mathcal{S}(g_i, p_i)$, $0 < p_i \leq 2$ and

$$|(f_i * e^{g_i})(z)| \leq M_i, \qquad (z \in \mathbb{U}) \tag{2.2}$$

then the integral operator $F_{n,\alpha}(f, g)(z)$ defined by (1.5) is in the class $\mathcal{S}$.

Proof. We begin by observing that the integral operator $F_{n,\alpha}(f, g)(z)$ in (1.5) can be rewritten as follows:

$$F_{n,\alpha}(f, g)(z) = \left((1 + n(\alpha - 1)) \int_0^z t^{n(\alpha-1)} \prod_{i=1}^{n} \left(\frac{(f_i * e^{g_i})(t)}{t} \right)^{\alpha-1} dt \right)^{\frac{1}{1+n(\alpha-1)}}.$$

Let us define the function $h(z)$ by

$$h(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{(f_i * e^{g_i})(t)}{t} \right)^{\alpha-1} dt$$

$$f_i, g_i \in \mathcal{A} \text{ for all } i \in \{1, 2, ..., n\}.$$

The function $h(z)$ is indeed regular in $\mathbb{U}$ and satisfies the following usual normalization condition:

$$h(0) = h'(0) - 1 = 0.$$

Now, calculating the derivatives of $h(z)$ of the first and second orders, we readily obtain

$$h'(z) = \prod_{i=1}^{n} \left(\frac{(f_i * e^{g_i})(z)}{z} \right)^{\alpha - 1}$$

and

$$\frac{zh''(z)}{h'(z)} = (\alpha - 1) \sum_{i=1}^{n} \left(\frac{z(f_i * e^{g_i})'(z)}{(f_i * e^{g_i})(z)} - 1 \right), \quad (z \in \mathbb{U})$$

which readily shows that

$$\left| c\,|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zh''(z)}{\alpha h'(z)} \right|$$

$$= \left| c\,|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{\alpha - 1}{\alpha} \sum_{i=1}^{n} \left(\frac{z(f_i * e^{g_i})'(z)}{(f_i * e^{g_i})(z)} - 1 \right) \right|$$

$$\leq |c| + \left| \frac{\alpha - 1}{\alpha} \right| \sum_{i=1}^{n} \left(\left| \frac{z(f_i * e^{g_i})'(z)}{(f_i * e^{g_i})(z)} \right| + 1 \right)$$

$$\leq |c| + \left| \frac{\alpha - 1}{\alpha} \right| \sum_{i=1}^{n} \left(\left| \frac{z^2(f_i * e^{g_i})'(z)}{[(f_i * e^{g_i})(z)]^2} \right| \left| \frac{(f_i * e^{g_i})(z)}{z} \right| + 1 \right). \tag{2.3}$$

Furthermore, from the hypothesis (2.2) of Theorem 3, we have

$$|(f_i * e^{g_i})(z)| \leq M_i \qquad (z \in \mathbb{U})$$

then by General Schwarz Lemma, we thus obtain

$$|(f_i * e^{g_i})(z)| \leq M_i\,|z| \qquad (z \in \mathbb{U})$$

for all $i \in \{1, 2, ..., n\}$.
Next, by making use of (2.3), we have

$$\left| c\,|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zh''(z)}{\alpha h'(z)} \right|$$

$$\leq |c| + \left| \frac{\alpha - 1}{\alpha} \right| \sum_{i=1}^{n} \left(\left(\left| \frac{z^2(f_i * e^{g_i})'(z)}{[(f_i * e^{g_i})(z)]^2} - 1 \right| + 1 \right) M_i + 1 \right)$$

$$\leq |c| + \left| \frac{\alpha - 1}{\alpha} \right| \sum_{i=1}^{n} [(p_i + 1)\, M_i + 1]$$

$$\leq 1 \qquad (z \in \mathbb{U}, c \in \mathbb{C}, M_i \geq 1)$$

where we have also used the hypothesis (2.1) of Theorem 3.
Finally, by applying Theorem 1, we conclude that the integral operator $F_{n,\alpha}(f,g)(z)$ defined by (1.5) is in the class $\mathcal{S}$. □

Setting $g_1 = g_2 = ... = g_n = 0$ in Theorem 3 we have

Corollary 4. *Let $f_i \in \mathcal{A}$ for all $i \in \{1,2,...,n\}$, $\alpha \in \mathbb{C}$ and $M_i \geq 1$ with*

$$|c| \leq 1 - \left|\frac{\alpha - 1}{\alpha}\right| \sum_{i=1}^{n} [(p_i + 1)M_i + 1], \qquad c \in \mathbb{C}, c \neq -1.$$

If for all $i \in \{1,2,...,n\}$, $f_i \in \mathcal{S}(p_i)$, $0 < p_i \leq 2$ satisfy the condition (1.4) and

$$|f_i(z)| \leq M_i \qquad (z \in \mathbb{U})$$

then the integral operator

$$F_{n,\alpha}(f)(z) = \left((1 + n(\alpha - 1)) \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha - 1} dt\right)^{\frac{1}{1+n(\alpha-1)}}$$

is in the class $\mathcal{S}$.

Setting $n = 1$ and $M = 1$ in Corollary 4 we have

Corollary 5. *Let $f \in \mathcal{A}$, $\alpha \in \mathbb{C}$ and*

$$|c| \leq 1 - \left|\frac{\alpha - 1}{\alpha}\right| (p + 2), \qquad c \in \mathbb{C}, c \neq -1.$$

If $f \in \mathcal{S}(p)$, $0 < p \leq 2$ satisfies the condition (1.4) and

$$|f(z)| \leq 1 \qquad (z \in \mathbb{U})$$

then the integral operator

$$F_{\alpha}(f)(z) = \left(\alpha \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha - 1} dt\right)^{\frac{1}{\alpha}}$$

is in the class $\mathcal{S}$.

Theorem 6. *Let $f_i \in \mathcal{A}$, $\alpha \in \mathbb{C}$ and $M_i \geq 1$ for all $i \in \{1,2,...,n\}$. If for all $i \in \{1,2,...,n\}$, $f_i \in \mathcal{S}(g_i, p_i)$, $0 < p_i \leq 2$ and*

$$|(f_i * g_i)(z)| \leq M_i \qquad (z \in \mathbb{U}) \tag{2.4}$$

then for any complex number β,

$$\mathrm{Re}\beta \geq |\alpha| \sum_{i=1}^{n} [(p_i + 1)M_i + 1] \tag{2.5}$$

the integral operator $G_{n,\beta}(f,g)(z)$ defined by (1.6) is in the class $\mathcal{S}$.

Proof. We define a function

$$h(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{(f_i * g_i)(t)}{t} \right)^{\alpha} dt \tag{2.6}$$

and we obtain that $h(0) = h'(0) - 1 = 0$. We calculate the derivatives of the first and second orders. From (2.6) we have

$$h'(z) = \prod_{i=1}^{n} \left(\frac{(f_i * g_i)(z)}{z} \right)^{\alpha}$$

and

$$\frac{zh''(z)}{h'(z)} = \alpha \sum_{i=1}^{n} \left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right)$$

which readily shows that

$$\begin{aligned} &\left| c\,|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| \\ &= \left| c\,|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{\alpha}{\beta} \sum_{i=1}^{n} \left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right) \right| \\ &\leq |c| + \frac{|\alpha|}{|\beta|} \sum_{i=1}^{n} \left(\left| \frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} \right| + 1 \right) \\ &\leq |c| + \frac{|\alpha|}{|\beta|} \sum_{i=1}^{n} \left(\left| \frac{z^2(f_i * g_i)'(z)}{[(f_i * g_i)(z)]^2} \right| \left| \frac{(f_i * g_i)(z)}{z} \right| + 1 \right). \end{aligned}$$

Since $|(f_i * g_i)(z)| \leq M_i$, $z \in \mathbb{U}$ and $f_i \in \mathcal{S}(g_i, p_i)$, $0 < p_i \leq 2$ for all $i \in \{1, 2, ..., n\}$, then from General Schwarz Lemma and (1.3), we obtain

$$\begin{aligned} &\left| c\,|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| \\ &\leq |c| + \frac{|\alpha|}{|\beta|} \sum_{i=1}^{n} \left(\left(\left| \frac{z^2(f_i * g_i)'(z)}{[(f_i * g_i)(z)]^2} - 1 \right| + 1 \right) M_i + 1 \right) \\ &\leq |c| + \frac{|\alpha|}{|\beta|} \sum_{i=1}^{n} [(p_i + 1)\, M_i + 1] \qquad (z \in \mathbb{U}) \end{aligned}$$

which, in the light of the hypothesis (2.5), we have

$$\left| c\,|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| \leq 1 \qquad (z \in \mathbb{U}).$$

Applying Theorem 1 for the function $h(z)$, we obtain that the integral operator $G_{n,\beta}(f, g)(z)$ defined by (1.6) is in the class $\mathcal{S}$. □

Setting $g_1 = g_2 = ... = g_n = 0$ in Theorem 6, we have

Corollary 7. *Let $f_i \in \mathcal{A}$, $\alpha \in \mathbb{C}$ and $M_i \geq 1$ for all $i \in \{1, 2, ..., n\}$. If for all $i \in \{1, 2, ..., n\}$, $f_i \in \mathcal{S}(p_i)$, $0 < p_i \leq 2$ satisfy the inequality (1.4) and*

$$|f_i(z)| \leq M_i \qquad (z \in \mathbb{U})$$

then for any complex number β,

$$\mathrm{Re}\beta \geq |\alpha| \sum_{i=1}^{n} [(p_i + 1)M_i + 1]$$

the integral operator

$$G_{n,\beta}(f, g)(z) = \left(\beta \int_0^z t^{\beta - n\alpha - 1} \prod_{i=1}^{n} (f_i(t))^{\alpha} \, dt \right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Setting $n = 1$ and $M = 1$ in Corollary 7, we have

Corollary 8. *Let $f \in \mathcal{A}$, $\alpha \in \mathbb{C}$ and $M \geq 1$. If $f \in \mathcal{S}(p)$, $0 < p \leq 2$ satisfies the inequality (1.4) and*

$$|f(z)| \leq M \qquad (z \in \mathbb{U})$$

then for any complex number β,

$$\mathrm{Re}\beta \geq |\alpha| \left[(p + 1)M + 1 \right]$$

the integral operator

$$G_{1,\beta}(f, g)(z) = \left(\beta \int_0^z t^{\beta - \alpha - 1} (f(t))^{\alpha} \, dt \right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

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