ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 8 (2013), 35 – 49

APPROXIMATE ANALYTICAL SOLUTION OF DIFFUSION EQUATION WITH FRACTIONAL TIME DERIVATIVE USING OPTIMAL HOMOTOPY ANALYSIS METHOD

S. Das, K. Vishal and P. K. Gupta

Abstract. In this article, optimal homotopy-analysis method is used to obtain approximate analytic solution of the time-fractional diffusion equation with a given initial condition. The fractional derivatives are considered in the Caputo sense. Unlike usual Homotopy analysis method, this method contains at the most three convergence control parameters which describe the faster convergence of the solution. Effects of parameters on the convergence of the approximate series solution by minimizing the averaged residual error with the proper choices of parameters are calculated numerically and presented through graphs and tables for different particular cases.

1 Introduction

Fractional diffusion equations model phenomena exhibiting anomalous diffusion have played an increasing role in the disciplines of Applied mathematics, Physics and Engineering for last few decades. Fractional diffusion equation is obtained from the classical diffusion equation in mathematical physics by replacing the first order time derivative by a fractional derivative of order $\alpha (0 < \alpha < 1)$, which is nowadays a field of growing interest. An important characteristic of these evolution equations is that they generate the fractional Brownian motion (FBM) which is a generalization of Brownian motion (BM). Fractional-order partial differential equations are used by researchers to model anomalous diffusion and Hamiltonian Chaos. These governing equations describe the asymptotic behavior of continuous time random walks. Stochastic solutions to the simplest governing equations are Levy motions, generalizing the Brownian motion solution to the classical diffusion equation. More generally, these equations invoke pseudo-differential operators that are non-local. Fractional kinetic equations have proved particularly useful in the context of anomalous subdiffusion (Metzler and Klafter [1]). The fractional diffusion equation, which demon-

2010 Mathematics Subject Classification: 26A33; 34A08; 60G22; 65Gxx.
Keywords: Diffusion equation; Caputo derivative; Error analysis; Homotopy analysis method.

strates the prevalence of anomalous sub-diffusion, has led to an intensive effort in recent years to find the solution accurately in straight forward manner (Langlands and Henry [2]). The fractional diffusion equation is valuable for describing reactions in the dispersive transport media ([3]-[4]). Anomalous diffusion processes occur in many physical systems for various reasons including disorder in terms of energy or space or both ([5]-[6]). Fractional reaction-diffusion equations or continuous time random walk models are also introduced for the description of nonlinear reactions, propagating fronts and two species reactions in sub-diffusive transport media (Henry and Wearne [7]). In this article we focus our attention to find the numerical solution of the fractional diffusion equation with external force given as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial(xu(x,t))}{\partial x}, 0 < \alpha \leq 1, (x,t) \in \Omega, \tag{1.1}$$

subject to the initial condition

$$u(x,0) = x^n. \tag{1.2}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo derivative of order α.

In 2000, Metzler and Klafter [8] in their research article suggested that the fractional Kinetic equations are useful in describing both sub- and super-diffusion processes. In 2007, Chen et al. [9] proposed an Implicit difference approximation scheme (ISAS) for solving fractional diffusion equation, where the stability and convergence of the method analyzed by Fourier method. Schot et al. [10] have given an approximate solution of the diffusion equation in terms of Fox H-function. Zahran [11] has offered a closed form solution in Fox H-function of the generalized fractional reaction-diffusion equation subjected to an external linear force field, one that is used to describe the transport processes in disorder systems. Recently Wang et al. [12] develop a fast finite difference method for fractional diffusion equation, where the authors claim that it requires storage and computational cost while retaining the same accuracy as the regular difference method. Sprouse et al. [13] have shown that the fractional diffusion equations are computationally intensive due to the effect of non-local derivatives in which every previous time derivative contribute to current iteration. It is to be noted that some work on fractional diffusion equations have already been done by Angulo et al. [14], Pezat and Zabczyk [15], Schneider and Wyss [16], Yu and Zhang [17], Mainardi [18], Mainardi et al. [19], Anh and Leonenko [20], Das and Gupta [21] etc. using various mathematical techniques. But to the best of authors' knowledge the solution of the above fractional diffusion equation using the upgraded version of the HAM whose reliability and effectiveness are much better than the useful mathematical tools, has not been explored by any researcher.
Homotopy Analysis Method (HAM) proposed by Liao [22] is based on homotopy,

a fundamental concept in topology and differential geometry. It is an analytical approach to get the series solution of linear and nonlinear partial differential equations (PDE's). The difference with the other perturbation methods is that this method is independent of small / large physical parameters. Another important advantage as compared to the other existing perturbation and non-perturbation method lies in the freedom to choose proper base function to get better approximate solution of the problems. It also provides a simple way to ensure the convergence of a series solution. Recently, Liao [23] has claimed that the difference with the other analytical methods is that one can ensure the convergence of series solution by means of choosing a proper value of convergence-control parameter. Recently, Das et al. [24] have successfully applied the method to investigate the influences of auxiliary parameter to find the region of convergence through h-curve analysis in solving the considered fractional diffusion equation. Still there are lot of restrictions of the method e.g., in usual HAM one cannot predict for which value of the convergence parameter c_0 gives better convergence even through the plotting of c_0-curve. To overcome this restriction the authors have used a new mathematical tool optimal homotopy analysis method, also proposed by Liao [25] to find the approximate analytical solution of our considered problem where the rate of convergence of the series solution is faster. The basic difference of the method from usual HAM is that here we have to consider at the most three parameters c_0, c_1, c_2 $(|c_1| \leq 1, |c_2| \leq 1)$, which are known as convergence control parameters whereas in usual HAM there was only one parameter c_0. The present approach contains special deformation functions which are determined completely by two parameters c_1 and c_2, thus depending on three parameters c_0, c_1 and c_2. The salient feature of the approach is the introduction of new type of residual error which helps to find out the optimal values of these parameters for getting better convergence of the solution.

2 Solution of the problem by optimal homotopy-analysis method

To solve equation (1.1) by optimal homotopy-analysis method, we choose the initial approximation

$$u_0(x,t) = x^n, \tag{2.1}$$

and the linear auxiliary operator,

$$L[\phi(x,t;q)] = \frac{\partial^\alpha \phi(x,t;q)}{\partial t^\alpha}, \tag{2.2}$$

with the property

$$L[c] = 0, \tag{2.3}$$

where c is integral constant, $\phi(x,t;q)$ is an unknown function. Furthermore, in the view of equation (1.1), we have defined the nonlinear operator as

$$N[\phi(x,t;q)] = \frac{\partial^\alpha \phi(x,t;q)}{\partial t^\alpha} - \frac{\partial^2 \phi(x,t;q)}{\partial x^2} - \frac{\partial (x\phi(x,t;q))}{\partial x}, \tag{2.4}$$

By means of the optimal homotopy analysis-method, Liao [25] constructs the so-called zeroth-order deformation equation as

$$(1 - B(q))L[\phi(x,t;q) - u_0(x,t)] = c_0 A(q) N[\phi(x,t;q)], \tag{2.5}$$

where $q \in [0,1]$ is the embedding parameter, c_0 is convergence control parameter, $A(q)$ and $B(q)$ are so called deformation function satisfying

$$A(0) = B(0) = 0 \quad \text{and} \quad A(1) = B(1) = 1. \tag{2.6}$$

The Taylor series of these functions are given by

$$A(q) = \sum_{m=1}^{\infty} \mu_m \, q^m, \tag{2.7}$$

$$B(q) = \sum_{m=1}^{\infty} \sigma_m \, q^m, \tag{2.8}$$

which exist and convergent for $|q| \leq 1$. As given by Liao [25] there exists a large number of deformations function satisfying these properties, but for the sake of computer efficiency, we use here so called one parameter deformation functions which are given as

$$A(q;c_1) = \sum_{m=1}^{\infty} \mu_m(c_1) \, q^m, \tag{2.9}$$

$$B(q;c_2) = \sum_{m=1}^{\infty} \sigma_m(c_2) \, q^m, \tag{2.10}$$

where $|c_1| \leq 1$ & $|c_2| \leq 1$ are constants, called the convergence control parameter. One can define μ_m and σ_m as

$$\mu_1(c_1) = (1 - c_1); \qquad \mu_m(c_1) = (1 - c_1)c_1^{m-1}, \quad m > 1 \tag{2.11}$$

$$\sigma_1(c_2) = (1 - c_2); \qquad \sigma_m(c_2) = (1 - c_2)c_2^{m-1}, \quad m > 1 \tag{2.12}$$

Thus the zeroth-order deformation equation (2.5) becomes

$$(1 - B(q; c_2))L[\phi(x, t; q) - u_0(x, t)] = c_0 A(q; c_1) N[\phi(x, t; q)], \tag{2.13}$$

In Fig. 1, $A(q; c_1)$ is plotted for different values of c_1, which can also be found in Liao [25]. It is obvious that for the embedding parameter $q = 0$ and $q = 1$, equation (2.13) becomes

$$\begin{aligned} \phi(x, t; 0) &= u_0(x, t), \\ \phi(x, t; 1) &= u(x, t), \end{aligned}$$

respectively. Thus, as q increases from 0 to 1, the solution $\phi(x, t; q)$ varies from the initial guess $u_0(x, t)$ to the solution $u(x, t)$. Expanding $\phi(x, t; q)$ in Taylor series with respect to q, one has

$$\phi(x, t; q) = u_0(x, t) + \sum_{k=1}^{\infty} q^k \ u_k(x, t), \tag{2.14}$$

$$\text{where} \quad u_k(x, t) = \frac{1}{k!} \left[\frac{\partial^k \phi(x, t; q)}{\partial q^k} \right]_{q=0}, \tag{2.15}$$

If the auxiliary linear operator, the initial guess and the convergence control parameters are properly chosen, the series (2.14) converges at $q = 1$. In this case one has

$$\phi(x, t; q) = u_0(x, t) + \sum_{k=1}^{\infty} u_k(x, t), \tag{2.16}$$

which must be one of the solutions of the original equation, as proven by Liao [25]. Let G denotes a function of $q \in [0, 1]$ and define the so called mth-order homotopy derivative as

$$D_m(G) = \frac{1}{m!} \left[\frac{\partial^m G}{\partial q^m} \right]_{q=0}, \tag{2.17}$$

Taking the above operation on both sides of zeroth- order equation (2.13), we have so called the mth-order deformation equations

$$L[u_m(x, t) - \chi_m \sum_{k=1}^{m-1} \sigma_{m-k}(c_2) u_m(x, t)] = c_0 \sum_{k=0}^{m-1} \mu_{m-k}(c_1) R_k(x, t), \tag{2.18}$$

where,

$$\chi_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 0 \end{cases} \tag{2.19}$$

$$\text{and } R_k(x,t) = D_k N[\phi(x,t;q)] = \frac{\partial^\alpha u_k(x,t)}{\partial t^\alpha} - \frac{\partial^2 u_k(x,t)}{\partial x^2} - \frac{\partial (x u_k(x,t))}{\partial x}, \quad (2.20)$$

Applying the idea of optimal homotopy- analysis method, we have

$$u_m(x,t) = \chi_m \sum_{k=1}^{m-1} \sigma_{m-k}(c_2) u_m(x,t) + c_0 \sum_{k=0}^{m-1} \mu_{m-k}(c_1) J_t^\alpha R_k(x,t) + c, \quad (2.21)$$

where, $J_t^\alpha(f(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi$, and the integration constant c is determined by the initial condition (2.1).
The m-th order approximation series solution is given as

$$u_m(x,t) = \sum_{k=0}^{m} u_k(x,t), \quad (2.22)$$

It is clear from equation (2.12) that $u_m(x,t)$ contains at most three unknown convergence-control parameters c_0, c_1andc_2, which determine the convergence region and rate of the homotopy-series solution.
As given by Liao [25], at the mth-order of approximation, one can define the exact square residual error

$$\Delta_m = \int \int_\Omega \left(N \left[\sum_{i=0}^{m} u_i(x,t) \right] \right)^2 dx\ dt, \quad (2.23)$$

However, it is proven by Liao [25] that the exact residual error Δ_m defined by equation (2.23) needs too much CPU time to calculate even if the order of approximation is not very high.
Thus, to overcome this difficulty i.e., to decrease the CPU time, we use here the so-called averaged residual error defined by

$$E_m = \frac{1}{5} \sum_{j=1}^{5} \sum_{k=1}^{5} \left(N \left[\sum_{i=0}^{m} u_i \left(\frac{j}{6}, \frac{k}{6} \right) \right] \right)^2. \quad (2.24)$$

3 Numerical Results and Discussion

In this section, the optimal values of convergence control parameters c_0, c_1, c_2 for better approximation of the series solution are determined through minimizing the square averaged residual error $(\sqrt{E_m})$ for standard motion $\alpha = 1$ and Brownian motions $\alpha = 0.9, 0.75$ for three specific cases for the initial conditions x and x^2. The numerical results for different particular cases are depicted through Tables 1-8 and Figs 2-7.

Order of approximation	c_0	$\sqrt{E_m}$	$\sqrt{E_m}$ at $c_0 = -1$
3	-1.330	1.57722×10^{-1}	9.77453×10^{-1}
6	-1.190	8.98950×10^{-4}	3.40575×10^{-2}
8	-1.118	4.44214×10^{-6}	1.65551×10^{-3}
10	-1.114	4.09087×10^{-8}	5.06753×10^{-5}

Table 1: Comparison of square averaged residual error for different values of c_0 at $\alpha = 1, n = 1$

Order of approximation	c_0	$\sqrt{E_m}$	$\sqrt{E_m}$ at $c_0 = -1$
3	-1.46	2.91315×10^{-1}	1.51736
6	-1.28	3.63406×10^{-3}	1.114801×10^{-1}
8	-1.19	1.03285×10^{-4}	1.027430×10^{-2}
10	-1.13	9.94831×10^{-7}	6.100700×10^{-4}

Table 2: Comparison of square averaged residual error for different values of c_0 at $\alpha = 0.9, n = 1$

Case I: optimal c_0 for the case of $c_1 = c_2 = 0$

In this case, we have only one convergence control parameter c_0. Figs. 2-4 are plotted for exact residual error Δ_m and averaged residual error E_m vs. c_0 for $n = 1$ and $\alpha = 1, 0.9, 0.75$. Tables 1- 3 show the comparison of the results of the averaged residual error for proper choices of c_0 with the increase in the order of approximations. It is clear from the tables that optimal values of c_0 are -1.114, -1.13, -1.31 for $\alpha = 1, 0.9, 0.75$ respectively for $n = 1$.

It is also observed from Tables 1, 2 and 3 that as the value of α decreases the optimal value of c_0 goes away from $c_0 = -1$, the case of usual HAM.

Next, Figs. 5-7 are plotted for exact residual error Δ_m and averaged residual

Order of approximation	c_0	$\sqrt{E_m}$	$\sqrt{E_m}$ at $c_0 = -1$
3	-1.78	8.20035×10^{-1}	2.80388
6	-1.31	1.09849×10^{-2}	6.36651×10^{-1}
8	-1.32	6.63514×10^{-4}	1.36230×10^{-1}
10	-1.31	3.10408×10^{-5}	2.09136×10^{-2}

Table 3: Comparison of square averaged residual error for different values of c_0 at $\alpha = 0.75, n = 1$

Order of approximation	c_0	$\sqrt{E_m}$	$\sqrt{E_m}$ at $c_0 = -1$
3	-1.58	3.43496	11.7748
6	-1.19	2.56197×10^{-2}	1.39696
8	-1.19	6.86627×10^{-4}	1.52833×10^{-1}
10	-1.09	1.14270×10^{-5}	1.05264×10^{-2}

Table 4: Comparison of square averaged residual error for different values of c_0 at $\alpha = 1, n = 2$

Order of approximation	c_0	$\sqrt{E_m}$	$\sqrt{E_m}$ at $c_0 = -1$
3	0.19	4.11286	18.2787
6	-1.28	9.77028×10^{-2}	4.70889
8	-1.19	4.30506×10^{-3}	9.48507×10^{-1}
10	-1.20	1.51957×10^{-4}	1.26725×10^{-1}

Table 5: Comparison of square averaged residual error for different values of c_0 at $\alpha = 0.9, n = 2$

Order of approximation	c_0	$\sqrt{E_m}$	$\sqrt{E_m}$ at $c_0 = -1$
3	0.19	3.35696	33.7766
6	-1.54	1.13749	26.1141
8	-1.37	8.81268×10^{-2}	12.5765
10	-1.38	6.45506×10^{-3}	4.34423

Table 6: Comparison of square averaged residual error for different values of c_0 at $\alpha = 0.75, n = 2$

Order of approx-imation	$c_1 = 0$ $c_2 = 0$ $c_0 = -1$	$c_1 = 0$ $c_2 = 0$ $c_0 = -1.1$	$c_1 = -0.1$ $c_2 = -0.1$ $c_0 = -1$	$c_1 = -0.09$ $c_2 = -0.09$ $c_0 = -1.067$
3	9.77453×10^{-1}	6.97991×10^{-1}	6.97991×10^{-1}	5.21029×10^{-1}
6	3.40575×10^{-2}	1.84513×10^{-3}	1.84513×10^{-3}	1.65726×10^{-3}
8	1.65551×10^{-3}	1.98049×10^{-5}	1.98049×10^{-5}	3.18427×10^{-5}
10	5.06753×10^{-5}	1.76189×10^{-7}	1.76189×10^{-7}	3.14832×10^{-7}

Table 7: Comparison of square averaged residual error $\sqrt{E_m}$ at $\alpha = 1, n = 1$

Order of approx-imation	$c_1 = 0$ $c_2 = 0$ $c_0 = -1$	$c_1 = 0$ $c_2 = 0$ $c_0 = -1.1$	$c_1 = -0.128$ $c_2 = -0.128$ $c_0 = -1$	$c_1 = -0.07$ $c_2 = -0.07$ $c_0 = -1.05$
3	11.7748	10.7968	10.3136	9.56604
6	1.39696	5.07662×10^{-1}	2.52837×10^{-1}	2.41372×10^{-2}
8	1.52833×10^{-1}	1.34014×10^{-2}	1.06664×10^{-3}	1.07976×10^{-3}
10	1.05264×10^{-2}	1.14270×10^{-5}	4.16127×10^{-6}	1.53751×10^{-5}

Table 8: Comparison of square averaged residual error $\sqrt{E_m}$ at $\alpha = 1, n = 2$

error E_m vs. c_0 for $n = 2$ and $\alpha = 1, 0.9, 0.75$ respectively. It is clear from Tables 4, 5 and 6 that optimal values of c_0 are -1.09, -1.20, -1.38 for $\alpha = 1, 0.9, 0.75$ respectively for the case of $n = 2$. It is also clear from the tables that as the value of α decreases the optimal value of c_0 goes away from $c_0 = -1$, which is similar to the case of $n = 1$.

Case II: optimal c_0 for the case of $c_1 = c_2 \neq 0$

In this case, we have at most two convergence parameters viz., c_0 and c_1. It is seen through Table 7 that optimal value of c_0 in the case of $c_1 = c_2 = -0.09$ is -1.067 for $n = 1$. Also it is observed from Table 8 that optimal value of c_0 in the case of $c_1 = c_2 = -0.07$ is -1.05 for $n = 2$.

Case III: optimal $c_1 = c_2$ for the case of $c_0 = -1$

Here, we have only one convergence control parameter c_1. Table 7 shows that optimal value of c_1 is -0.1 for $n = 1$ and Table 8 shows that optimal value of c_1 is -0.128 for $n = 2$. Table 7 clearly demonstrates that the values of the square averaged residual error $\sqrt{E_m}$ is the same for two cases $c_0 = -1.1, c_1 = c_2 = 0$ and $c_0 = -1, c_1 = c_2 = -0.1$, which proves that there is freedom to chose any set of parameters for better approximation of the solution.

Conclusion

From the numerical computation given in Tables, it is clear that optimal homotopy analysis method gives better approximation than the usual HAM ($c_1 = c_2 = 0$). Faster convergence of the series solution with the proper choices of the parameters renders the procedure appropriate for solving fractional diffusion equations in different dimension. Thus we may conclude that the study of finding the 'best' deformation function among all the existing ones for getting faster convergent series solution has been very useful.

Applying the method successfully in solving the diffusion equation with fractional time derivative, we may also conclude that the present method is very effective and efficient even for solving fractional PDEs.

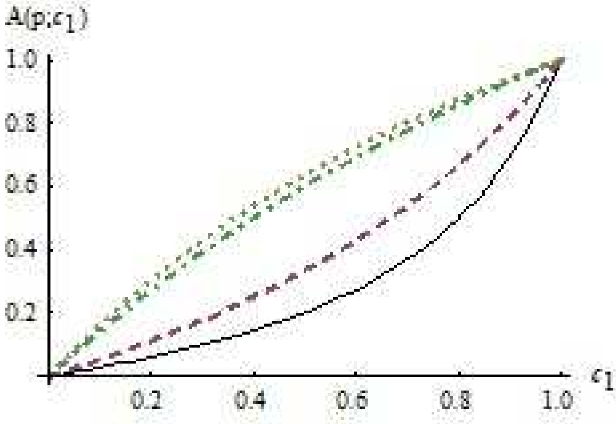


Figure 1: plot of $A(p; c_1)$ for different value of c_l Solid line : $c_l = 3/4$; Dashed line: $c_l = 1/2$; Dotted line: $c_l = -3/4$; Dash-dotted line: $c_l = -1/2$.

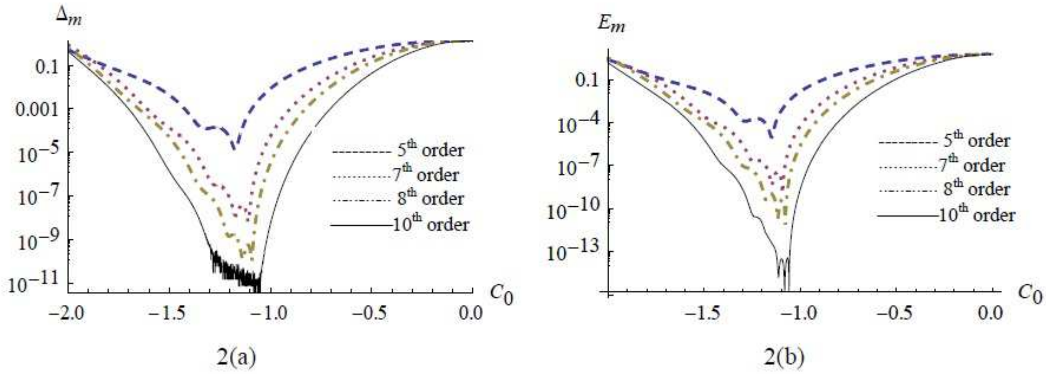
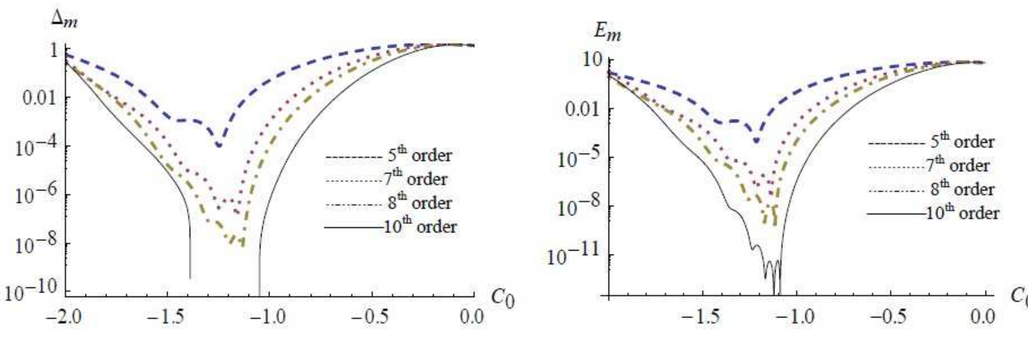
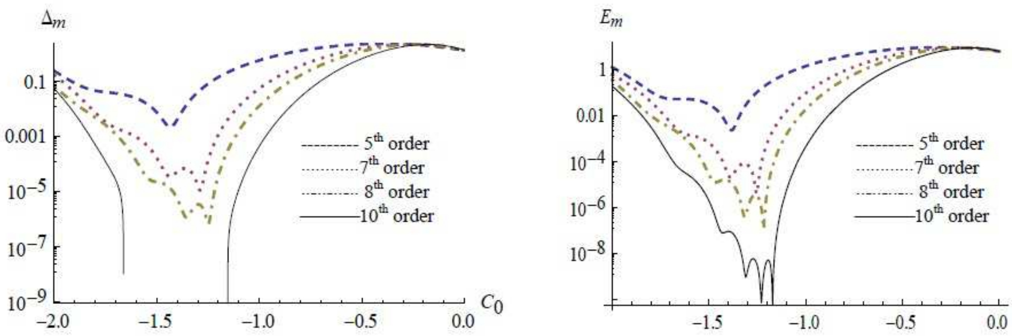


Figure 2: Plots of exact residual error $\triangle_m$ and averaged residual error E_m vs. c_0 for $\alpha = 1$ and $n = 1$ for different order of approximation



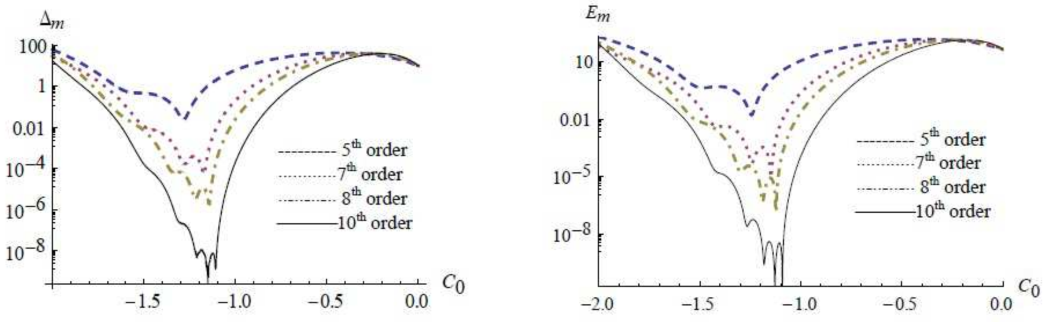
3(a) 3(b)

Figure 3: Plots of exact residual error $\triangle_m$ and averaged residual error E_m vs. c_0 for $\alpha = 0.9$ and $n = 1$ for different order of approximation



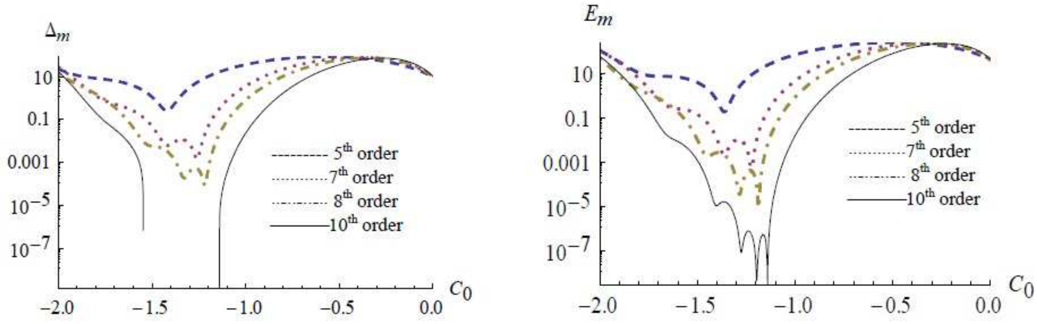
4(a) 4(b)

Figure 4: Plots of exact residual error $\triangle_m$ and averaged residual error E_m vs. c_0 for $\alpha = 0.75$ and $n = 1$ for different order of approximation



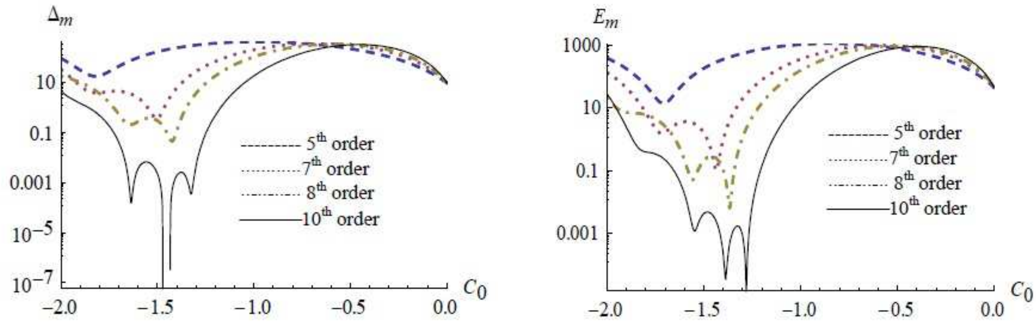
5(a) 5(b)

Figure 5: Plots of exact residual error $\triangle_m$ and averaged residual error E_m vs. c_0 for $\alpha = 1$ and $n = 2$ for different order of approximation



6(a) 6(b)

Figure 6: Plots of exact residual error $\triangle_m$ and averaged residual error E_m vs. c_0 for $\alpha = 0.9$ and $n = 2$ for different order of approximation



7(a) 7(b)

Figure 7: Plots of exact residual error $\triangle_m$ and averaged residual error E_m vs. c_0 for $\alpha = 0.75$ and $n = 2$ for different order of approximation

References

[1] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep. **339** (2000), 1-77. MR1809268. Zbl 0984.82032.

[2] T. A. M. Langlands and B. I. Henry, *The accuracy and stability of an implicit solution method for the fractional diffusion equation*, J. Comp. Phys. 205 (2005), 719-736. MR2135000. Zbl 1072.65123.

[3] S. B. Yuste and K. Lindenberg, *Subdiffusion-Limited A+A Reactions*, Phys. Rev. Lett. **87** (2001), 118301.

[4] J. Sung, E. Barkai, R. J. Silbey and S. Lee, *Fractional dynamics approach to diffusion-assisted reactions in disordered media*, J. Chem. Phys., **116** (2002), 2338-2341.

[5] G. H. Weiss, *Aspects and Applications of the Random Walk*, North-Holland, Amsterdam, 1994. MR MR1280031. Zbl 0925.60079.

[6] B. D. Hughes, *Random Walks and Random Environments*, vol.1, Clarendon Press, Oxford, 1995. MR1911759. Zbl 0820.60053.

[7] B. I. Henry and S. L. Wearne, *Fractional reaction-diffusion*, Physica A **276** (2000), 448-455. MR1780373.

[8] R. Metzler and J. Klafter, *Accelerating Brownian motion: A fractional dynamics approach to fast diffusion*, Europhys. Lett., **61** (2000), 492-496.

[9] C. M. Chen, F. Liu, I. Turner and V. Anh, *A Fourier method for the fractional diffusion equation describing sub-diffusion*, J. Compu. Phys., **227** (2007), 886-897. MR2442379. Zbl 1165.65053.

[10] A. Schot, M. K. Lenzi, L. R. Evangelista, L. C. Malacarne, R. S. Mendis and E. K. Lenzi, *Fractional diffusion equation with an absorbent term and a linear external force: Exact solution*, Phys. Lett. A, **366** (2007), 346-350.

[11] M. A. Zahran, *On the derivation of fractional diffusion equation with an absorbent term and a linear external force*, App. Math. Model., **33** (2009), 3088-3092.

[12] H. Wang, K. Wang and T. Sircar, *A direct O(N log2N) finite difference method for fractional diffusion equations*, J. Comp. Phy., 229 (2010), 8095-8114. MR2719162. Zbl 1198.65176.

[13] B. P. Sprouse, C. L. MacDonald and G. A. Silva, *Computational efficiency of fractional diffusion using adaptive time step*, math-ph, 2010. arXiv:1004.5128v1.

[14] J. M. Angulo, M. D. Ruiz-Medina, V. V. Anh and W. Grecksch, *Fractional diffusion and fractional heat equation*, Adv. Appl. Prob. **32** (2000), 1077-1099. Zbl 0986.60077.

[15] S. Pezat and J. Zabczyk, *Nonlinear stochastic wave and heat equations*, Prob. Theory Relat. Fields **116** (2000), 421-443. MR1749283. Zbl 0959.60044.

[16] W. R. Schneider and W. Wyss, *Fractional diffusion and wave equations*, J. Math. Phys. **30** (1989), 134-144. MR0974464. Zbl 0692.45004.

[17] R. Yu and H. Zhang, *New function of Mittag-Leffler type and its application in the fractional diffusion-wave equation*, Chaos Solitons & Fractals **30** (2006), 946-955. MR2250265. Zbl 1142.35479.

[18] F. Mainardi, *The fundamental solutions for the fractional diffusion-wave equation*, Appl. Math. Lett. **9** (1996), 23-28. MR1419811. Zbl 1080.47054.

[19] F. Mainardi, Y. Luchko and G. Pagnini, *The fundamental solution of the space-time fractional diffusion equation*, Fract. Calculus Appl. Anal. **4** (2001), 153-192. MR1829592. Zbl 1054.35156.

[20] V. V. Anh and N. N. Leonenko, *Harmonic analysis of fractional diffusion-wave equations*, Appl. Math. Comput. **141** (2003), 77-85. Zbl 1053.60064.

[21] S. Das and P. K. Gupta, *An approximate analytical solution of the fractional diffusion equation with absorbent term and external force by Homotopy Perturbation Method*, Zeitschrift fr Naturforschung, **65**(3) (2010), 182-190.

[22] S. J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*, Ph.D. Thesis, Shanghai Jiao Tong University, 1992.

[23] S. J. Liao, *Notes on the homotopy analysis method: Some definition and theorems*, Comm. in Nonlin. Sci. Numer. Simul., **14** (2009), 983-997. MR2468931. Zbl 1221.65126.

[24] S. Das, K. Vishal, P.K. Gupta and S.S. Ray, *Homotopy Analysis Method for solving Fractional Diffusion equation*, Int. J. Appl. Math. Mech., **7**(9) (2011), 28–37. Zbl 1252.65174.

[25] S. J. Liao, *An optimal homotopy-analysis approach for strongly nonlinear differential equations*, Comm. in Nonlin. Sci. Numer. Simul., **15** (2010), 2003-2016. MR2592613. Zbl 1222.65088.

S. Das
Department of Applied Mathematics,
Indian Institute of Technology,
Banaras Hindu University,
Varanasi - 221 005, India.
e-mail: subir_das08@hotmail.com

K. Vishal
Department of Applied Mathematics,
Indian Institute of Technology,
Banaras Hindu University,
Varanasi - 221 005, India.

P. K. Gupta
Indian Institute of Technology,
Banaras Hindu University,
Varanasi - 221 005, India.