

## ON APPROXIMATION OF FUNCTIONS BY PRODUCT OPERATORS

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**Abstract.** In the present paper, two quite new results on the degree of approximation of a function  $f$  belonging to the class  $Lip(\alpha, r)$ ,  $1 \leq r < \infty$  and the weighted class  $W(L_r, \xi(t))$ ,  $1 \leq r < \infty$  by  $(C, 2)(E, 1)$  product operators have been obtained. The results obtained in the present paper generalize various known results on single operators.

### 1 Introduction

Let  $\sum_{n=0}^{\infty} u_n$  be a given infinite series with the sequence of its  $n^{th}$  partial sum  $\{s_n\}$ .

The  $(C, 2)$  transform is defined as the  $n^{th}$  partial sum of  $(C, 2)$  summability and is given by

$$t_n = \frac{1}{(n+1)(n+2)} \sum_{k=0}^n (n-k+1)s_k \rightarrow s \text{ as } n \rightarrow \infty$$

then the series  $\sum_{n=0}^{\infty} u_n$  is summable to  $s$  by  $(C, 2)$  method.

If

$$(E, 1) = E_n^1 = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} s_k \rightarrow s \text{ as } n \rightarrow \infty \quad (1.1)$$

then the series  $\sum_{n=0}^{\infty} u_n$  with the  $n^{th}$  partial sum  $s_n$  is said to be summable  $(E, 1)$  to  $s$  [3].

The  $(C, 2)$  transform of the  $(E, 1)$  transform defines  $(C, 2)(E, 1)$  product transform and we denote it by  $C_n^2 E_n^1$ .

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Thus if

$$C_n^2 E_n^1 = \frac{2}{(n+1)(n+2)} \sum_{k=0}^n (n-k+1) E_k^1 \rightarrow s \quad \text{as } n \rightarrow \infty, \quad (1.2)$$

where  $E_n^1$  denotes the (E,1) transform of  $s_n$  and  $C_n^2$  denotes the (C,2) transform of  $s_n$ , then the series  $\sum_{n=0}^{\infty} u_n$  is said to be summable by (C,2)(E,1) means or summable (C,2)(E,1) to  $s$ .

Let  $f(x)$  be periodic with period  $2\pi$  and integrable in the sense of Lebesgue. The Fourier series of  $f(x)$  is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1.3)$$

with  $n^{\text{th}}$  partial sum  $s_n(f; x)$ .

$L_{\infty}$ -norm of a function  $f : R^n \rightarrow R$  is defined by

$$\|f\|_{\infty} = \sup\{|f(x)| : x \in R\} \quad (1.4)$$

$L_r$ -norm is defined by

$$\|f\|_r = \left( \int_0^{2\pi} |f(x)|^r dx \right)^{\frac{1}{r}}, \quad r \geq 1 \quad (1.5)$$

The degree of approximation of a function  $f : R^n \rightarrow R$  by a trigonometric polynomial  $t_n$  of order  $n$  under sup norm  $\|\cdot\|_{\infty}$  is defined by

$$\|t_n - f\|_{\infty} = \sup\{|t_n(x) - f(x)| : x \in R\} \quad \text{Zygmund [14]} \quad (1.6)$$

and  $E_n(f)$  of a function  $f$  belongs to  $L_r$  is given by

$$E_n(f) = \min \|t_n - f\|_r \quad (1.7)$$

One says that a function  $f$  belongs to the class  $Lip\alpha$  if

$$f(x+t) - f(x) = O(|t|^{\alpha}) \quad \text{for } 0 < \alpha \leq 1 \quad (1.8)$$

and that  $f$  belongs to the class  $Lip(\alpha, r)$  if

$$\left( \int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{\frac{1}{r}} = O(|t|^{\alpha}), \quad 0 < \alpha \leq 1 \quad \text{and } r \geq 1 \quad (1.9)$$

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(Mc Fadden [6], definition 5.38)

Given a positive increasing function  $\xi(t)$  and some  $r$ ,  $1 \leq r < \infty$ , one says that a function  $f$  belongs to the class  $Lip(\xi(t), r)$  if

$$\left( \int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{\frac{1}{r}} = O(\xi(t)) \quad (1.10)$$

and that  $f$  belongs to the class  $W(L_r, \xi(t))$ ,  $1 \leq r < \infty$  if

$$\left( \int_0^{2\pi} | \{f(x+t) - f(x)\} \sin^\beta x |^r dx \right)^{\frac{1}{r}} = O(\xi(t)), \quad \beta \geq 0. \quad (1.11)$$

If  $\beta = 0$ , our newly defined class  $W(L_r, \xi(t))$  reduces to the class  $Lip(\xi(t), r)$  and if  $\xi(t) = t^\alpha$  then  $Lip(\xi(t), r)$  class reduces to the class  $Lip(\alpha, r)$  and if  $r \rightarrow \infty$  then  $Lip(\alpha, r)$  class reduces to the class  $Lip\alpha$ .

We use the following notations throughout this paper:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$K_n(t) = \frac{1}{\pi (n+1)(n+2)} \sum_{k=0}^n \left\{ \left( \frac{n-k+1}{2^k} \right) \sum_{\nu=0}^k \binom{k}{\nu} \frac{\sin(\nu + \frac{1}{2})t}{\sin \frac{t}{2}} \right\}$$

## 2 Main Theorems

Several researchers ([1], [2], [4], [5], [7], [8], [9], [10], [11], [12]) initiated the studies of error estimates of function belonging to the classes  $Lip\alpha$ ,  $Lip(\alpha, r)$ ,  $Lip(\xi(t), r)$ ,  $W(L_r, \xi(t))$  using different linear operators. But till now nothing seems to have been in the direction of present work. Therefore, in this paper, two quite new theorems on degree of approximation of a function belonging to the class  $Lip(\alpha, r)$  and the weighted class  $W(L_r, \xi(t))$  by  $(C, 2)(E, 1)$  summability means on Fourier series have been established. In fact, we proved the following theorems.

**Theorem 1.** *If  $f$  is a  $2\pi$ -periodic function, Lebesgue integrable on  $[0, 2\pi]$  belonging to the class  $Lip(\alpha, r)$ ,  $1 \leq r < \infty$ , then its degree of approximation by  $(C, 2)(E, 1)$  means on Fourier series (1.3) is given by*

$$\| C_n^2 E_n^1 - f \|_r = O \left\{ \frac{1}{(n+1)^{\alpha - \frac{1}{r}}} \right\} \text{ for } 0 < \alpha \leq 1, \quad (2.1)$$

where  $\delta$  is an arbitrary number such that  $0 < \delta < \frac{1}{r} - \alpha$ , where  $r^{-1} + s^{-1} = 1$ ,  $1 \leq r < \infty$  and  $C_n^2 E_n^1$  is  $(C, 2)(E, 1)$  means of the series (1.3).

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**Theorem 2.** *If a positive increasing function  $\xi(t)$  satisfies the conditions*

$$\left\{ \int_0^{\frac{1}{n+1}} \left( \frac{t |\phi(t)|}{\xi(t)} \right)^r \sin^{\beta r} t \, dt \right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \quad (2.2)$$

and

$$\left\{ \int_{\frac{1}{n+1}}^{\pi} \left( \frac{t^{-\delta} |\phi(t)|}{\xi(t)} \right)^r dt \right\}^{\frac{1}{r}} = O\{(n+1)^\delta\} \quad (2.3)$$

uniformly in  $x$ , in which  $\delta$  is an arbitrary number with  $s(1-\delta) - 1 > 0$ , where  $r^{-1} + s^{-1} = 1$ ,  $1 \leq r < \infty$  and

$$\left\{ \frac{\xi(t)}{t} \right\} \text{ is non increasing in } t, \quad (2.4)$$

then the degree of approximation of a  $2\pi$ -periodic function  $f$  belonging to the weighted class  $W(L^r, \xi(t))$ ,  $1 \leq r < \infty$ , by  $(C, 2)(E, 1)$  means on Fourier series (1.3) is given by

$$\| C_n^2 E_n^1 - f \|_r = O \left\{ (n+1)^{\beta + \frac{1}{r}} \xi \left( \frac{1}{n+1} \right) \right\}, \quad (2.5)$$

where  $C_n^2 E_n^1$  denote the sequence of  $(C, 2)(E, 1)$  means of the series (1.3).

### 3 Lemmas

Following lemmas are required for the proof of our theorems.

**Lemma 3.**

$$|K_n(t)| = O(n+1), \text{ for } 0 \leq t \leq \frac{1}{n+1}$$

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*Proof.* For  $0 \leq t \leq \frac{1}{n+1}$ ,  $\sin nt \leq n \sin t$

$$\begin{aligned}
 |K_n(t)| &= \frac{1}{\pi (n+1)(n+2)} \left| \sum_{k=0}^n \left[ \left( \frac{n-k+1}{2^k} \right) \sum_{v=0}^k \binom{k}{v} \frac{\sin(v + \frac{1}{2})t}{\sin \frac{t}{2}} \right] \right| \\
 &\leq \frac{1}{\pi (n+1)(n+2)} \left| \sum_{k=0}^n \left[ \left( \frac{n-k+1}{2^k} \right) \sum_{v=0}^k \binom{k}{v} \frac{(2v+1) \sin \frac{t}{2}}{\sin \frac{t}{2}} \right] \right| \\
 &\leq \frac{1}{\pi (n+1)(n+2)} \left| \sum_{k=0}^n \left[ \left( \frac{n-k+1}{2^k} \right) (2k+1) \sum_{v=0}^k \binom{k}{v} \right] \right| \\
 &= \frac{1}{\pi (n+1)(n+2)} \left| \sum_{k=0}^n (n-k+1)(2k+1) \right| \\
 &= \frac{n+1}{\pi (n+1)(n+2)} \sum_{k=0}^n (2k+1) - \frac{1}{\pi (n+1)(n+2)} \sum_{k=0}^n k(2k+1) \\
 &= \frac{1}{\pi (n+2)} \sum_{k=0}^n (2k+1) - \frac{1}{\pi (n+1)(n+2)} \left[ 2 \sum_{k=0}^n k^2 + \sum_{k=0}^n k \right] \\
 &= \frac{(n+1)^2}{\pi (n+2)} - \frac{1}{\pi (n+1)(n+2)} \left[ \frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2} \right] \\
 &= \frac{(n+1)^2}{\pi (n+2)} - \frac{n(2n+1)}{3\pi (n+2)} - \frac{n}{2\pi (n+2)} \\
 &= \frac{2n^2 + 7n + 6}{6\pi (n+2)} \\
 &= O(n+1)
 \end{aligned}$$

□

**Lemma 4.**

$$|K_n(t)| = O\left(\frac{1}{t}\right), \text{ for } \frac{1}{n+1} \leq t \leq \pi$$

*Proof.* For  $\frac{1}{n+1} \leq t \leq \pi$ , by applying Jordan's lemma  $\sin \frac{t}{2} \geq \frac{t}{\pi}$  and  $\sin nt \leq 1$

$$\begin{aligned}
 |K_n(t)| &= \frac{1}{\pi (n+1)(n+2)} \left| \sum_{k=0}^n \left[ \left( \frac{n-k+1}{2^k} \right) \sum_{v=0}^k \binom{k}{v} \frac{\sin(v + \frac{1}{2})t}{\sin \frac{t}{2}} \right] \right| \\
 &\leq \frac{n+1}{\pi (n+1)(n+2)} \left| \sum_{k=0}^n \left[ \frac{1}{2^k} \sum_{v=0}^k \binom{k}{v} \frac{1}{\pi} \right] \right| - \frac{1}{\pi (n+1)(n+2)} \left| \sum_{k=0}^n \left[ \frac{k}{2^k} \sum_{v=0}^k \binom{k}{v} \frac{1}{\pi} \right] \right|
 \end{aligned}$$

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$$\begin{aligned}
&= \frac{1}{t(n+2)} \left| \sum_{k=0}^n \left[ \frac{1}{2^k} \sum_{v=0}^k \binom{k}{v} \right] \right| - \frac{1}{t(n+1)(n+2)} \left| \sum_{k=0}^n \left[ \frac{k}{2^k} \sum_{v=0}^k \binom{k}{v} \right] \right| \\
&= \frac{1}{t(n+2)} \sum_{k=0}^n 1 - \frac{1}{t(n+1)(n+2)} \sum_{k=0}^n k \\
&= \frac{n+1}{t(n+2)} - \frac{1}{t(n+1)(n+2)} \left\{ \frac{n(n+1)}{2} \right\} \\
&= \frac{n+1}{t(n+2)} - \frac{n}{2t(n+2)} \\
&= \frac{1}{2t} \\
&= O\left(\frac{1}{t}\right)
\end{aligned}$$

□

**Lemma 5.** (Mc Fadden [6], lemma 5.40) If  $f(x)$  belongs to  $Lip(\alpha, r)$  on  $[0, \pi]$  then  $\phi(t)$  belongs to  $Lip(\alpha, r)$  on  $[0, \pi]$ .

## 4 Proof of the main theorems

### 4.1 Proof of theorem 1

Following Titchmarsh [13] and using Riemann-Lebesgue theorem,  $s_n(f; x)$  of the series (1.3) is given by

$$s_n(f; x) - f(x) = \frac{1}{2\pi} \int_0^\pi \phi(t) \frac{\sin(n + \frac{1}{2})t}{\sin \frac{t}{2}} dt$$

Therefore using (1.3), the (E,1) transform  $E_n^1$  of  $s_n(f; x)$  is given by

$$E_n^1 - f(x) = \frac{1}{2\pi} \frac{1}{2^n} \int_0^\pi \phi(t) \sum_{k=0}^n \binom{n}{k} \frac{\sin(k + \frac{1}{2})t}{\sin \frac{t}{2}} dt$$

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Now denoting (C,2)(E,1) transform of  $s_n(f; x)$  by  $C_n^2 E_n^1$ , we write

$$\begin{aligned}
 & C_n^2 E_n^1 - f(x) \\
 &= \frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^n \left[ \frac{(n-k+1)}{2^k} \int_0^\pi \frac{\phi(t)}{\sin \frac{t}{2}} \left\{ \sum_{v=0}^k \binom{k}{v} \sin \left( v + \frac{1}{2} \right) t \right\} dt \right] \\
 &= \int_0^\pi \phi(t) K_n(t) dt \\
 &= \left[ \int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^\pi \right] \phi(t) K_n(t) dt \\
 &= I_{1.1} + I_{1.2} \text{ (say)}
 \end{aligned} \tag{4.1}$$

We consider,

$$|I_{1.1}| \leq \int_0^{\frac{1}{n+1}} |\phi(t)| |K_n(t)| dt$$

Using Hölder's inequality and lemma 3,

$$\begin{aligned}
 |I_{1.1}| &\leq \left[ \int_0^{\frac{1}{n+1}} \left\{ \frac{t |\phi(t)|}{t^\alpha} \right\}^r dt \right]^{\frac{1}{r}} \left[ \int_0^{\frac{1}{n+1}} \left\{ \frac{|K_n(t)|}{t^{1-\alpha}} \right\}^s dt \right]^{\frac{1}{s}} \\
 &= \left( \frac{1}{n+1} \right) \left[ \int_0^{\frac{1}{n+1}} \left\{ \frac{|K_n(t)|}{t^{1-\alpha}} \right\}^s dt \right]^{\frac{1}{r}} \\
 &= O \left[ \int_0^{\frac{1}{n+1}} \left\{ \frac{1}{t^{1-\alpha}} \right\}^s dt \right]^{\frac{1}{s}} \text{ by lemma 1} \\
 &= O \left[ \int_0^{\frac{1}{n+1}} t^{\alpha s - s} dt \right]^{\frac{1}{s}} \\
 &= O \left[ \left( \frac{1}{n+1} \right)^{\frac{\alpha s - s + 1}{s}} \right] \\
 &= O \left[ \left( \frac{1}{n+1} \right)^{\alpha - 1 + \frac{1}{s}} \right] \\
 &= O \left[ \left( \frac{1}{n+1} \right)^{\alpha - (1 - \frac{1}{s})} \right] \\
 I_{1.1} &= O \left[ \left( \frac{1}{n+1} \right)^{\alpha - \frac{1}{r}} \right]
 \end{aligned} \tag{4.2}$$

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Similarly as above, we have

$$\begin{aligned}
|I_{1.2}| &\leq \left[ \int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{t^{-\delta} |\phi(t)|}{t^{\alpha}} \right\}^r dt \right]^{\frac{1}{r}} \left[ \int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{|K_n(t)|}{t^{-\delta-\alpha}} \right\}^s dt \right]^{\frac{1}{s}} \\
&= O \left[ \int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{t^{-\delta} t^{\alpha-\frac{1}{r}}}{t^{\alpha}} \right\}^r dt \right]^{\frac{1}{r}} \left[ \int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{|K_n(t)|}{t^{-\delta-\alpha}} \right\}^s dt \right]^{\frac{1}{s}} \\
&= O \left[ \int_{\frac{1}{n+1}}^{\pi} \left\{ t^{-\frac{1}{r}-\delta} \right\}^r dt \right]^{\frac{1}{r}} \left[ \int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{1}{t^{1-\delta-\alpha}} \right\}^s dt \right]^{\frac{1}{s}} \\
&= O \left[ \int_{\frac{1}{n+1}}^{\pi} t^{-1-\delta r} dt \right]^{\frac{1}{r}} \left[ \int_{\frac{1}{n+1}}^{\pi} t^{s\alpha+s\delta-s} dt \right]^{\frac{1}{s}} \\
&= O \left[ (n+1)^{\delta} \left\{ (n+1)^{-s\alpha-s\delta+s-1} \right\}^{\frac{1}{s}} \right] \\
&= O \left[ (n+1)^{\delta} (n+1)^{-\alpha-\delta+1-\frac{1}{s}} \right] \\
I_{1.2} &= O \left[ \frac{1}{(n+1)^{\alpha-\frac{1}{r}}} \right] \tag{4.3}
\end{aligned}$$

Combining (4.1), (4.2) and (4.3) we get

$$\| C_n^2 E_n^1 - f \|_r = O \left[ \frac{1}{(n+1)^{\alpha-\frac{1}{r}}} \right]$$

This completes the proof of theorem 1.

## 4.2 Proof of theorem 2

Following the proof of theorem 1

$$\begin{aligned}
C_n^2 E_n^1 - f(x) &= \left[ \int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right] \phi(t) K_n(t) dt \\
&= I_{2.1} + I_{2.2} \text{ (say)} \tag{4.4}
\end{aligned}$$

we have

$$| \phi(x+t) - \phi(x) | \leq | f(u+x+t) - f(u+x) | + | f(u-x-t) - f(u-x) |$$

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Hence, by Minkowski's inequality,

$$\begin{aligned} \left[ \int_0^{2\pi} |\{\phi(x+t) - \phi(x)\} \sin^\beta x|^r dx \right]^{\frac{1}{r}} &\leq \left[ \int_0^{2\pi} |\{f(u+x+t) - f(u+x)\} \sin^\beta x|^r dx \right]^{\frac{1}{r}} \\ &+ \left[ \int_0^{2\pi} |\{f(u-x-t) - f(u-x)\} \sin^\beta x|^r dx \right]^{\frac{1}{r}} = O\{\xi(t)\}. \end{aligned}$$

Then  $f \in W(L_r, \xi(t)) \Rightarrow \phi \in W(L_r, \xi(t))$ .

Using Hölder's inequality and the fact that  $\phi(t) \in W(L_r, \xi(t))$  condition (2.2),  $\sin t \geq \frac{2t}{\pi}$ , lemma 1 and second mean value theorem for integrals, we have

$$\begin{aligned} |I_{2.1}| &\leq \left[ \int_0^{\frac{1}{n+1}} \left\{ \frac{t |\phi(t) \sin^\beta t|}{\xi(t)} \right\}^r dt \right]^{\frac{1}{r}} \left[ \int_0^{\frac{1}{n+1}} \left\{ \frac{\xi(t) |K_n(t)|}{t \sin^\beta t} \right\}^s dt \right]^{\frac{1}{s}} \\ &= O\left(\frac{1}{n+1}\right) \left[ \int_0^{\frac{1}{n+1}} \left\{ \frac{(n+1) \xi(t)}{t^{1+\beta}} \right\}^s dt \right]^{\frac{1}{s}} \\ &= O\left\{ \xi\left(\frac{1}{n+1}\right) \right\} \left[ \int_\epsilon^{\frac{1}{n+1}} \frac{dt}{t^{(1+\beta)s}} \right]^{\frac{1}{s}} \text{ for some } 0 < \epsilon < \frac{1}{n+1} \\ I_{2.1} &= O\left\{ (n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right) \right\} \end{aligned} \quad (4.5)$$

Now using Hölder's inequality,  $|\sin t| \leq 1$ ,  $\sin t \geq \frac{2t}{\pi}$ , conditions (2.3) and (2.4), lemma 2 and second mean value theorem for integrals, we have

$$\begin{aligned} |I_{2.2}| &\leq \left[ \int_{\frac{1}{n+1}}^\pi \left\{ \frac{t^{-\delta} |\phi(t) \sin^\beta t|}{\xi(t)} \right\}^r dt \right]^{\frac{1}{r}} \left[ \int_{\frac{1}{n+1}}^\pi \left\{ \frac{\xi(t) |K_n(t)|}{t^{-\delta} \sin^\beta t} \right\}^s dt \right]^{\frac{1}{s}} \\ &= O\{(n+1)^\delta\} \left[ \int_{\frac{1}{n+1}}^\pi \left\{ \frac{\xi(t)}{t^{\beta+1-\delta}} \right\}^s dt \right]^{\frac{1}{s}} \\ &= O\{(n+1)^\delta\} \left[ \int_{\frac{1}{\pi}}^{n+1} \left\{ \frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-1-\beta}} \right\}^s \frac{dy}{y^2} \right]^{\frac{1}{s}} \\ &= O\left\{ (n+1)^\delta \xi\left(\frac{1}{n+1}\right) \right\} \left[ \int_\eta^{n+1} \frac{dy}{y^{s(\delta-1-\beta)+2}} \right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi} < \eta < n+1 \\ &= O\left\{ (n+1)^\delta \xi\left(\frac{1}{n+1}\right) \right\} \left[ \int_1^{n+1} \frac{dy}{y^{s(\delta-1-\beta)+2}} \right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi} < 1 < n+1 \end{aligned}$$

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$$\begin{aligned}
&= O \left\{ (n+1)^\delta \xi \left( \frac{1}{n+1} \right) \right\} \left[ (n+1)^{(1+\beta-\delta)-\frac{1}{s}} \right] \\
I_{2.2} &= O \left\{ (n+1)^{\beta+\frac{1}{r}} \xi \left( \frac{1}{n+1} \right) \right\} \tag{4.6}
\end{aligned}$$

Now combining (4.4), (4.5) and (4.6), we get

$$|C_n^2 E_n^1 - f(x)|_r = O \left\{ (n+1)^{\beta+\frac{1}{r}} \xi \left( \frac{1}{n+1} \right) \right\}.$$

Hence,

$$\begin{aligned}
\|C_n^2 E_n^1 - f(x)\|_r &= \left\{ \int_0^{2\pi} |C_n^2 E_n^1 - f(x)|^r dx \right\}^{\frac{1}{r}} \\
&= O \left[ \left\{ \int_0^{2\pi} \left\{ (n+1)^{\beta+\frac{1}{r}} \xi \left( \frac{1}{n+1} \right) \right\}^r dx \right\}^{\frac{1}{r}} \right] \\
&= O \left\{ (n+1)^{\beta+\frac{1}{r}} \xi \left( \frac{1}{n+1} \right) \right\} \left[ \left\{ \int_0^{2\pi} dx \right\}^{\frac{1}{r}} \right] \\
&= O \left\{ (n+1)^{\beta+\frac{1}{r}} \xi \left( \frac{1}{n+1} \right) \right\}
\end{aligned}$$

This completes the proof of theorem 2.

## 5 Applications

Following corollaries can be derived from our main theorems:

**Corollary 6.** *If  $\xi(t) = t^\alpha$  then the weighted class  $W(L_r, \xi(t))$ ,  $1 \leq r < \infty$  reduces to the class  $Lip(\alpha, r)$  and then the degree of approximation of a function  $f$  belonging to the class  $Lip(\alpha, r)$ ,  $r^{-1} \leq \alpha \leq 1$  is given by*

$$\|C_n^2 E_n^1 - f(x)\|_r = O \left\{ \frac{1}{(n+1)^{\alpha-\frac{1}{r}}} \right\}.$$

*Proof.* The result follows by setting  $\beta = 0$  in (2.5) □

**Corollary 7.** *If  $r \rightarrow \infty$  in corollary 6, then the class  $f \in Lip(\alpha, r)$  reduces to the class  $Lip\alpha$  and the degree of approximation of a function  $f$  belonging to the class  $Lip\alpha$ ,  $0 < \alpha < 1$  is given by*

$$\|C_n^2 E_n^1 - f(x)\|_r = O \left\{ \frac{1}{(n+1)^\alpha} \right\}$$

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## References

- [1] G. Alexits, *Convergence problems of orthogonal series*. Translated from German by I Földes. International series of Monographs in Pure and Applied Mathematics, vol. **20**, Pergamon Press, New York-Oxford-Paris, 1961. [MR0218827](#)(36:1911).
- [2] Prem Chandra, *Trigonometric approximation of functions in  $L^p$  norm*, J. Math. Anal. Appl. **275**(2002), No. 1, 13-26. [MR1941769](#)(2003h:42003). [Zbl 1011.42001](#).
- [3] G. H. Hardy, *Divergent series*, first edition, Oxford University Press, 1949, 70.
- [4] H. H. Khan, *On degree of approximation of functions belonging to the class  $Lip(\alpha, p)$* , Indian J. Pure Appl. Math., **5**(1974), no.2, 132-136. [MR0380230](#)(52:1130). [Zbl 0308.41010](#).
- [5] László Leindler, *Trigonometric approximation in  $L^p$  norm*, J. Math. Anal. Appl., **302**(2005). [MR2107350](#)(2005g:42004). [Zbl 1057.42004](#).
- [6] Leonard McFadden, *Absolute Nörlund summability*, Duke Math. J., **9**(1942), 168-207. [MR0006379](#)(3,295g). [Zbl 0061.12106](#).
- [7] K. Qureshi, *On the degree of approximation of a periodic function  $f$  by almost Nörlund means*, Tamkang J. Math., **12**(1981), No. 1, 35-38. [MR0714492](#)(85h:42002). [Zbl 0502.42002](#).
- [8] K. Qureshi, *On the degree of approximation of a function belonging to the class  $Lip\alpha$* , Indian J. pure Appl. Math., **13**(1982), no. 8, 898-903. [MR0670332](#)(84c:41013). [Zbl 0498.42002](#).
- [9] K. Qureshi, *On the degree of approximation of a function belonging to weighted  $W(L(r), \xi(t))$  class*, Indian J. pure Appl. Math., **13**(1982), no. 4, 471-475-903. [MR0653599](#)(83f:41014b).
- [10] K. Qureshi, and H. K. Neha, *A class of functions and their degree of approximation*, Ganita, **41**(1990), no. 1, 37-42. [MR1155949](#). [Zbl 0856.42005](#).
- [11] B. E. Rhaodes, *On the degree of approximation of functions belonging to Lipschitz class by Hausdorff means of its Fourier series*, Tamkang Journal of Mathematics, **34**(2003), no. 3, 245-247. [MR2001920](#)(2004g:41023). [Zbl 1039.42001](#).
- [12] B.N. Sahney, and D.S. Goel, *On the degree of continuous functions*, Ranchi University Math. Jour., **4**(1973), 50-53. [MR0352846](#)(50:5332). [Zbl 0296.41008](#).
- [13] E.C. Titchmarsh, *The Theory of functions*, Oxford University Press, 1939, 402-403.

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- [14] A. Zygmund, *Trigonometric series*, 2nd rev. ed., Vol. 1, Cambridge Univ. Press, Cambridge, New York, 1959. [MR0107776\(21:6498\)](#). [Zbl 0085.05601](#).

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