Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 14 (2019), 307 – 325

NONLOCAL FRACTIONAL DIFFERENTIAL INCLUSIONS WITH IMPULSES AT VARIABLE TIMES

Abdelghani Ouahab and Sarah Seghiri

Abstract. In this paper, we study the existence of mild solutions for a fractional semi-linear differential inclusions posed in a Banach space with nonlocal conditions and impulses at variable times. The main existence result is obtained by using fractional calculus, measure of noncompactness, and multivalued fixed point theory. We study also the topological properties of the solution set.

1 Introduction

Differential equations and inclusions of fractional order appear in many physical phenomena of engineering science, such as problems in electro-chemistry, electro-magnetic, . . . (see, e.g., [24], [31], [35]). Many evolution processes in physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations. These perturbations may be seen as impulses. Differential equations with impulses were first considered by Milman and Myshkis [34]. Since then, several research works have been published. The monograph by Halanay and Wexler [22] presents the first impulsive problems.
Particular attention has been given to differential equations and inclusions with impulses at variable moments (see the papers of Bajo and Liz [8], Belarbi and Benchohra [10], Benchohra *et al.* [12], Agarwal et al [3] and Benchohra and Slimani [13] have considered impulsive fractional differential equations at variable moments. The results was extended to the multivalued case by Ait dads *et al.* [4]. More recently Cardinaly and Rubbioni [17] studied a nonlocal Cauchy problem in the present of impulses governed by a nonautonomous semi-linear differential inclusion. The study of semi-linear nonlocal initial value problem was initiated by Byszewski [15], and then followed by many works (see, e.g., [16], [21], [32], [18]).
In this paper, we are concerned with the following fractional semi-linear differential

2010 Mathematics Subject Classification: 47H10; 26A33; 34A60; 34B37; 14F45.
Keywords: Fixed point theorems; Fractional derivatives; Differential inclusions; Boundary value problems with impulses; Topological properties.

inclusion:

$$^cD^\alpha y(t) \in Ay(t) + F(t, y(t)), \quad t \in J,\, t \neq \tau_k(y(t)),\, k = \overline{1,m} \tag{1.1}$$

$$y(t^+) = I_k(y(t)), \quad t = \tau_k(y(t)),\ k = \overline{1,m} \tag{1.2}$$

$$y(0) = g(y), \tag{1.3}$$

where $J = (0, T)$ and $\overline{1,m} = \{1, 2, \ldots, m\}$. $0 < \alpha < 1$, $^cD^\alpha$ is the Caputo fractional derivative, $A : D(A) \subset X \longrightarrow X$ is the infinitesimal generator of a strongly continuous semigroup (C_0-semigroup) $\{T(t)\}_{t\geqslant 0}$ on X with $D(A)$ representing the domain of the linear operator A. $F : J \times X \longrightarrow X$ is a Carathéodory multi-function, X is an ordered reflexive Banach space with norm $\|\cdot\|$, and the nonlocal term g is a given function.

Finally $\tau_k : X \longrightarrow \mathbb{R}$ and $I_k : X \longrightarrow X$ for $k = \overline{1,m}$. Since $\{T(t)\}_{t\geqslant 0}$ is strongly continuous, there exists a constant M such that $M = \sup\limits_{t\in J} \|T(t)\| < \infty$. $\overline{1,m}$ stands for the set $\{1, 2, \ldots, \}$ and $y(t^+) = \lim\limits_{s\to t^+} y(s)$.

Differential inclusions of the form (1.1) were first considered by Aizicovici and Gao [6] when g and $T(t)$ are compact. In [7] and [33], the authors discussed (1.1) when A generates a compact semigroup. Finally, we mention Lian *et al.* [32] who studied the existence of solutions to problem (1.1)-(1.3) without impulses.

In the study of the topological structure of the solution sets of differential equations and inclusions, an important aspect is the $R_\delta-$ property, which includes acyclicity (in particular, compactness and connectedness). An R_δ-set may not be a singleton but, from the point of view of algebraic topology, it is equivalent to a point, in the sense that it has the same cohomology groups as one point space. The topological structure of solution sets of differential inclusions on compact intervals has been recently investigated by many authors, see Aronszajn [2], Deimling [18], Hu and Papageorgiou [25], and Peng and Zhou [40].

The aim of this paper is to extend the results of Lian *et al.*[32] when impulses at variable times are involved. In section 2 we start with some backgrounds on multivalued analysis, fractional derivatives, and measure of noncompactness. In section 3, we define a generalized Cauchy operator and give some related properties. Section 4 is devoted to the existence of solutions for problem (1.1)-(1.3). The topological structure of the solution set is investigated in section 5, where some elements from algebraic topology are used.

2 Preliminaries

In this section we introduce some background material used throughout this paper. For more definitions and details about the multivalued mappings, we refer, e.g., to

[5], and [26]. In order to define the solution of problem (1.1)-(1.3) we shall consider the space of functions

$$\Sigma = \{y : \bar{J} \longrightarrow X : \text{ there exist } 0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T \\ \text{such that } t_k = \tau_k(y(t_k)),\ y(t_k^+) \text{ exists } k = \overline{1,m} \text{ and } y_k \in C((t_{k-1}, t_k], X), \\ k = \overline{1, m+1}\},$$

where X is an ordered reflexive Banach space, y_k is the restriction of y over $(t_{k-1}, t_k]$, for $k = \overline{1, m+1}$, and $y(t_k^+) = \lim\limits_{t \to t_k^+} y(t)$.

$L^1(J, X)$ will denote the Banach space of measurable functions from J into X which are Bochner integrable and $\mathcal{L}(X)$ denote the space of bounded linear operator from X into X. Consider the following subsets of X:

$$\mathcal{P}_{cl}(X) = \{Y \in P(X) : Y \text{ is closed }\}$$
$$\mathcal{P}_{cp}(X) = \{Y \in P(X) : Y \text{ is compact }\}$$
$$\mathcal{P}_{cv}(X) = \{Y \in P(X) : Y \text{ is convex }\}$$
$$\mathcal{P}_{cp,cv}(X) = P_{cp}(X) \cap P_{cv}(X).$$

Definition 1. *Let X and Y be two topological spaces and $F : X \longrightarrow \mathcal{P}(Y)$ a multivalued function.*
(1) F is said to be compact (convex) valued if $F(x)$ is compact (convex) in Y for all $x \in X$.
(2) F is said to be upper semi-continuous (u.s.c.) on X if $F^{-1}(V) = \{x \in X / F(x) \subset V\}$ is an open subset of X for every open subset V of Y.
(3) F is said to be closed if its graph $G_F = \{(x, y) \in X \times Y : y \in F(x)\}$ is a closed subset of the topological space $X \times Y$, that is $x_n \to x$, $y_n \to y$ and $y_n \in F(x_n)$ imply $y \in F(x)$.
(4)If $Y = X$, a point x of X is said to be fixed point of F if $x \in F(x)$.
(5) A function $f : X \longrightarrow Y$ is said to be selection of F if $f(x) \in F(x)$ for every $x \in X$.

Definition 2. *A sequence $\{f_n\}_{n=1}^{\infty} \subset L^1(J, X)$ is said to be semi-compact if:*
(i) it is integrably bounded, that is, there exists $\omega \in L^1(J, X)$ such that $\|f_n(t)\| \leq \omega(t)$, for a.e. $t \in J$ and every $n \geq 1$,
(ii) the set $\{f_n(t)\}_{n=1}^{\infty}$ is relatively compact in X for a.e. $t \in J$.

Lemma 3. *[28] Every semi-compact sequence in $L^1(J, X)$ is weakly compact in $L^1(J, X)$.*

Definition 4. *A multivalued map $F : J \times X \longrightarrow \mathcal{P}(X)$ is said to be Carathéodory if:*
(i) $t \longmapsto F(t, y)$ is measurable for each $y \in X$,
(ii) $y \longmapsto F(t, y)$ is u.s.c. for almost all $t \in J$.

It is further an L^1-Carathéodory if it is locally integrably bounded, i.e. for each positive r, there exists some $h_r \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t,z)\| \leq h_r(t) \ \text{ for a.e. } \ t \in J \ \text{ and all } \|z\| \leq r.$$

For each $y \in \Sigma$, define the set of selections of F by

$$\mathcal{S}_F(y) = \{f \in L^1(J,X) : f(t) \in F(t, y(t)), \ \text{ for a.e. } \ t \in J\}.$$

When F is an L^1-Carathéodory multi-valued mapping, we know from a result due to Lasota and Opial [30] that for each $y \in C((t_{k-1}, t_k)$, the set $S_F(y)$ contains functions $f_k \in L^1((t_{k-1}, t_k)$, $k = \overline{1, m}$.

Lemma 5. *[30] Let $F : J \times X \longrightarrow \mathcal{P}_{cp,cv}(X)$ be a Carathéodory multivalued map and let G be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$, then the operator*

$$G \circ S_F : C(J,X) \longrightarrow \mathcal{P}_{cp,cv}(X),$$

where $(G \circ S_F)(y) = G(S_F(y))$, is a closed graph operator in $C(J,X) \times C(J,X)$.

Next some properties related to measure of non-compactness are recalled [28].

Definition 6. *Let X be a Banach space and $(\mathcal{A}, \succeq)$ a partially ordered set. A function $\gamma : \mathcal{P}(X) \longrightarrow A$ is called a measure of non-compactness (for short M.N.C) in X if:*

$$\gamma(\overline{co}\,\Omega) = \gamma(\Omega), \ \ \text{for every } \ \Omega \in P(X),$$

where $co\Omega$ is the convex hull of Ω. A measure of non-compactness γ is called:
(a) monotone if for $\Omega_0, \Omega_1 \in \mathcal{P}(X)$, $\Omega_0 \subset \Omega_1 \Longrightarrow \gamma(\Omega_0) \leqslant \gamma(\Omega_1)$,
(b) nonsingular if $\gamma(\{a\} \bigcup \Omega) = \gamma(\Omega)$, for every $a \in X$ and $\Omega \in \mathcal{P}(X)$,
(c) real if $A = [0, \infty]$ with natural ordering and $\Omega \in \mathcal{P}(X)$,
(d) regular if $\gamma(\Omega) = 0$ is equivalent to the relative compactness of Ω.

We recall that for a bounded subset Ω of X, the Hausdorff M.N.C β is defined by

$$\beta(\Omega) = \{\varepsilon > 0 : \ \Omega \ has \ a \ finite \ \varepsilon - net \ in \ X\}.$$

We note that the Hausdorff MNC satisfies the above properties. Moreover, we have

Lemma 7. *[9] If $w \subset C(J,X)$ is bounded and equicontinuous, then $\beta(w(t))$ is continuous on J and*

$$\beta(w) = \sup_{t \in J} \beta(w(t)).$$

Lemma 8. *[23] If $\{u_n\}_{n=1}^{\infty} \subset L^1(J,X)$ is integrably bounded, then $\beta(\{U_n(t)\}_{n=1}^{\infty})$ is measurable and*

$$\beta\left(\left\{\int_0^t u_n(s)\right\}_{n=1}^{\infty}\right) \leqslant 2\int_0^t \beta(\{u_n(s)\}_{n=1}^{\infty})ds.$$

Lemma 9. *[36] If $B \subset X$ is bounded, then for each $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^{\infty}$ in B such that*

$$\beta(B) \leqslant 2\beta(\{u_n\}_{n=1}^{\infty}) + \varepsilon.$$

Definition 10. *Let W be a closed subset of a Banach space X and γ a measure of non-compactness on X. A multi-mapping $F : W \longrightarrow \mathcal{P}_{cp}(X)$ is said to be γ-condensing if for every $\Omega \subset W$, the relation $\gamma(F(\Omega)) \geqslant \gamma(\Omega)$, implies the relative compactness of Ω.*

We will make use of the following fixed point theorem.

Theorem 11. *[38] If M is a closed bounded and convex subset of a Banach space X and $F : M \longrightarrow \mathcal{P}_{cp}(M)$ is a closed $\gamma-$condensing multi-mapping, where γ is a monotone MNC defined on subsets of M. Then the fixed point set $Fix\, F = \{x \in M : \ x \in F(x)\}$ is nonempty and compact.*

Definition 12. *Let $\alpha > 0$ and $f \in L^1(J, X)$, then the fractional order integral of f of order α is defined by*

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \ t > 0,$$

where $\Gamma(.)$ is the Euler gamma function.

The basic definitions of fractional derivative and fractional integral are presented below. For more details on the fractional calculus, we refer the reader to [29] and [37].

Definition 13. *The Caputo derivative of order $\alpha > 0$ of a function $f : J \longrightarrow X$ is defined as*

$${}^cD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \ t > 0,$$

where $n = [\alpha] + 1$. If $0 < \alpha < 1$, then

$${}^cD_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds.$$

Before considering problem (1.1)-(1.3), let us start with the following problem wheach is already discussed in [32]

$${}^cD^{\alpha} y(t) \in Ay(t) + F(t, y(t)), \quad t \in J = [0, T], \tag{2.1}$$

$$y(0) = g(y). \tag{2.2}$$

Definition 14. *A function $y \in C(J,X)$ is said to be a mild solution of problem (2.1)-(2.2) if $y(0) = g(y)$ and there exists $f \in L^1(J,X)$ such that $f \in \mathcal{S}_F(y)$ and*

$$y(t) = S_\alpha(t)g(y) + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s)ds, \quad \text{for } t \in J,$$

where

$$S_\alpha(t) = \int_0^\infty h_\alpha(\theta) T(t^\alpha \theta) d\theta, \tag{2.3}$$

$$P_\alpha(t) = \alpha \int_0^\infty \theta h_\alpha(\theta) T(t^\alpha \theta) d\theta. \tag{2.4}$$

Here h_α is the the probability density function on $(0,\infty)$ given by

$$h_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}} \overline{\omega_\alpha}(\theta^{-\frac{1}{\alpha}}), \tag{2.5}$$

where

$$\overline{\omega_\alpha}(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(\pi n\alpha), \; \theta \in (0,\infty). \tag{2.6}$$

Note that $h_\alpha(\theta) \geqslant 0$ for $\theta \in (0,\infty)$ and

$$\int_0^\infty h_\alpha(\theta) d\theta = 1, \tag{2.7}$$

$$\int_0^\infty \theta^\delta h_\alpha(\theta) d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\alpha\delta)}, \; \delta \in [0,1]. \tag{2.8}$$

Lemma 15. *[27] The linear operators $S_\alpha(t)$ and $P_\alpha(t)$ have the following properties:*

(1) For any fixed $t > 0$, $S_\alpha(t)$ and $P_\alpha(t)$ are bounded operators. More precisely for any $x \in X$, we have

$$\|S_\alpha(t)x\| \leqslant M\|x\|, \tag{2.9}$$

$$\|P_\alpha(t)x\| \leqslant \frac{M\alpha}{\Gamma(1+\alpha)}\|x\|, \tag{2.10}$$

where $M = \sup_{t\in J}\|T(t)\|$.
(2) Operators $S_\alpha(t)$ and $P_\alpha(t)$ are equicontinuous for $t \in J$ if $\{T(t)\}_{t\geqslant 0}$ is equicontinuous.

The following result is easily checked.

Lemma 16. *For $\theta \in (0,1)$ and $0 < a \leqslant b$, we have*

$$|a^\theta - b^\theta| \leqslant (b-a)^\theta.$$

3 Main existence result

Definition 17. *A function y of Σ is said to be a solution of problem (1.1)-(1.3) if there exists a function $h \in L^1(J, X)$ such that*

$$h(t) \in Ay(t) + F(t, y(t)), \quad \textit{for a.e. } \ t \in J$$

and satisfies the equation

$$^cD^{\alpha}_{t_k}(y(t)) = h(t), \quad \textit{for a.e. } \ t \in (t_k, t_{k+1}],\ t \neq \tau_k(y(t)),\ k = \overline{1, m}$$

and the conditions $y(t^+) = I_k(y(t))$, $t = \tau_k(y(t))$, $k = \overline{1, m}$ and $y(0) = g(y)$ are satisfied.

To prove the existence of mild solution for the problem (1.1)-(1.3), we list some assumptions:

(H1) The C_0-semigroup $\{T(t)\}_{t\geqslant 0}$ generated by the linear infinitesimal operator A is equicontinuous.

(H2) The operator $g : \Sigma \longrightarrow X$ is continuous and compact.

(H3) The multivalued mapping $F : J \times X \longrightarrow P(X)$ is Carathéodory, has compact and convex values, and satisfies:

  (1) there exist a nondecreasing continuous function $\psi : [0, \infty) \longrightarrow [0, \infty)$ and $q \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, y)\| \leqslant q(t)\psi(\|y\|), \quad \text{a.e. } \ t \in J \ \text{ and all } \ y \in X,$$

  (2) there exists a function $l \in L^1(J, X)$ such that for every bounded $D \subset X$: $\beta(F(t, D)) \leqslant l(t)\beta(D)$, $t \in J$.

(H4) The functions $\tau_k \in C(X, \mathbb{R})$ $(k = \overline{1, m})$ satisfy

$$0 < \tau_1(z) < \tau_2(z) < \ldots < \tau_m(z) < T, \quad \text{for all } \ z \in X.$$

(H5) The functions $I_k : X \longrightarrow X$, $k = \overline{1, m}$ are continuous nondecreasing and verify:

$$\tau_k(I_k(z)) < \tau_k(z) < \tau_{k+1}(I_k(z)), \quad \text{for all } \ z \in X.$$

(H6) For all $y \in C(J, X)$ and $k \in \overline{1, m}$, the set $E_k = E_k(y) = \{t \in [0, T] : \tau_k(y(t)) = t\}$ is finite, for all $k = \overline{1, m}$.

Theorem 18. *[32] Assume that hypotheses (H1)-(H3) are satisfied and suppose that*

$$\lim_{k\to\infty} \sup_{k} \left\{ \frac{M}{k} \left(\mu(k) + \frac{\Psi(k)}{\Gamma(\alpha+1)} q^0 T^\alpha \right) \right\}) < 1, \tag{3.1}$$

where $\mu(k) = \sup\{\|g(y)\| \ /\|y\| \leqslant k\}$, $q^0 = \sup\{q(t) : t \in J\}$. *Then the fractional differential inclusion (2.1)-(2.2) has at least one mild solution on* J *and a compact solution set.*

Theorem 19. *Assume that hypotheses of theorem 3.1 are satisfies. if the conditions (H4)-(H6) hold, then the problem (1.1)-(1.3) has a nonempty compact mild solution set.*

Proof. The proof will be given in several steps.

Step 1 . Using Theorem 3.1, the problem (2.1)-(2.2) has at least one mild solution.

Step 2 . Let y_1 be a solution of problem (2.1)-(2.2). For $k = \overline{1,m}$, define the function

$$r_{k,1}(t) = \tau_k(y_1(t)) - t, \quad \text{for} \ \ t > 0.$$

The condition $(H4)$ implies that $r_{k,1}(0) \neq 0$ for all $k = \overline{1,m}$. If $r_{k,1}(t) \neq 0$ on J for all $k = \overline{1,m}$, then y_1 is a solution of problem (1.1)-(1.3).
Consider the case when $r_{1,1}(t) = 0$ for some $t \in (0,T]$. Since $r_{1,1}(0) \neq 0$ and $r_{1,1}$ is continuous, then the set $E_1 = \{t_1^i, i \in I\}$ is nonempty and from $(H6)$, E_1 is finite. We distinguish between two cases:

Case 1. If $E_1 = \{t_1\}$ then $r_{1,1}(t_1)) = 0$ and $r_{1,1}(t) \neq 0, \forall t \in (0,t_1)$. By $(H3)$, $r_{k,1}(t) \neq 0, \forall t \in (0,t_1]$ and $k = \overline{1,m}$. Hence y_1 is a solution of the problem

$$\begin{array}{rcl} {}^cD^\alpha y(t) & \in & Ay(t) + F(t,y(t)), \ t \in [0,t_1] \\ y(0) & = & g(y). \end{array}$$

Case 2. If $E_1 = \{t_1^i, i \in I\}$ is finite, take $t_1 = \max E_1$ and consider the problem

$${}^cD^\alpha_{t_1} y(t) \in Ay(t) + F(t,y(t)), \quad \text{for a.e.} \ \ t \in [0,t_1], \ t \neq t_1^i, \ i \in I,$$

with the impulsive conditions $y(t_1^{i+}) = I_1(y(t_1^i))$, $i \in I$ and the initial condition $y(0) = g(y)$. The solution for the above problem reads

$$\overline{y_1}(t) = \begin{cases} y_1(t), \ \text{if } t \in [0,t_1^1] \\ S_\alpha(t-t_1^i)I_1(y(t_1^i)) + \int_{t_1^i}^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s)ds, \\ \qquad \text{if } t \in (t_1^i, t_1^{i+1}], \ i \in I. \end{cases}$$

We have

$$r_{1,1}(t_1) = 0 \text{ and } r_{1,1}(t) \neq 0 \text{ for } t \in (0, t_1).$$

By $(H4)$,

$$r_{k,1}(t) \neq 0 \; for \; all \; t \in [0, t_1) \; and \; k = \overline{1, m}$$

and this $\overline{y_1}$ is a solution on $[0, t_1]$.

Step 3. Consider the problem

$$^cD^{\alpha}_{t_1} y(t) \in Ay(t) + F(t, y(t)), \text{ for a.e. } t \in [t_1, T], \tag{3.2}$$

$$y(t_1^+) = I_1(y_1(t_1)) \tag{3.3}$$

and the operator $R_1 : C([t_1, T], X) \longrightarrow P(C([t_1, T], X))$ defined by

$$\begin{aligned} R_1(y) = \; & \{h \in C([t_1, T], X) : \\ & h(t) = S_\alpha(t - t_1)I_1(y_1(t_1)) + \textstyle\int_{t_1}^t (t - s)^{\alpha - 1} P_\alpha(t - s) f(s) ds, \\ & f \in \mathcal{S}_F(y)\}. \end{aligned}$$

Operator R_1 is well defined. Indeed for $t = t_1$, $h(t_1) = I_1(y(t_1))$. As in Step 1, we can show that R_1 satisfies the assumptions of Theorem 11 and deduce that problem (3.2)-(3.3) has a nonempty compact solution set. Denote a solution of (3.2)-(3.3) by y_2. and consider the map $r_{k,2}(t) = \tau_k(y_2(t)) - t$ for $t \geqslant t_1$. If $r_{k,2}(t) \neq 0$ for $t \in (t_1, T]$ and $k = \overline{1, k}$, then

$$y(t) = \begin{cases} \overline{y_1}(t), & \text{for } t \in [0, t_1] \\ y_2(t), & \text{for } t \in (t_1, T]. \end{cases}$$

is a solution of problem (1.1)-(1.3). Moreover, when $r_{2,2}(t) = 0$ for some $t \in (t_1, T]$, by $(H5)$ we have

$$\begin{aligned} r_{2,2}(t_1^+) &= \tau_2(y_2(t_1^+)) - t_1 \\ &= \tau_2(I_1(y_1(t_1))) - t_1 \\ &> \tau_1(y_1(t_1)) - t_1 \\ &= r_{1,1}(t_1) \\ &= 0. \end{aligned}$$

Since $r_{2,2}$ is continuous and $r_{2,2}(t_1) > 0$, the set E_2 is nonempty and from $(H6)$ E_2 is finite. Let $E_2 = \{t_2^i, \; i \in I'\}$. Then we consider two cases:

Case 1. If $E_2 = \{t_2\}$ then $r_{2,2}(t_2) = 0$ and $r_{2,2}(t) \neq 0$ for $t \in (t_1, t_2)$. We have $r_{k,2}(t) =\neq 0$ for all $t \in (t_1, t_2)$ and $k = \overline{2, m}$. For $k = 1$, we have

$$\begin{aligned} r_{1,2}(t_1) &= \tau_1(y_2(t_1)) - t_1 \\ &= \tau_1(I_1(y_1(t_1))) - t_1 \\ &\leqslant \tau_1(y_1(t_1)) - t_1 \\ &= r_{1,1}(t_1) \\ &= 0, \end{aligned}$$

i.e., $r_{1,2}(t_1) < 0$. Furthermore, from $(H4)$ we have

$$\begin{array}{rcl} r_{1,2}(t_2) & = & \tau_1(y_2(t_2)) - t_2 \\ & < & \tau_2(y_2(t_2)) - t_2 \\ & = & r_{2,2}(t_2) = 0, \end{array}$$

i.e., $r_{1,2}(t_2) < 0$ and we know that $\forall t > t_1$, $\tau_1(y(t)) \neq t$. Then for $t \in (t_1, t_2)$ $\tau_1(y_2(t)) \neq t$ i.e., $r_{1,2}(t) \neq 0$. Moreover

$$r_{1,2}(t) < 0, \quad \text{for } \ t \in (t_1, t_2).$$

We conclude that $r_{k,2}(t) \neq 0$, for $t \in (t_1, t_2)$ and $k = \overline{1, m}$.

Case 2 If $E_2 = \{t_2^i, i \in I'\}$ is finite, let $t_2 = \max E_2$. Then the solution of problem (3.2)-(3.3) over $(t_1, t_2]$ is

$$\overline{y_2}(t) = \begin{cases} y_2(t), \ \text{if } t \in [t_1, t_2^1] \\ S_\alpha(t - t_2^i) I_2(y(t_2^i)) + \int_{t_2^i}^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s) ds, \\ \qquad \text{if } t \in (t_2^i, t_2^{i+1}], \ i \in I'. \end{cases}$$

Step 4. We continue this process taking into account that $\overline{y_{m+1}} = y_{/[t_m, T]}$ is a solution to the problem

$$^cD_{t_m}^\alpha y(t) \in Ay(t) + F(t, y(t)), \ \text{ for a.e. } \ t \in (t_m, T], \tag{3.4}$$

$$y(t_m^+) = I_m(y_{m-1}(t_m^-)). \tag{3.5}$$

Finally the solution y of problem (1.1)-(1.3) is then defined by

$$y(t) = \begin{cases} \overline{y_1}(t), & \text{if } t \in [0, t_1] \\ \overline{y_2}(t), & \text{if } t \in (t_1, t_2]. \\ \dots & \\ \overline{y_{m+1}}(t), & \text{if } t \in (t_m, T]. \end{cases}$$

In addition the solution set of problem (1.1)-(1.3) is compact.

□

Example 20. *We consider the following fractional partial differential inclusion:*

$$\begin{cases} \partial_t^\alpha y(t,x) \in \partial_x^2 y(t,x) + G(t, y(t,x)), \ \textit{if } t \in [0,1] \textit{and } t \neq \tau_k(y(t,x)) \textit{for} \quad k = \overline{1,m}, \\ y(t^+, x) = I_k(y(t,x)), \quad \textit{if } t \neq \tau_k(y(t,x)) \quad \textit{for } k = \overline{1,m}, \\ y(t,0) = y(t,\pi) = 0, \\ y(0,x) = \int_0^1 h(s) \sin(1+ \mid y(s,x) \mid)) ds. \end{cases} \tag{3.6}$$

Where $X = L^2([0,\pi];\mathbb{R})$, ∂^α *is the Caputo fractional partial derivative of order* α *with* $0 < \alpha < 1$, $h \in L^1([0,1];\mathbb{R})$, *and* $G : [0,1] \times X \longrightarrow \mathcal{P}(X)$.
We define the operator $\mathcal{A}$ *by the Laplace operator, i.e.* $\mathcal{A} = \frac{\partial^2}{\partial x^2}$ *on the domain*

$$\mathcal{D}(\mathcal{A}) = \{w \in X,\ w, w^{'} \text{ are absolutly continuous and } w^{''} \in X,\ \ w(0) = w(\pi) = 0\}.$$

Clearly, $\mathcal{A}$ *generates a strongly continuous semigroup* $\{T(t),\ t \in [0,1]\}$.
Then the system above can be reformed as

$$\begin{cases} {}^cD^\alpha y(t) \in \mathcal{A}y(t) + F(t,\ y(t)), \quad t \in J = [0,1],\, t \neq \tau_k(y(t)),\, k = \overline{1,m}, \\ y(t^+) = I_k(y(t)), \quad t = \tau_k(y(t)),\ k = \overline{1,m}, \\ y(0) = g(y), \end{cases} \tag{3.7}$$

where $y(t)(x) = y(t,x)$, $t \in [0,1]$, $x \in [0,\pi]$, $\tau_k(y(t,x)) = \tau_k(y(t))(x)$, $I_k(y(t,x)) = I_k(y(t))(x)$ *and* $F(t,y(t))(x) = G(t,y(t,x))$ *Now we assume that* $F(t,y(t)) = \{f(t,y(t))\}$ *such that* $f : [0,1] \times X \longrightarrow X$ *is a defined continuous function. Assume that there are* $q \in L^1([0,1];\mathbb{R}^+)$ *and* $\psi : [0,\infty) \longrightarrow [0,\infty)$ *continuous and nondecreasing such that*

$$\| f(t,y) \| \leqslant q(t)\psi(\| y \|),$$

and assume that there exist $l \in L^1([0,1];X)$ *such that, for every bounded* $D \subset X$ *:* $\beta(f(t,D)) \leqslant l(t)\beta(D)$,
the function $g : [0,1] \times X \longrightarrow X$ *is given by* $g(y)(x) = \int_0^1 h(s)\sin(1+ \mid y(s,x) \mid)ds$ *is continuous and compact.*
Consider the functions

$$\tau_k(y(t)) = t^2 - \frac{1}{e^k(1+\|y\|_{L^2})} \text{ and } I_k(y) = b_k y \text{ where } b_k \in [\tfrac{1}{e},1], \text{ for } k = \overline{1,m}.$$

Both τ_k *and* I_k *are continuous for* $k = \overline{1,m}$ *and we have*

$$\tau_{k+1}(y(t)) - \tau_k(y(t)) = \frac{e-1}{e^{k+1}(1+ \| y \|_{L^2})} > 0, \text{ for each } k = \overline{1,m},$$

$$\tau_k(y(t)) - \tau_k(I_k(y)(t)) = \frac{(1-b_k) \| y \|_{L^2}}{e^k(1+b_k \| y \|_{L^2})(1+ \| y \|_{L^2})} \geqslant 0, \text{ for each } k = \overline{1,m}$$

and

$$\tau_{k+1}(I_k(y)(t)) - \tau_k(y(t)) = \frac{e-1+(eb_k-1) \| y \|_{L^2}}{e^{k+1}(1+ \| y \|_{L^2})(1+b_k \| y \|_{L^2})} > 0 \text{ for each } k = \overline{1,m}.$$

Suppose that there exists $k_0 > 0$ *such that*

$$M\left(\mu(k_0) + \frac{\Psi(k_0)}{\Gamma(\alpha+1)}q^0\right) < 1,$$

where $\mu(k_0) = \sup\{\|g(y)\| \ /\|y\| \leqslant k_0\}$, *and* $q^0 = \sup\{q(t), t \in J\}$ *We can verify easly that* F *satisfies the hypothesis* $(H3)$. *The equation* $\tau_k(y(t)) = t$ *is equivalent to* $t^2 - t - \frac{1}{e^k(1+\|y\|_{L^2})} = 0$ *wich admis two solution at maximum (finite solution set), then by theorem 4.1, the problem (3.6) admis at least one solution on* $[0,1]$.

4 Topological structure of solution set

In this section, we prove that the solution set is in fact R_δ, hence acyclicity. First, we recall the general theory (see, e.g., for more details [19]). Let (X,d) and (Y,d') be two metric spaces

Definition 21. *A set A of X is called a contractible space if there exists a continuous homotopy $h : A \times [0,1] \longrightarrow Y$ and $x_0 \in A$ such that*
(a) $h(x,0) = x$, for every $x \in A$,
(b) $h(x,1) = x_0$, for every $x \in A$,
i.e., if the identity map $id_A : A \longrightarrow A$is homotopic to a constant map.
In particular any closed convex subset of X is contractible.

Definition 22. *We say that a compact nonempty metric space X is an R_δ-set if there exists a decreasing sequence of compact nonempty contractible metric spaces $(X_n)_{n\in\mathbb{N}^*}$ such that $X = \bigcap_{n=1}^{\infty} X_n$.*

Let $H^n(X)$ denote the Čech cohomology in the space X with coefficients in a group G.

Definition 23. *A space A is called G-acyclic if $\bar{H}^n(A) = 0$, for every $n \geqslant 0$.*

Intuitively, acyclic set has no holes.

Proposition 24. *If A is R_δ-set then it is acyclic.*

An u.s.c map $F : X \longrightarrow P(Y)$ is called acyclic if for each $x \in X$, $F(x)$ is a compact acyclic set.

Theorem 25. *[19] Let $\varphi : X \longrightarrow \mathcal{P}_{cp,cv}(X)$ be an u.s.c multivalued map from metric space X to a Banach space E. If $\overline{\varphi(X)}$ is a compact set, then there exists a sequence of u.s.c mappings φ_n from X to $\overline{co}(\varphi(X))$ which approximates φ in the sense that, for all $x \in X$, we have:*

$$\varphi(x) \subset \ldots \subset \varphi_{n+1}(x) \subset \varphi_n(x) \subset \ldots \subset \varphi_0(x), \quad \text{for all } n \geqslant 0, \tag{4.1}$$

for all $\epsilon > 0$ there exists $n_0 = n_0(\epsilon, x)$ such that

$$\varphi_n(x) \subset \overline{\mathcal{O}_\epsilon(\varphi(x))}, \quad \text{for all } n \geqslant n_0. \tag{4.2}$$

Theorem 26. *Under conditions $(H1)-(H6)$, the solution set of problem (1.1)-(1.3) is an R_δ-set.*

Proof. First let us denote the set of mild solutions of problem (1.1)-(1.3) by $\mathsf{S}(g)$. By Theorem 5.1, there exists a sequence $(F_n)_{n\geqslant 0}$ that verifies (4.1) and (4.2). For every $n \geqslant 0$, consider the following semi-linear evolution inclusion:

$$^cD^\alpha y(t) \in Ay(t) + F_n(t, y(t)), \qquad t \in J = [0,T],\ t \neq \tau_k(y(t)),\ k = \overline{1,m} \tag{4.3}$$

$$y(t^+) = I_k(y(t)), \qquad t = \tau_k(y(t)),\ k = \overline{1,m} \tag{4.4}$$

$$y(0) = g(y). \tag{4.5}$$

Denote by $\mathsf{S}_n(g)$, the set of mild solutions of this problem. By Theorem 4.1, problem (4.3)-(4.5) has a nonempty compact solution set. Let $y^* = y^*[f^*]$, $(f^* \in \mathcal{S}_{F_n,y})$ be an element of $\mathsf{S}_n(g)$ and for $\lambda \in [0,1]$, let the problem

$$^cD^\alpha y(t) \in Ay(t) + F_n(t, y(t)), \qquad t \in [\lambda T, T],\ t \neq \tau_k(y(t)),\ k = \overline{1,m} \tag{4.6}$$

$$y(t^+) = I_k(y(t)), \qquad t = \tau_k(y(t)),\ k = \overline{1,m} \tag{4.7}$$

$$y(0) = y^*(t) \qquad t \in [0, \lambda T]. \tag{4.8}$$

We know that problem (4.6)-(4.8) has a nonempty compact solution set for every $y^* \in \mathsf{S}_n(g)$. Moreover the solution depends continuously on (λ, y^*). Denote this solution by $y[y^*, \lambda](t)$. Consider $h : \mathsf{S}_n(g) \times [0,1] \longrightarrow \mathsf{S}_n(g)$ be the mapping given by

$$h(y,\lambda)(t) = \begin{cases} y^*(t) & \text{if } t \in [0, \lambda T] \\ y[y^*, \lambda](t) & \text{if } t \in [\lambda T, T]. \end{cases}$$

Clearly $h(y,\lambda)(.)$ is a solution of the problem (4.3)-(4.5). In fact, note that for any $y \in \mathsf{S}_n(g)$, there exists $\widetilde{f} \in \mathcal{S}_{F,y}$ such that $y = y[\widetilde{f}]$. Let

$$\hat{f}(t) = \widetilde{f}\chi_{[\lambda T, T]}(t) + f^*(t)\chi_{[0,\lambda T]}(t),$$

for each $t \in [0,T]$ such that χ is the characteristic function. It is clear that $\hat{f} \in \mathcal{S}_{F_n,h}$ and it is checked that $y[\widetilde{f}] = y[\hat{f}]$ is a solution to (4.3)-(4.5)for $t \in [\lambda T, T]$ and $y[\widetilde{f}] = y^*[f^*]$ is the solution for $t \in [0, \lambda T]$. Hence $h(\lambda, y^*) \in \mathsf{S}_n(g)$.

To show that h is continuous, let $(y_k, \lambda_k) \in \mathsf{S}_n(g) \times [0,1]$ a sequence such that $(y_k, \lambda_k) \to (y, \lambda)$, as $k \to \infty$. Then

$$h(y_k,\lambda_k)(t) = \begin{cases} y_k(t), & \text{if } t \in [0, \lambda_k T], \\ y[y_k, \lambda_k](t), & \text{if } t \in [\lambda_k T, T]. \end{cases}$$

We check that $h(y_k, \lambda_k) \to h(y, \lambda)$, as $k \to \infty$. Without loss of generality, assume that $\lambda_k \leqslant \lambda$ and distinguish between three cases.

- If $t \in [\lambda T, T]$, then

$$\begin{aligned} \|h(y_k,\lambda_k) - h(y^*,\lambda)\|_{[\lambda T, T]} &= |y[y_k,\lambda](t) - y[y^*,\lambda](t)| \\ &= \sup_{t\in[\lambda T,T]} |y[y_k,\lambda](t) - y[y^*,\lambda](t)|, \end{aligned}$$

  which tends to 0, as $k \to \infty$ for $y[y^*, \lambda](t)$ depends continuously on (y^*, λ).

- If $t \in [0, \lambda_k T]$, then

$$\|h(y_m, \lambda_m) - h(y^*, \lambda)\| = |y_k(t) - y^*(t)|,$$

which tends to 0, as $k \to \infty$.

- If $t \in [\lambda_k T, \lambda T]$, then

$$\begin{array}{rcl} |h(y_k, \lambda_k)(t) - h(y^*, \lambda)(t)| & = & |y[y_k, \lambda_k](t) - y^*(t)|, \\ & \leqslant & |y[y_k, \lambda_k](t) - y(t)| + |y_k(t) - y^*(t)|, \\ & \longrightarrow & 0 \text{ as } k \to \infty. \end{array}$$

Moreover, for all $y \in \mathsf{S}_n(g)$, we have that

$$\begin{cases} h(y, 0) = y^*, \\ h(y, 1) = y[y^*, 1](t). \end{cases}$$

Hence the set $\mathsf{S}_n(g)$ is contractible for every $n \geqslant 0$. By Theorem 5.1, we have

$$F(t, y) \subseteq F_{n+1}(t, y) \subseteq F_n(t, y) \subseteq \ldots \subseteq F_1(t, y).$$

Hence

$$\mathsf{S}(g) \subseteq \mathsf{S}_{n+1}(g) \subseteq \mathsf{S}_n(g) \subseteq \ldots \subseteq \mathsf{S}_1(g),$$

which implies that

$$\mathsf{S}(g) \subseteq \bigcap_{n \in \mathbb{N}^*} \mathsf{S}_n(g).$$

To prove the converse, let $y \in \bigcap_{n \in \mathbb{N}^*} \mathsf{S}_n(g)$. Then there exists a sequence of selections $\{f_n\}_{n \in \mathbb{N}^*} \subset L^1([0, T], X)$ such that $f_n \in \mathcal{S}_{F_n, y}$ and $y = y[f_n]$ for all $n \in \mathbb{N}^*$. Let $\varepsilon > 0$. From (4.2), there exists $n_0 = n_0[\varepsilon, y]$ such that

$$F_n(x) \subset \overline{\mathcal{O}_\epsilon(F(x))}, \text{ for all } n \geqslant n_0.$$

Then

$$\|F_n(t, y)\| \leqslant \|F(t, y)\| + 2\varepsilon,$$

and without loss of generality

$$|f_n(t)| \leqslant q(t)\psi(\|y\|) + 2\varepsilon, \text{ for a. e. } t \in [0, T] \text{ and } n \geqslant n_0.$$

Hence the sequence $\{f_n\}_{n \in \mathbb{N}^*}$ is integrably bounded. From the reflexivity of X, there exists a subsequence of $\{f_n\}$ still denoted by $\{f_n\}$ such that $f_n \rightharpoonup f$ weakly , $f \in L^1([0, T], X)$. By Mazur's convexity theorem (see [14]), we obtain a sequence

$$\{\widetilde{f_n}\} \subset \overline{co}\{f_n : n \geqslant 1\}$$

such that $\widetilde{f}_n \to f$. Moreover $\widetilde{f}_n(t) \to f(t)$, for a. e. $t \in [0,T]$ and $f_n(t) \in F_n(t,y)$. Dfine by $\mathcal{N}$ the subset of $[0,T]$:

$$\mathcal{N} = \{t \in [0,T] : \widetilde{f}_n(t) \to f(t)\}.$$

For $t \in \mathcal{N}$, we have

$$\begin{aligned} \|\widetilde{f}_n(t)\| &\leqslant \|F_n(t,y(t))\|, \\ &\leqslant \|F(t,y(t))\| + 2\varepsilon. \end{aligned}$$

Since F has convex closed values, we conclude that $f(t) \in F(t,y(t))$ for $t \in \mathcal{N}$. Moreover $y(t) = \overline{y}_k(t)$ (for some $k = \overline{1,m}$) and $y = \overline{y}k[f_n]$, then from Theorem 3.1 we get $Gf_n \to Gf$, which implies that $\overline{y_k}[f_n] \to \overline{y_k}[f]$. We deduce that $y \in \mathsf{S}(g)$ and that $\mathsf{S}(g) = \bigcap_{n\in\mathbb{N}^*} \mathsf{S}_n(g)$. Finally $\mathsf{S}(g)$ is R_δ-set. □

Corollary 27. *The solution set for problem (1.1)-(1.3) is acyclic.*

References

[1] A. Ambrosetti, *Un teorema di esistenza per le equazioni differenziali negli spazi di Banach* (Italian), Rend. Sem. Mat. Univ. Padova **39** (1967), 349–361. MR0222426. Zbl 0174.46001.

[2] N. Aronszajn, *Le correspondant topologique de l'unicité dans la théorie des équations differentielles* (French) Ann. of Math. (2) **43** (1942), 730–738. MR0007195. Zbl 0061.17106

[3] R. P. Agarwal, M. Benchohra and B.A. Slimani, *Existence results for differential equations with fractional order and impulses*, Mem. Differential Equations Math. Phys. **44** (2008), 1–21. MR2527032. Zbl 1178.26006.

[4] E. Ait Dads, M. Benchohra, and S. Hamani, *Impulsive fractional differential inclusions involving the Caputo fractional derivative*, Fract. Calc. Appl. Anal. **12** (2009), no. 1, 15–38. MR2494428. Zbl 1179.26012.

[5] J.P. Aubin, H. Frankowska, *Set-Valued Analysis*, Reprint of the 1990 edition. Modern Birkhäuser Classics. Birkh?user Boston, Inc., Boston, MA, 2009. MR2458436. Zbl 1168.49014.

[6] S. Aizicovici, Y. Gao, *Functional-differential equations with nonlocal initial conditions,* J. Appl. Math. Stochastic Anal. **10** (1997), no. 2, 145–156. MR1453467. Zbl 0883.34065.

[7] S. Aizicovici, H. Lee, *Nonlinear nonlocal Cauchy problems in Banach spaces,* Appl. Math. Lett. **18** (2005), no. 4, 401–407. MR2124297. Zbl 1084.34002.

[8] I. Bajo and E. Liz, *Periodic boundary value problem for first order differential equations with impulses at variable times*, J. Math. Anal. Appl. **204** (1996), no. 1, 65–73. MR1418522. Zbl 0876.34020.

[9] J. Banas and K. Goebel, *Measure of Noncompactness in Banach Spaces*, Lect. Notes Pure Appl. Math., vol. **60**. Dekker, New York, 1980. MR0591679. Zbl 0441.47056.

[10] A. Belarbi and M. Benchohra, *Existence theory for perturbed impulsive hyperbolic differential inclusions with variable times*, J. Math. Anal. Appl. **327** (2007), no. 2, 1116–1129. MR2279992. Zbl 1122.35148.

[11] M. Benchohra, F. Berhoun and J.J. Nieto, *Fractional differential inclusions with impulses at variable times*, Adv. Dyn. Syst. Appl. **7** (2012), no. 1, 1–15. MR2911988.

[12] M. Benchohra, J.R. Graef, S.K. Ntouyas and A. Ouahab, *Upper and lower solutions method for impulsive differential inclusions with nonlinear boundary conditions and variable times*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **12** (2005), no. 3-4, 383–396. MR2127026. Zbl 1085.34007.

[13] M. Benchohra and B.A. Slimani, *Existence and uniqueness of solutions to impulsive fractional differential equations*, Electron. J. Differential Equations 2009, No. 10, 11 pp. MR2471119. Zbl 1178.34004.

[14] H. Brezis, *Functional analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2011. MR2759829. Zbl 1220.46002.

[15] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl. **162** (1991), no. 2, 494–505. MR1137634. Zbl 0748.34040.

[16] L. Byszewski and V. Lakshmikantham, *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space*, Appl. Anal. **40** (1991), no. 1, 11–19. MR1121321. Zbl 0694.34001.

[17] T. Cardinali and P. Rubbioni, *Impulsive mild solution for semilinear differential inclusions with nonlocal conditions in Banach spaces*, Nonlinear Anal. **75** (2012), no. 2, 871–879. MR2847463. Zbl 1252.34068.

[18] K. Deimling, *Multivalued Differential Equations*, De Gruyter Series in Nonlinear Analysis and Applications, 1. Walter de Gruyter & Co., Berlin, 1992. MR1189795. Zbl 0559.47040.

[19] S. Djebali, L. Górniewicz and A. Ouahab, *Solution Sets for Differential Equations and Inclusions*, De Gruyter Series in Nonlinear Analysis and

Applications, **18** Walter de Gruyter & Co., Berlin, 2013. MR2987468. Zbl 1258.34002.

[20] M.M. El-Borai, *Some probability densities and fundamental solutions of fractional evolution equations*, Chaos Solitons Fractals **14** (2002), no. 3, 433–440. MR1903295. Zbl 1005.34051.

[21] Z. Fan and G. Li, *Existence results for semilinear differential equations with nonlocal and impulsive conditions*, J. Funct. Anal. **258** (2010), no. 5, 1709–1727. MR2566317. Zbl 1193.35099.

[22] A. Halanay and D. Wexler, *Teoria calitativa a sistemelor cu impulsuri*, Editura Republicii Socialiste Romania, Bucuresti, 1968.

[23] H.P. Heinz, *On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Anal. TMA **7** (1983) 1351-1371. MR2726478. Zbl 0528.47046.

[24] R. Hilfer, *Applications of Fractional Calculus in Physics,* World Scientific Publishing Co., Inc., River Edge, NJ, Singapore, 2000. MR1890104. Zbl 0998.26002.

[25] S. Hu and N. S. Papageorgiou, *On the topological regularity of the solution set of differential inclusions with constraints,* J. Differential Equations **107** (1994), no. 2, 280–289. MR1264523. Zbl 0796.34017.

[26] S. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis*, Vol. I. Theory. Mathematics and its Applications, **419** Kluwer Academic Publishers, Dordrecht, (1997). MR1485775. Zbl 0887.47001.

[27] S. Ji and G. Li, *Solutions to nonlocal fractional differential equations using a noncompact semigroup*, Electron. J. Differential Equations (2013), No. 240, 14 pp. MR3138815. Zbl 1295.34010.

[28] M. Kamenskii, V. Obukhovskii and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, De Gruyter Series in Nonlinear Analysis and Applications, vol. **7** de Gruyter, Berlin (2001). MR1831201. Zbl 0988.34001.

[29] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, **204** Elsevier Science B.V., Amsterdam, (2006). MR2218073. Zbl 1092.450031.

[30] A. Lasota and Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. **13** (1965), 781-786. MR2196178. Zbl 0151.10703.

[31] V. Lakshmikantham and A.S. Vatsala, *Basic theory of fractional differential equations,* Nonlinear Anal. **69** (2008), no. 8, 2677–2682. MR2446361. Zbl 1161.340013.

[32] T. Lian, C. Xue and S. Deng, *Mild solutions to fractional differential inclusions with nonlocal conditions*, Bound. Value Probl. 2016, Paper No. 219, 16 pp. MR3581676. Zbl 1361.35197.

[33] J. Liang, J. Liu and T. Xiao, *Nonlocal Cauchy problems governed by compact operator families,* Nonlinear Anal. **57** (2004), no. 2, 183–189. RM2056425. Zbl 1083.34045

[34] V.D. Milman and A.A. Myshkis, *On the stability of motion in the presence of impulses (Russian)*, Sibirsk. Mat. Ž. **1** (1960), 233–237 MR0126028. Zbl 1358.34022.

[35] K.B. Oldham and J. Spanier, *The Fractional Calculus. Theory and applications of differentiation and integration to arbitrary order*, With an annotated chronological bibliography by Bertram Ross. Mathematics in Science and Engineering, Vol. **111** Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974. MR0361633. Zbl 0292.26011.

[36] S. Plaskacz, *Periodic solutions of differential inclusions on compact subsets of* $\mathbb{R}^n$, J. Math. Anal. Appl., **148** (1990), 202–212. MR1052055. Zbl 0705.34040.

[37] I. Podlubny, *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Mathematics in Science and Engineering, **198** Academic Press, Inc., San Diego, CA, 1999. MR1658022. Zbl 0924.34008.

[38] R. Skiba, *Fixed points of multivalued weighted maps*, Lecture Notes in Nonlinear Analysis, **9** Juliusz Schauder Center for Nonlinear Studies, Toruń, 2007. MR2361669. Zbl 1129.54029.

[39] J. Wang and Y. Zhou, *Existence and controllability results for fractional semilinear differential inclusions*, Nonlinear Anal. Real World Appl. **12** (2011), no. 6, 3642–3653. MR2361669. Zbl 1231.34108.

[40] Y. Zhou and L. Peng, *Topological properties of solution sets for partial functional evolution inclusions,* C. R. Math. Acad. Sci. Paris **355** (2017), no. 1, 45–64. MR3590286. Zbl 1418.341401.

Abdelghani Ouahab
Department of Mathematics and Informatics,
The African University Ahmed Draia of Adrar,

Adrar, Algeria.
e-mail: agh_ouahab@univ-adrar.dz
and
Laboratory of Mathematics,
Sidi-Bel-Abbès University,
PoBox 89, 22000 Sidi-Bel-Abbès, Algeria.

Sarah Seghiri
Departement of Mathematics,
Normal High School,
B.P N 92, Vieux Kouba, Algiers,
Algeria.
e-mail: saramer29@live.com
and
Laboratory of Fixed Point Theory and Its Applications,
Normal High School,
B.P N 92, Vieux Kouba, Algiers,
Algeria.

License