UPPER BOUND ON HANKEL DETERMINANT FOR BOUNDED TURNING FUNCTION ASSOCIATED WITH SALÂGEAN-DIFFERENCE OPERATOR

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Abstract. By making use of Salâgean-difference operator we introduce a new function class $R_\alpha^\beta$ which generalizes the class of functions of bounded turning of order alpha. We investigate upper bounds on the third Hankel determinant for the class $R_\alpha^\beta$. Our results generalize the results of earlier researchers in this direction.

1 Introduction and Definitions

Let $\mathbb{R}$ and $\mathbb{C}$ be denote the set of real and complex numbers respectively. Denote by $\mathcal{A}$, the class of of all functions $h$ of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the open unit disc $\Delta = \{ z \in \mathbb{C} : |z| < 1 \}$ satisfying condition $h(0) = h'(0) - 1 = 0$. Let $S$ be the subclass of $\mathcal{A}$ consist of univalent functions. A function $h \in \mathcal{A}$ said to be of bounded turning if and only if $\Re(h'(z)) > 0$ for any $z \in \Delta$. We denote such class of functions by $\mathbb{R}$. For a function $h \in \mathcal{A}$, we define a linear operator $D_\lambda^\beta: \mathcal{A} \rightarrow \mathcal{A}$ as follows:

$$D_\lambda^0 h(z) = h(z),$$

$$D_\lambda^1 h(z) = z h'(z) + \frac{\lambda}{2} [h(z) - h(-z) - 2z] \quad (\lambda \in \mathbb{R})$$

$$= z + \sum_{n=2}^{\infty} \left[ n + \frac{\lambda}{2} (1 + (-1)^{n+1}) \right] a_n z^n,$$

$$D_\lambda^2 h(z) = D_1^1 (D_\lambda^1 h(z)).$$
In general, for \( \beta \in \mathbb{N}_0 = \{0, 1, 2, 3, \cdots \} \),

\[
D_\lambda^\beta h(z) = D_\lambda^1(D_\lambda^{\beta-1}h(z)) = z + \sum_{n=2}^{\infty} \left( n + \frac{\lambda}{2}(1 + (-1)^{n+1}) \right) a_n z^n \quad (z \in \Delta). \quad (1.2)
\]

The operator \( D_\lambda^\beta \) is known as the S\( \lambda \)-gean-difference operator in literature (see [17, 18]). This operator is a modified Dunkel operator of complex variables (see [8, 16]). When \( \lambda = 0 \), \( D_\lambda^\beta = D_0^\beta = D^\beta \) is known as the S\( \lambda \)-gean differential operator (see [44]).

**Example 1.** Let

\[
h(z) = ze^z = z + \infty \sum_{n=2}^{\infty} \frac{z^n}{2^{n-1}(n-1)!}.
\]

Then

\[
D_1^1 h(z) = z + z^2 + \frac{z^3}{2} + \frac{z^4}{12} + \frac{z^5}{64} + \cdots.
\]

**Example 2.** Let

\[
h(z) = z \left( 1 - \frac{z}{5} \right)^{-2} = z + \infty \sum_{n=2}^{\infty} \frac{n}{5^n-1} z^n.
\]

Then

\[
D_1^1 h(z) = z + 4 \frac{z^2}{5} + 12 \frac{z^3}{25} + 16 \frac{z^4}{125} + \cdots.
\]

In 1976, Noonan and Thomas [32] defined the \( q \)-th Hankel determinant of function \( h \) for \( q \geq 1 \) and \( n \geq 1 \) as

\[
H_q(n) = \begin{vmatrix}
  a_n & a_{n+1} & \cdots & a_{n+q-1} \\
  a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}.
\]

The Hankel determinant plays an important role in the study of singularities (see [7, 10]). It is useful in showing that a function of bounded characteristic in \( \Delta \), i.e. a function which is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational (see [3]). Pommerenke [37] proved that the Hankel determinant of univalent functions satisfy \( |H_q(n)| < Kn^{-\left(\frac{q+1}{2}\right)+\frac{\beta}{2}} \), \( (n = 1, 2, \cdots; \ q = 2, 3, \cdots) \), where \( \beta > \frac{1}{4000} \) and \( K \) depends only on \( q \). Later, Hayman [15] proved that \( |H_2(n)| < An^{\frac{1}{2}}, \) \( (n = 1, 2, \cdots) \), \( (A \) is an absolute constant) for areally mean univalent functions. The study of \( |H_q(n)| \) for various subfamilies of \( A \) are of interest for many researchers (see [11, 32, 38]. Finding the upper bounds of the Hankel determinants whose elements are the coefficients of univalent and multivalent functions for different values of \( q \) and \( n \) is an interesting
area of research in the geometric function theory. For $q = 2$, $n = 1$, $a_1 = 1$ and $q = n = 2$, $a_1 = 1$, the Hankel determinant respectively reduce to

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$ 

The Hankel determinant $H_2(1)$ is popularly known as the Fejér-Szegő functional. Fejér-Szegő [12] gave a sharp estimate of non-linear functional $|a_3 - \mu a_2^2|$ for $\mu$ real. It is a combination of the two coefficients which describes the area problem posed earlier by Gronwall [14]. The problem of calculating $\max_{h \in \mathbb{F}} |H_2(1)|$ for various compact subfamilies $\mathbb{F}$ of $\mathcal{A}$ was considered by various researchers (see [4, 22, 23, 29, 34, 35, 36]).

Recent research has focused on $H_2(2)$ for various subclasses of $S$. Janteng et al. [19, 20] derived the exact bounds for $H_2(2)$ for the class of starlike functions ($S^*$), the class of convex functions ($C$) and the class of functions whose derivatives have positive real parts ($\mathbb{R}^T$) in $\Delta$. The bounds obtained for these three classes are $|H_2(2)| \leq 1$, $|H_2(2)| \leq \frac{1}{8}$, $|H_2(2)| \leq \frac{2}{9}$ respectively. Lee et al. [25] investigated $H_2(2)$ in the general class $S^*(\phi)$ of starlike functions with respect to a given function $\phi$ and in particular obtained the results when $f \in S^*(\alpha)$, the class of starlike functions of order $\alpha$ ($|H_2(2)| \leq (1 - \alpha)^2$), the class $S^*_L$ of lemniscate starlike functions ($|H_2(2)| \leq \frac{1}{4\alpha^2}$) and the class $S^*_S$ of strongly starlike functions of order $\beta$ ($|H_2(2)| \leq \beta^2$). Krishna and Ramreddy [24] generalized the result of Janteng et al. [20] giving the sharp bound of $H_2(2)$ in the class of starlike and convex functions of $\alpha$.

Zaprawa [50] showed that if $f \in \mathbb{T}$, the class of typically real functions, then $|H_2(2)| \leq 9$. Ramreddy and Krishna [40] obtained the Hankel determinant for starlike and convex functions with respect to symmetric points. Using Owa and Srivastava operator $\Omega^\delta_z$ ($0 \leq \delta \leq 1$), Mishra and Gochhayat [28] introduced the class

$$\mathbb{R}_\delta(\gamma, \alpha) = \left\{ h \in \mathcal{A} : \mathbb{R} \left( e^{i\gamma} \frac{\Omega^\delta_z h(z)}{z} \right) > \alpha \cos \gamma, \ (|\gamma| < \frac{\pi}{2}, \ 0 \leq \alpha \leq 1) \right\}$$

and obtained the sharp upper bounds for $H_2(2)$. Apart from these, many researchers obtained the upper bounds for various subclasses of univalent analytic functions (see [2, 6, 21, 30, 42, 43, 47, 49]).

In this paper, we focus on third Hankel determinant for $q = 3$ and $n = 1$, denoted by $H_3(1)$ given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$ 

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http://www.utgjiu.ro/math/sma
For \( h \in \mathcal{A} \) and \( a_1 = 1 \), we have
\[
H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).
\]
An application of triangle inequality yields
\[
|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| - |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|.
\]

Recently, Babalola (see [1]) obtained the sharp upper bound of \( H_3(1) \) for the functions belongs to the class \( S^* \), \( C \) and \( RT \). Krishna et al. [48] introduced the generalized class \( RT(\alpha) \) as \( \Re(h'(z)) > \alpha \) \((z \in \Delta)\) and obtained the bound on \( H_3(1) \). Further, Bansal et al. [3] and Raza and Malik [41] obtained the bound \( H_3(1) \) for certain subclass of univalent functions. Very recently, making use of Hohlov operator, Gochhayat et al. [13] introduced the class \( R_{\alpha, \beta} \) and obtained the sharp bounds for \( H_2(2) \) and \( H_3(1) \) in terms of Gauss hypergeometric function. For recent results on third Hankel determinant see [31, 39, 45, 46].

Motivated by the above researchers, we introduced the subclass of univalent function by making use of Salagean-difference operator \( D_{\beta, \lambda} \) as follows:

**Definition 3.** A function \( h \in \mathcal{A} \) given by (1.1) is in the class \( R_{\alpha, \beta} \) if it satisfy the condition
\[
\Re\left[\frac{D_{\beta, \lambda}h(z)}{z}\right] > \alpha \quad (0 \leq \alpha \leq 1, \ \beta \in \mathbb{N}_0, \ \lambda \in \mathbb{R}, \ z \in \Delta).
\]

It may be noted that by taking \( \lambda = 0 \) and \( \beta = 1 \) the class \( R_{\alpha} \) studied by Krishna et al. [48] (also, see [28]). Also, letting \( \lambda = 0, \ \beta = 1 \) and \( \alpha = 0 \) we obtain the class \( \mathcal{R}_0 \) studied by Babalola [1]. Further, if we take \( \beta = 0 \) in the class \( R_{\alpha, \lambda} \), we get the class \( \mathcal{R}_0(0, \alpha) = \mathcal{R}(\alpha) \) studied by Mishra and Gochhayat [28].

In this paper, following a method of classical analysis derived by Libera and Zlotkiewicz [26, 27], we obtain the upper bounds of \( H_3(1) \) for the function belonging to the class \( R_{\alpha, \lambda} \).

## 2 Preliminary Lemmas

Let \( \mathcal{P} \) denote the class of functions denoted by \( p \) such that
\[
p(z) = 1 + d_1z + d_2z^2 + \cdots
\]
which are regular in the open unit disk \( \Delta \) and satisfy \( \Re(p(z)) > 0 \) for any \( z \in \Delta \). Here \( p(z) \) is called the Caratheodory function (see [9]).

To prove our main results we need the following lemmas.
Lemma 4. (see[9]) If \( p \in \mathcal{P} \) is of the form (2.1), then
\[
|d_n| \leq 2 \quad (n \in \mathbb{N}).
\] (2.2)
The equality holds for the function
\[
\phi(z) = \frac{1 + z}{1 - z} = 1 + 2 \sum_{n=1}^{\infty} z^n.
\]

Lemma 5. (see [26, 27]) If \( p \in \mathcal{P} \) is of the form (2.1), then
\[
2d_2 = d_1^2 + (4 - d_1^2)x,
\] (2.3)
and
\[
4d_3 = d_1^3 + 2(4 - d_1^2)d_1x - (4 - d_1^2)d_1x^2 + 2(4 - d_1^2)(1 - |x|^2)z,
\] (2.4)
for some complex numbers \( x, z \) satisfying \( |x| \leq 1 \) and \( |z| \leq 1 \).

3 Main Results

Theorem 6. Let the function \( h \) given by (1.1) be in the class \( R_\lambda^\beta(\alpha) \) \( (\lambda \in \mathbb{R}, \beta \in \mathbb{N}_0, 0 \leq \alpha \leq 1) \). Then
\[
|a_2| \leq \frac{2(1 - \alpha)}{2^\beta}, \quad |a_3| \leq \frac{2(1 - \alpha)}{(3 + \lambda)^\beta},
\]
\[
|a_4| \leq \frac{2(1 - \alpha)}{4^\beta}, \quad |a_5| \leq \frac{2(1 - \alpha)}{(5 + \lambda)^\beta}.
\]

Proof. Let \( h \in R_\lambda^\beta(\alpha) \). Then there exists an analytic function \( p \in \mathcal{P} \) in the unit disk \( \Delta \) with \( p(0) = 1 \) and \( \Re(p(z)) > 0 \) such that
\[
\frac{D_\lambda^\beta h(z)}{z} = \alpha + (1 - \alpha)p(z).
\] (3.1)
Using the series expansion for \( D_\lambda^\beta h(z) \) and \( p(z) \) in (3.1), we get
\[
1 + 2^\beta a_2z + (3 + \lambda)^\beta a_3z^2 + 4^\beta a_4z^3 + (5 + \lambda)^\beta a_5z^4 + \cdots
\]
\[
= 1 + (1 - \alpha)d_1z + (1 - \alpha)d_2z^2 + (1 - \alpha)d_3z^3 + (1 - \alpha)d_4z^4 + \cdots.
\] (3.2)
Equating the coefficient of various powers of \( z \), \( z^2 \), \( z^3 \) and \( z^4 \) on both the sides of (3.2), we obtain
\[
a_2 = (1 - \alpha)d_1, \quad a_3 = (1 - \alpha)d_2, \quad a_4 = (1 - \alpha)d_3, \quad a_5 = (1 - \alpha)d_4.
\] (3.3)
Application of triangle inequality to (3.3) and followed by the Lemma 4 give the desire estimate. This completes the proof of Theorem 6.
Remark 7. Taking $\lambda = 0$, $\beta = 1$ in the above theorem we obtain the coefficient bounds for the class $R_\alpha^2(\beta)$ as $|a_n| \leq \frac{2(1-\alpha)}{n}$ $(n \geq 2)$.

Theorem 8. Let the function $h(z)$ given by (1.1) be in the class $R_\alpha^2(\alpha)$. Then

$$|a_2a_4 - a_3^2| \leq \frac{4(1-\alpha)^2}{(3+\lambda)^{2\beta}}.$$ 

Proof. Putting the values of $a_2, a_3$ and $a_4$ from (3.3) in the functional $|a_2a_4 - a_3^2|$, we obtain

$$|a_2a_4 - a_3^2| = \left| \frac{(1-\alpha)d_1(1-\alpha)d_2}{2^\beta} \cdot \frac{(1-\alpha)d_3}{4^\beta} - \frac{(1-\alpha)^2d_2^2}{(3+\lambda)^{2\beta}} \right|$$

$$= \frac{(1-\alpha)^2}{2^\beta(3+\lambda)^{2\beta}} \left| (3+\lambda)^{2\beta}d_1d_3 - 2^{3\beta}d_2^2 \right|$$

$$= \frac{(1-\alpha)^2}{2^\beta(3+\lambda)^{2\beta}} \left| e_1d_1d_3 - e_2d_2^2 \right|,$$

where

$$e_1 = (3+\lambda)^{2\beta} \quad \text{and} \quad e_2 = 2^{3\beta}. \tag{3.5}$$

Substituting the values of $d_2$ and $d_3$ from (2.3) and (2.4) of Lemma 5 on the right hand side of (3.4), we get

$$|e_1d_1d_3 - e_2d_2^2| = \left| \frac{e_1d_1}{4} \left\{ d_1^2 + 2d_1(4 - d_1^2)x - d_1(4 - d_1^2)x^2 + 2(4 - d_1^2)(1 - |x|^2)z \right\} \right|$$

$$= \left| \frac{e_1d_1}{4} + \frac{e_1d_1(4 - d_1^2)x}{2} - \frac{e_1d_1^2(4 - d_1^2)x^2}{4} + \frac{e_1d_1(4 - d_1^2)(1 - |x|^2)z}{2} \right|$$

$$- \left| \frac{e_2d_1^4}{4} - \frac{e_2(4 - d_1^2)^2x^2}{4} - \frac{e_2d_1^2(4 - d_1^2)x}{2} \right|.$$

Therefore,

$$4|e_1d_1d_3 - e_2d_2^2| = |(e_1 - e_2)d_1^2 + 2e_1d_1(4 - d_1^2)x + 2(e_1 - e_2)d_1^2x(4 - d_1^2) - (4 - d_1^2)x^2(e_1d_1^2 + e_2(4 - d_1^2)) - 2e_1d_1(4 - d_1^2)|x|^2z|. \tag{3.6}$$

Using the fact that $|z| \leq 1$ and $|xa + yb| \leq |x||a| + |y||b|$ where $x, y, a, b \in \mathbb{R}$ in the
expression (3.6), after simplifying we get

\[ 4|e_1d_1d_3 - e_2d_2^2| \leq |(e_1 - e_2)d_1^4 + 2e_1d_1(4 - d_1^2) + 2(e_1 - e_2)d_1^2(4 - d_1^2)|x| \\
- \{e_1d_1^2 + e_2(4 - d_1^2) + 2e_1d_1\}(4 - d_1^2)|x|^2 \]

By Lemma 4, \(|d_1| \leq 2\). Suppose that \(d_1 = d\) and we may assume without restriction that \(d \in [0, 2]\). Using the well-known results \((d_1 + a)(d_1 + b) \geq (d_1 - a)(d_1 - b)\) where \(a, b \geq 0\) on the right hand side of (3.7) upon simplification give

\[ 4|e_1d_1d_3 - e_2d_2^2| \leq |(e_1 - e_2)d_1^4 + 2e_1d_1(4 - d_1^2) + 2(e_1 - e_2)d_1^2(4 - d_1^2)|x| \\
- (d_1 - 2)\{(e_1 - e_2)d_1 - 2e_2\}(4 - d_1^2)|x|^2 \]. 

(3.8)

Applying triangle inequality to the right hand side of (3.8), replacing \(|x|\) by \(\rho\) and putting the values of \(e_1\) and \(e_2\) from (3.5) in (3.8) we get

\[ 4|e_1d_1d_3 - e_2d_2^2| \leq [(3 + \lambda)^{2\beta} - 2^{3\beta}]d^4 + 2(3 + \lambda)^{2\beta}d(4 - d^2) + 2[(3 + \lambda)^{2\beta} - 2^{3\beta}]d^2(4 - d^2) \rho \\
+ (d - 2)\{(3 + \lambda)^{2\beta} - 2^{3\beta}|d - 2^{3\beta+1}|\}(4 - d^2) \rho_2^2 \\
= (d - 2)\{(3 + \lambda)^{2\beta} - 2^{3\beta}|d - 2^{3\beta+1}|\rho^2\}(4 - d^2) \\
= G(d, \rho) (\text{say}) \quad (0 \leq \rho = |x| \leq 1). \]

(3.9)

Now, we maximize the function \(G(d, \rho)\) on the close interval region \([0, 2] \times [0, 1]\). Differentiating \(G\) partially with respect to \(\rho\) we get

\[ \frac{\partial G}{\partial \rho} = 2\left[ (3 + \lambda)^{2\beta} - 2^{3\beta} \right]d^2 + (d - 2)\left\{ (3 + \lambda)^{2\beta} - 2^{3\beta} \right\}d - 2^{3\beta+1} \rho \right] (4 - d^2). \quad (3.10) \]

For \(0 < \rho < 1\) and for fixed \(d\) with \(0 < d < 2\) we observe from (3.10) that \(\frac{\partial G}{\partial \rho} > 0\). Therefore \(G(d, \rho)\) is an increasing function of \(\rho\) and hence it cannot have the maximum value in the interior of the close region \([0, 2] \times [0, 1]\). Hence, for fixed \(d \in [0, 2]\), we have

\[ \max G(d, \rho) = G(d, 1) = H(d) (\text{say}), \]

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From (3.12), we observe that
\[ H = 0 \]
d is a decreasing function of \( No \).
Now, from (3.9) and (3.13) we have
\[ \text{Remark 11.} \]
For (3.14) in (3.4) we obtain
\[ \left| a_2 a_4 - a_3^2 \right| \leq \frac{(1 - \alpha)^2 e_2}{2\beta(3 + \lambda)^{2\beta}} = \frac{4(1 - \alpha)^2}{(3 + \lambda)^{2\beta}}. \]
This completes the proof of Theorem 8.
\[
\text{Remark 9. For } \lambda = 0 \text{ and } \beta = 1, \text{ our result in Theorem 8 coincides with the result of Krishan and Ramreddy [48] (also, see [28]).}
\]
\[
\text{Remark 10. For } \lambda = 0, \beta = 1 \text{ and } \alpha = 0 \text{ our result coincides to that of Janteng et. al. (see [19]).}
\]
\[
\text{Remark 11. Letting } \beta = 0 \text{ in Theorem 8 we get the result due to Mishra and Gochhayat [28].}
\]
Theorem 12. If the function $h(z)$ defined by (1.1) belongs to the class $R^3_\lambda(\alpha)$, then for $0 \leq \alpha \leq \frac{2^{\beta+1}-(3+\lambda)\beta}{2^{\beta+1}}$, we have

$$|a_2a_3 - a_4| \leq \frac{2(1 - \alpha)}{3\sqrt{34}(3 + \lambda)^\beta} \left| \frac{[3(3 + \lambda)\beta - 2^{\beta+1}(1 + \alpha)]^3}{[(3 + \lambda)\beta - 2\beta]^3} \right|.$$  \hfill (3.16)

Proof. Let the function $h(z)$ given by (1.1) be in the class $R^3_\lambda(\alpha)$. Proceeding as in Theorem 6 and putting the values of $a_2$, $a_3$ and $a_4$ in the functional $|a_2a_3 - a_4|$ we get

$$|a_2a_3 - a_4| = \left| \frac{(1 - \alpha)}{4\beta(3 + \lambda)\beta} \right| k_1d_1d_2 + k_2d_3$$ \hfill (3.17)

where

$$k_1 = 2\beta(1 - \alpha), \quad k_2 = -(3 + \lambda)\beta.$$ \hfill (3.18)

Substituting the values of $d_2$ and $d_3$ from (2.2) and (2.3) of Lemma 5 on the right hand side of (3.17) we have

$$|k_1d_1d_2 + k_2d_3| = \left| \frac{k_1d_1}{2} \left\{ d_1^2 + x(4 - d_1^2) \right\} \right|$$

$$+ \frac{k_2}{4} \left\{ d_1^2 + 2d_1x(4 - d_1^2) - d_1x^2(4 - d_1^2) + 2(4 - d_1^2)(1 - |x|^2)z \right\}$$

$$= \left| \frac{1}{2} \left( k_1d_1^2 + k_1d_1x(4 - d_1^2) \right) + \frac{1}{4} \left( k_2d_1^2 + 2k_2d_1(4 - d_1^2)x \right) - k_2d_1(4 - d_1^2)x + 2k_2(4 - d_1^2)(1 - |x|^2)z \right|,$$

which implies

$$4|k_1d_1d_2 + k_2d_3| = \left| 2k_1d_1^3 + 2k_1d_1(4 - d_1^2)x + k_2d_1^2 + 2k_2d_1(4 - d_1^2)x \right.$$

$$- k_2d_1(4 - d_1^2)x^2 + 2k_2(4 - d_1^2)z - 2k_2(4 - d_1^2)|x|^2z \right|.$$  \hfill (3.19)

Using the triangle inequality and the fact that $|z| \leq 1$ in the above equation, we have

$$4|k_1d_1d_2 + k_2d_3| \leq (2k_1 + k_2)d_1^3 + 2(k_1 + k_2)d_1|x|(4 - d_1^2) + 2k_2(4 - d_1^2) + k_2(d_1 + 2)(4 - d_1^2)|x|^2$$

$$= \left| \left[ 2^{\beta+1}(1 - \alpha) - (3 + \lambda)^\beta \right] d_1^3 + 2(2\beta(1 - \alpha) - (3 + \lambda)^\beta) d_1|x|(4 - d_1^2) \right.$$ \hfill (3.19)

$$- 2(3 + \lambda)^\beta(4 - d_1^2) + (3 + \lambda)^\beta(d_1 + 2)(4 - d_1^2)|x|^2 \right|.$$

Since $|d_1| < 2$ by Lemma 4 we may assume without any restriction $d_1 = d \in [0, 2]$. Using the well-known result that $d_1 + a > d_1 - a$ for $a \geq 0$, replacing $|x|$ by $\rho$ and $0 \leq \rho < 1$...
\( \rho \leq 1 \) and assuming \( \alpha \leq \frac{2^{\beta+1}-(3+\lambda)^{\beta}}{2^{\beta+1}} \) on the right hand side of the above inequality (3.19) we get

\[
4|k_1d_1d_2+k_2d_3| \leq \left[ 2^{\beta+1}(1-\alpha) - (3+\lambda)^{\beta} \right] d^3 + 2 \left[ -2^{\beta}(1+\alpha) + (3+\lambda)^{\beta} \right] d(4-d^2)\rho \\
+ 2(3+\lambda)^{\beta}(4-d^2) + (3+\lambda)^{\beta}(d-2)(4-d^2)\rho^2 = K(d, \rho) \text{(say)},
\]

(3.20)

where

\[
K(d, \rho) = \left[ 2^{\beta+1}(1-\alpha) - (3+\lambda)^{\beta} \right] d^3 + 2 \left[ -2^{\beta}(1+\alpha) + (3+\lambda)^{\beta} \right] d(4-d^2)\rho \\
+ 2(3+\lambda)^{\beta}(4-d^2) + (3+\lambda)^{\beta}(d-2)(4-d^2)\rho^2.
\]

(3.21)

Now, we have to maximize the function \( K(d, \rho) \) over closed region \([0, 2] \times [0, 1] \). Differentiating \( K \) partially with respect to \( \rho \) we get

\[
\frac{\partial K}{\partial \rho} = 2[(3+\lambda)^{\beta} - 2^{\beta}(1+\alpha)]d(4-d^2) + 2(3+\lambda)^{\beta}(d-2)(4-d^2)\rho.
\]

(3.22)

For \( 0 < \rho < 1 \), for fixed \( d \) with \( 0 < d < 2 \) and for \( \alpha \) with \( 0 \leq \alpha \leq \frac{2^{\beta+1}-(3+\lambda)^{\beta}}{2^{\beta+1}} \), we observe from (3.22) that \( \frac{\partial K}{\partial \rho} \geq 0 \) which implies the function \( K(d, \rho) \) is an increasing function of \( \rho \) and hence it cannot have a maximum at any point in the interior of the closed region \([0, 2] \times [0, 1] \). Thus for fixed \( d \), \( 0 \leq d \leq 2 \), we have

\[
\max_{0 \leq \rho \leq 1} K(d, \rho) = K(d, 1) = L(d) \text{ (say)}
\]

where

\[
L(d) = \left[ 2^{\beta+1}(1-\alpha) - (3+\lambda)^{\beta} \right] d^3 + 2(3+\lambda)^{\beta}(4-d^2) \\
+ 2\left[ (3+\lambda)^{\beta} - 2^{\beta}(1+\alpha) \right] d(4-d^2) + (3+\lambda)^{\beta}(d-2)(4-d^2) \\
= \left[ 2^{\beta+2} - 4(3+\lambda)^{\beta} \right] d^2 + 4\left[ 3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha) \right] d.
\]

(3.23)

A function \( L(d) \) to be maximum or minimum on the interval \([0, 2] \), we have

\[
L'(d) = 3\left[ 2^{\beta+2} - 4(3+\lambda)^{\beta} \right] d^2 + 4\left[ 3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha) \right] = 0,
\]

which implies

\[
d = \sqrt{\frac{[3(3+\lambda)^{\beta} - 2^{\beta+1}(1+\alpha)]}{3(3+\lambda)^{\beta} - 2^{\beta+1}}} \in [0, 2], \quad \left( 0 \leq \alpha \leq \frac{2^{\beta+1}-(3+\lambda)^{\beta}}{2^{\beta+1}} \right).
\]

(3.24)

Also,

\[
L''(d) = 6\left[ 2^{\beta+1} - 4(3+\lambda)^{\beta} \right] d.
\]

(3.25)

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Substitute the value of \( d \) from (3.24) in (3.25) we get

\[
L''(d) = \frac{12}{\sqrt{3}} \left[ 2^{\beta+1} - 4(3 + \lambda)\beta \right] \sqrt{\left[ 2^{\beta+1}(1 + \alpha) - 3(3 + \lambda)\beta \right] \left[ 2^{\beta+2} - 4(3 + \lambda)\beta \right]} < 0,
\]

for \( 0 \leq \alpha \leq \frac{2^{\beta+1} - (3 + \lambda)\beta}{2^{\beta+1}} \). Hence it follows from differential calculus, the function \( L(d) \) is maximum at the point \( d \) given by (3.24). Putting the value of \( d \) from (3.24)

in (3.23) and simplifying we get

\[
L_{\text{max}} = \left[ 2^{\beta+2} - 4(3 + \lambda)\beta \right] \left[ 2^{\beta+1}(1 + \alpha) - 3(3 + \lambda)\beta \right] \frac{2}{3^{\beta} - (3 + \lambda)\beta}
\]

\[
+ 4[3(3 + \lambda)\beta - 2^{\beta+1}(1 + \alpha)] \left[ 2^{\beta+1}(1 + \alpha) - 3(3 + \lambda)\beta \right] \frac{1}{3^{\beta} - (3 + \lambda)\beta}
\]

\[
= \frac{-4[2^{\beta+1}(1 + \alpha) + 3(3 + \lambda)\beta]}{3^{\beta} - 2^{\beta} + (3 + \lambda)\beta} + \frac{4[-2^{\beta+1}(1 + \alpha) + 3(3 + \lambda)\beta]}{\sqrt{3}(3 + \lambda)\beta - 2^{\beta} \frac{1}{2}}
\]

\[
= \frac{-4[3(3 + \lambda)\beta - 2^{\beta+1}(1 + \alpha)]}{3\sqrt{3}(3 + \lambda)\beta - 2^{\beta} \frac{1}{2}} + 12[3(3 + \lambda)\beta - 2^{\beta+1}(1 + \alpha)]\frac{3}{2}
\]

\[
= \frac{8}{3\sqrt{3}} \left[ 3(3 + \lambda)\beta - 2^{\beta+1}(1 + \alpha) \right] \frac{3}{2} \sqrt{(3 + \lambda)\beta - 2^{\beta}}.
\]

(3.26)

From (3.20) and (3.26) we have

\[
|k_1d_1d_2 + k_2d_3| \leq \frac{2}{3\sqrt{3}} \left[ 3(3 + \lambda)\beta - 2^{\beta+1}(1 + \alpha) \right] \frac{3}{2} \frac{1}{\sqrt{(3 + \lambda)\beta - 2^{\beta}}}.
\]

(3.27)

The relations (3.17) and (3.27) give

\[
|a_2a_3 - a_4| \leq \frac{1 - \alpha}{4^{\beta}(3 + \lambda)\beta} \frac{2}{3\sqrt{3}} \left[ 3(3 + \lambda)\beta - 2^{\beta+1}(1 + \alpha) \right] \frac{3}{2} \frac{1}{\sqrt{(3 + \lambda)\beta - 2^{\beta}}}
\]

\[
= \frac{2(1 - \alpha)}{3\sqrt{3}(3 + \lambda)\beta} \frac{3(3 + \lambda)\beta - 2^{\beta+1}(1 + \alpha)}{\frac{3}{2} \sqrt{(3 + \lambda)\beta - 2^{\beta}}}
\]

This completes the proof of the Theorem 12. \( \square \)

**Remark 13.** Putting \( \lambda = 0, \beta = 1 \) in the above theorem we get the following results due to Vamshree Krishna et al.[48].

**Corollary 14.** (see[48]) Let \( f \in RT(\alpha) \) \((0 \leq \alpha \leq \frac{1}{2})\). Then

\[
|a_2a_3 - a_4| \leq \left( \frac{1 - \alpha}{6} \right) \left( \frac{5 - 4\alpha}{3} \right)^{\frac{3}{2}}.
\]
Putting $\alpha = 0$ in Corollary 14 we get $|a_{2a3} - a_4| \leq \frac{5}{18} \sqrt{\frac{5}{3}}$. This result is coincide with Babalola [see1].

**Theorem 15.** Let $h(z) \in R^β_λ(\alpha)$ \(0 \leq \alpha \leq \frac{2(\lambda + \beta - 22\beta)}{2(\lambda + \alpha)^2}\). Then

$$|a_3 - a_2^2| \leq \frac{2(1 - \alpha)}{(3 + \alpha)^2}.$$  

**Proof.** Substituting the values of $a_2$ and $a_3$ from (3.3) in coefficient functional $|a_3 - a_2^2|$ we obtain

$$|a_3 - a_2^2| = \left| \frac{(1 - \alpha)d_2}{(3 + \lambda)^2} - \frac{(1 - \alpha)^2d_1^2}{2^2\beta} \right|$$

$$= \frac{(1 - \alpha)}{(3 + \lambda)^2(2\beta)} \left| 2^{2\beta}d_2 - (3 + \lambda)^2(1 - \alpha)d_1^2 \right|$$

$$= \frac{1 - \alpha}{(3 + \lambda)^2(2\beta)} \left| l_1d_2 + l_2d_1^2 \right|,$$

where

$$l_1 = 2^{2\beta}, \quad l_2 = -(3 + \lambda)^2(1 - \alpha).$$

Putting the value of $d_2$ from (2.2) of Lemma 5 in the right hand side of (3.28) we obtain

$$|l_1d_2 + l_2d_1^2| = \left| \frac{l_1}{2} \left( d_1^2 + x(4 - d_1^2) \right) + l_2d_1^2 \right| = \left| \frac{l_1d_1^2 + l_1x(4 - d_1^2) + 2l_2d_1^2}{2} \right|,$$

which implies

$$2|l_1d_2 + l_2d_1^2| = \left| (l_1 + 2l_2)d_1^2 + l_1x(4 - d_1^2) \right|$$

$$= \left| 2^{2\beta} - 2(3 + \lambda)^2(1 - \alpha)|d_1^2 + 2^{2\beta}x(4 - d_1^2) \right|.$$  

(3.29)

Choosing $d_1 = d \in [0,2]$, applying triangle inequality, replacing $|x|$ by $\rho$ on the right hand side of (3.29) and assume that $\alpha \leq \frac{2(\lambda + \lambda)^2}{2(3 + \lambda)^2}$, we have

$$2|l_1d_2 + l_2d_1^2| \leq \left| 2(3 + \lambda)^2(1 - \alpha) - 2^{2\beta} \right| d^2 + 2^{2\beta}(4 - d^2) \rho$$

$$= M(d, \rho) \text{(say)} \quad (0 \leq \rho = |x| \leq 1),$$

(3.30)

where

$$M(d, \rho) = [2(3 + \lambda)^2(1 - \alpha) - 2^{2\beta}] d^2 + 2^{2\beta}(4 - d^2) \rho.$$ 

In order to determine the maximum value of the function $M(d, \rho)$ differentiating $M(d, \rho)$, partially with respect to $\rho$, we get

$$\frac{\partial M}{\partial \rho} = 2^{2\beta}(4 - d^2) > 0 \quad \text{for} \quad d \in [0,2]$$

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For $0 < d < 2$, $\frac{\partial M}{\partial \rho} > 0$. Hence the function $M(d, \rho)$ is an increasing function of $\rho$.

The maximum value of $M$ occurs at $\rho = 1$ and given by

$$\max_{0 \leq \rho \leq 1} M(d, \rho) = M(d, 1) = N(d)$$

where

$$N(d) = \left[ 2(3 + \lambda)^\beta (1 - \alpha) - 2^{2\beta} \right] d^2 + 2^{2\beta} (4 - d^2)$$

$$= \left[ 2(3 + \lambda)^\beta (1 - \alpha) - 2^{2\beta+1} \right] d^2 + 2^{2\beta+2}. \quad (3.31)$$

Now

$$N'(d) = 4\left[ (3 + \lambda)^\beta (1 - \alpha) - 2^{2\beta} \right] d \leq 0, \quad \forall \quad d \in [0, 2], \quad \alpha \in \left[ 0, \frac{2(3 + \lambda)^\beta - 2^{2\beta}}{2(3 + \lambda)^\beta} \right].$$

Therefore $N(d)$ becomes a decreasing function of $d$ whose maximum value occur at $d = 0$. From (3.31) we get

$$\max_{0 \leq d \leq 2} N(d) = N(0) = 2^{2\beta+2}. \quad (3.32)$$

It follows from (3.30) and (3.32) that

$$|l_1d_2 + l_2d_1^2| \leq 2^{2\beta+1}. \quad (3.33)$$

Using (3.33) in (3.28) gives

$$|a_3 - a_2^2| \leq \frac{1 - \alpha}{(3 + \lambda)^\beta 2^{2\beta+1}} \frac{2(1 - \alpha)}{(3 + \lambda)^\beta}$$

The proof of Theorem 15 is thus completed.

\[\square\]

**Remark 16.** Taking $\lambda = 0, \beta = 1$ in Theorem 15 we get the estimate of $|a_3 - a_2^2| \leq \frac{2}{3}(1 - \alpha)$, studied by Vamshree Krishna et al [48].

**Remark 17.** Letting $\lambda = 0, \beta = 1$ and $\alpha = 0$ we get the results $|a_3 - a_2^2| \leq \frac{2}{3}$ due to Babalola [1]

**Theorem 18.** Let $f \in R_\lambda^\beta(\alpha)$. Then

$$|h_3(1)| \leq \frac{4(1 - \alpha)^2}{(3 + \lambda)^\beta} \left[ \frac{2(1 - \alpha)}{(3 + \lambda)^{2\beta}} + \frac{[3(3 + \lambda)^\beta - 2^{2\beta+1}(1 + \alpha)]^2}{3\sqrt{34^\beta}(3 + \lambda)^\beta - 2^{2\beta}} + \frac{1}{(5 + \lambda)^\beta} \right]$$

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Proof. Using the results from Theorem 6, 8, 12 and 15 in (1.3) we obtain

\[
H_3(1) \leq \frac{2(1 - \alpha) 4(1 - \alpha)^2}{(3 + \lambda)^3 (3 + \lambda)^{2\beta}} + \frac{2(1 - \alpha) 2(1 - \alpha) [3(3 + \lambda)^\beta - 2^{\beta+1}(1 + \alpha)]^{\frac{3}{2}}}{3\sqrt{3}4^\beta(3 + \lambda)^\beta [(3 + \lambda)^\beta - 2^\beta]^{\frac{1}{2}}}
\]

\[
+ \frac{2(1 - \alpha) 2(1 - \alpha)}{(5 + \lambda)^2 (3 + \lambda)^3}
\]

\[
= 8(1 - \alpha)^3 + \frac{4(1 - \alpha)^2 [3(3 + \lambda)^\beta - 2^{\beta+1}(1 + \alpha)]^{\frac{3}{2}}}{4^\beta 3\sqrt{3}[(3 + \lambda)^\beta - 2^\beta]^{\frac{1}{2}}} + \frac{4(1 - \alpha)^2}{(3 + \lambda)^2 (5 + \lambda)^3}
\]

\[
= \frac{4(1 - \alpha)^2}{(3 + \lambda)^3} \left[ \frac{2(1 - \alpha)}{(3 + \lambda)^{2\beta}} + \frac{3(3 + \lambda)^\beta - 2^{\beta+1}(1 + \alpha)]^{\frac{3}{2}}}{3\sqrt{3}4^\beta[(3 + \lambda)^\beta - 2^\beta]^{\frac{1}{2}}} + \frac{1}{(5 + \lambda)^3} \right].
\]

This complete the proof of Theorem 18.

Remark 19. Putting \( \lambda = 0, \beta = 1 \) in above theorem we get the results due to Vamshhee Krishan (see [48]).

Remark 20. Putting \( \lambda = 0, \beta = 1, \alpha = 0 \) in Theorem 18 we get the results of Babalola (see [1]).

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