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SUBORDINATING RESULTS FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY HADAMARD PRODUCT AND ATSHAN AND RAFID OPERATOR

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Abstract. In this paper, we defined a class of analytic functions defined by Hadamard product and Atshan and Rafid operator and obtained some subordinating results for functions in this class.

1 Introduction

Let S be the class of analytic univalent functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; z \in \mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\} \tag{1.1}$$

and C be the subclass of $f(z) \in S$ which are convex in $\mathbb{U}$ and let $M(\beta)$ and $N(\beta)$ denote the subclasses of S consisting of $f(z)$ which satisfying (see [2,3,15,17,18,21])

$$M(\beta) = \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \beta, \; \beta > 1 \tag{1.2}$$

and

$$N(\beta) = \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta, \; \beta > 1. \tag{1.3}$$

Then, we note that $f(z) \in N(\beta) \Leftrightarrow zf'(z) \in M(\beta)$.

For $f \in S$ given by (1.1) and $g \in S$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (b_k \geq 0), \tag{1.4}$$

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then the Hadamard product $f*g$ of f and g is defined by

$$(f*g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g*f)(z). \tag{1.5}$$

Let f and g be analytic in $\mathbb{U}$, then f is subordinate to g $(f(z) \prec g(z))$ if there exists an analytic function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$ (see [14]), and if g is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 1. *(Subordinating factor sequence) [22] A sequence $\{d_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f of the form (1.1) is analytic, univalent and convex in $\mathbb{U}$, we have the subordination given by*

$$\sum_{k=1}^{\infty} a_k d_k z^k \prec f(z) \qquad (z \in \mathbb{U},\ a_1 = 1). \tag{1.6}$$

For $f(z) \in S$, Atshan and Rafid [4] defined the operator R_μ^θ for $0 \le \mu < 1$, $0 \le \theta \le 1$ by

$$\begin{aligned} R_\mu^\theta(f(z)) &= \frac{1}{(1-\mu)^{1+\theta}\Gamma(\theta+1)} \int_0^\infty t^{\theta-1} e^{-(\frac{t}{1-\mu})} f(zt)dt \\ &= z + \sum_{k=2}^{\infty} K(k,\mu,\theta) a_k z^k, \end{aligned} \tag{1.7}$$

where

$$K(k,\mu,\theta) = \frac{(1-\mu)^{k-1}\Gamma(\theta+k)}{\Gamma(\theta+1)}. \tag{1.8}$$

Definition 2. *By using the operator R_μ^θ and for $\alpha \le 0$, $\beta > 1$, $0 \le \lambda \le 1$, $0 \le \mu < 1$, $0 \le \theta \le 1$, let $M_\mu^\theta(f, g, \lambda, \beta, \alpha)$ be the class consisting of functions $f, g \in S$ satisfying*

$$\begin{aligned} &\operatorname{Re}\left\{ \frac{z(R_\mu^\theta(f*g)(z))' + \lambda z^2 (R_\mu^\theta(f*g)(z))''}{(1-\lambda)R_\mu^\theta(f*g)(z) + \lambda z (R_\mu^\theta(f*g)(z))'} - \beta \right\} \\ < \ &\alpha \left| \frac{z(R_\mu^\theta(f*g)(z))' + \lambda z^2 (R_\mu^\theta(f*g)(z))''}{(1-\lambda)R_\mu^\theta(f*g)(z) + \lambda z (R_\mu^\theta(f*g)(z))'} - 1 \right|. \end{aligned} \tag{1.9}$$

For suitable choices of λ and g, we obtain the new classes:

(i) $M_\mu^\theta(f, \frac{z}{1-z}, \lambda, \beta, \alpha) = M_\mu^\theta(f, \lambda, \beta, \alpha) =$

$$\left\{ f \in S : \operatorname{Re}\left\{ \frac{z(R_\mu^\theta f(z))' + \lambda z^2 (R_\mu^\theta f(z))''}{(1-\lambda)R_\mu^\theta f(z) + \lambda z (R_\mu^\theta f(z))'} - \beta \right\} < \alpha \left| \frac{z(R_\mu^\theta f(z))' + \lambda z^2 (R_\mu^\theta f(z))''}{(1-\lambda)R_\mu^\theta f(z) + \lambda z (R_\mu^\theta f(z))'} - 1 \right|, \alpha \le 0, \beta > 1, 0 \le \lambda \le 1, 0 \le \theta \le 1, 0 \le \mu < 1, z \in \mathbb{U} \right\},$$

(ii) $M_\mu^\theta(f, \frac{z}{1-z}, 0, \beta, \alpha) = EM_\mu^\theta(f, \beta, \alpha) =$

$$\left\{ f \in S : \operatorname{Re}\left\{ \frac{z(R_\mu^\theta f(z))'}{R_\mu^\theta f(z)} - \beta \right\} < \alpha \left| \frac{z(R_\mu^\theta f(z))'}{R_\mu^\theta f(z)} - 1 \right|, \alpha \le 0, \beta > 1, 0 \le \theta \le 1, 0 \le \mu < 1, z \in \mathbb{U} \right\},$$

(iii) $M_\mu^\theta(f, \frac{z}{1-z}, 1, \beta, \alpha) = M_\mu^\theta(f, \beta, \alpha) =$

$$\left\{ f \in S : \operatorname{Re}\left\{ 1 + \frac{z(R_\mu^\theta f(z))''}{(R_\mu^\theta f(z))'} - \beta \right\} < \alpha \left| \frac{z(R_\mu^\theta f(z))''}{(R_\mu^\theta f(z))'} \right| \alpha \le 0, \beta > 1, 0 \le \theta \le 1, 0 \le \mu < 1, z \in \mathbb{U} \right\},$$

(iv) $M_\mu^\theta(f, g, 0, \beta, \alpha) = M_\mu^\theta(f, g, \beta, \alpha) =$

$$\left\{ f \in S : \operatorname{Re}\left\{ \frac{z(R_\mu^\theta (f*g)(z))'}{R_\mu^\theta (f*g)(z)} - \beta \right\} < \alpha \left| \frac{z(R_\mu^\theta (f*g)(z))'}{R_\mu^\theta (f*g)(z)} - 1 \right|, \alpha \le 0, \beta > 1, 0 \le \theta \le 1, 0 \le \mu < 1, z \in \mathbb{U} \right\},$$

(v) $M_\mu^\theta(f, g, 1, \beta, \alpha) = TM_\mu^\theta(f, g, \beta, \alpha)$

$$\left\{ f \in S : \operatorname{Re}\left\{ 1 + \frac{z(R_\mu^\theta (f*g)(z))''}{(R_\mu^\theta (f*g)(z))'} - \beta \right\} < \alpha \left| \frac{z(R_\mu^\theta (f*g)(z))''}{(R_\mu^\theta (f*g)(z))'} \right|, \alpha \le 0, \beta > 1, 0 \le \theta \le 1, 0 \le \mu < 1, z \in \mathbb{U} \right\}.$$

Also, we note that:

(1) if $g(z) = z + \sum\limits_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k$ (or $b_k = \Gamma_k(\alpha_1)$), where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1} ......(\alpha_q)_{k-1}}{(\beta_1)_{k-1}...(\beta_s)_{k-1}(k-1)!} \tag{1.10}$$

$(\alpha_i > 0, i = 1, ..., q; \beta_j > 0, j = 1, ..., s; q \leq s+1; q, s \in \mathbb{N}_0 = \mathbb{N}, \mathbb{N} = \{1, 2, ...\})$ then

$R_\mu^\theta(f * g)(z) = M_{\mu,q,s}^\theta(\alpha_1) f(z) = z + \sum\limits_{k=2}^{\infty} K(k, \mu, \theta) \Gamma_k(\alpha_1) a_k z^k$, where $K(k, \mu, \theta)$ is given by (1.8) and $M_\mu^\theta(f, g, \lambda, \beta, \alpha) = M_{\mu,q,s}^\theta([\alpha_1], \lambda, \beta, \alpha)$

$$= \left\{ f \in S : \operatorname{Re}\left\{ \frac{z(M_{\mu,q,s}^\theta(\alpha_1) f(z))' + \lambda z^2 (M_{\mu,q,s}^\theta(\alpha_1) f(z))''}{(1-\lambda) M_{\mu,q,s}^\theta(\alpha_1) f(z) + \lambda z (M_{\mu,q,s}^\theta(\alpha_1) f(z))'} - \beta \right\} \right.$$
$$< \alpha \left| \frac{z(M_{\mu,q,s}^\theta(\alpha_1) f(z))' + \lambda z^2 (M_{\mu,q,s}^\theta(\alpha_1) f(z))''}{(1-\lambda) M_{\mu,q,s}^\theta(\alpha_1) f(z) + \lambda z (M_{\mu,q,s}^\theta(\alpha_1) f(z))'} - 1 \right|,$$
$$\left. \alpha \leq 0, \beta > 1, 0 \leq \theta \leq 1, 0 \leq \mu < 1, 0 \leq \lambda \leq 1, \alpha_i, \beta_j > 0, q, s \in \mathbb{N}_0, z \in \mathbb{U} \right\},$$

where $g(z)$ is the Dziok-Srivastava operator (see [10, 11]) which contains well known operators (see [5, 6, 8, 9, 12, 13, 16, 19] and [20]);

(2) if $g(z) = z + \sum\limits_{k=2}^{\infty} \left[\frac{l+1+\delta(k-1)}{l+1}\right]^m z^k$ (or $b_k = \left[\frac{l+1+\delta(k-1)}{l+1}\right]^m$, $m \in \mathbb{N}_0$; $\delta \geq 0$; $l \geq 0$), then $R_\mu^\theta(f * g)(z) = N_\mu^{\theta,m}(\delta, l) f(z) = z + \sum\limits_{k=2}^{\infty} K(k, \mu, \theta) \left[\frac{l+1+\delta(k-1)}{l+1}\right]^m a_k z^k$ and $M_\mu^\theta(f, g, \lambda, \beta, \alpha) = N_\mu^\theta(l, m, \delta, \lambda, \beta, \alpha)$

$$= \left\{ f \in S : \operatorname{Re}\left\{ \frac{z(N_\mu^{\theta,m}(\delta, l) f(z))' + \lambda z^2 (N_\mu^{\theta,m}(\delta, l) f(z))''}{(1-\lambda) N_\mu^{\theta,m}(\delta, l) f(z) + \lambda z (N_{\mu,f}^{\theta,m}(\delta, l) f(z))'} - \beta \right\} \right.$$
$$< \alpha \left| \frac{z(N_\mu^{\theta,m}(\delta, l) f(z))' + \lambda z^2 (N_\mu^{\theta,m}(\delta, l) f(z))''}{(1-\lambda) N_\mu^{\theta,m}(\delta, l) f(z) + \lambda z (N_\mu^{\theta,m}(\delta, l) f(z))'} - 1 \right|,$$
$$\left. \alpha \leq 0, \beta > 1, 0 \leq \theta \leq 1, 0 \leq \mu < 1, 0 \leq \lambda \leq 1, m \in \mathbb{N}_0; \delta \geq 0; l \geq 0, z \in \mathbb{U} \right\},$$

where $g(z)$ is the Catas operator introduced and studied by Catas et al. [7].

In this paper the techniques used are similar to those of Aouf et al. [1–3], Nishiwaki and Owa [15], Owa and Nishiwaki [17] and Owa and Srivastava [18].

2 Main results

Unless indicated, we assume that $\alpha \leq 0$, $\beta > 1$, $0 \leq \lambda \leq 1$, $0 \leq \theta \leq 1$, $0 \leq \mu < 1$, $k \geq 2$, $K(k, \mu, \theta)$ is defined by (1.8), $b_k \geq 0$ and $z \in \mathbb{U}$.

To prove our results we need the following lemma.

Lemma 3. *[22] The sequence $\{d_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1+2\sum_{k=1}^{\infty} d_k z^k\right\} > 0, \quad (z \in \mathbb{U}). \tag{2.1}$$

Theorem 4. *If $f(z)$ defined by (1.1) satisfies the following condition:*

$$\sum_{k=2}^{\infty} K(k,\mu,\theta)\left[1+\lambda(k-1)\right]\left[(k-1)(1-2\alpha)+|k-2\beta+1|\right] b_k \left|a_k\right| \leq 2(\beta-1). \tag{2.2}$$

Then $f(z) \in M_{\mu}^{\theta}(f,g,\lambda,\beta,\alpha)$.

Proof. Assume that (2.2) holds. It suffices to show that

$$\left|\frac{\frac{z(R_\mu^\theta(f*g)(z))'+\lambda z^2(R_\mu^\theta(f*g)(z))''}{(1-\lambda)R_\mu^\theta(f*g)(z)+\lambda z(R_\mu^\theta(f*g)(z))'}-\alpha\left|\frac{z(R_\mu^\theta(f*g)(z))'+\lambda z^2(R_\mu^\theta(f*g)(z))''}{(1-\lambda)R_\mu^\theta(f*g)(z)+\lambda z(R_\mu^\theta(f*g)(z))'}-1\right|-1}{\frac{z(R_\mu^\theta(f*g)(z))'+\lambda z^2(R_\mu^\theta(f*g)(z))''}{(1-\lambda)R_\mu^\theta(f*g)(z)+\lambda z(R_\mu^\theta(f*g)(z))'}-\alpha\left|\frac{z(R_\mu^\theta(f*g)(z))'+\lambda z^2(R_\mu^\theta(f*g)(z))''}{(1-\lambda)R_\mu^\theta(f*g)(z)+\lambda z(R_\mu^\theta(f*g)(z))'}-1\right|-(2\beta-1)}\right| < 1.$$

We have

$$\begin{aligned}
&\left|\frac{\frac{z(R_\mu^\theta(f*g)(z))'+\lambda z^2(R_\mu^\theta(f*g)(z))''}{(1-\lambda)R_\mu^\theta(f*g)(z)+\lambda z(R_\mu^\theta(f*g)(z))'}-\alpha\left|\frac{z(R_\mu^\theta(f*g)(z))'+\lambda z^2(R_\mu^\theta(f*g)(z))''}{(1-\lambda)R_\mu^\theta(f*g)(z)+\lambda z(R_\mu^\theta(f*g)(z))'}-1\right|-1}{\frac{z(R_\mu^\theta(f*g)(z))'+\lambda z^2(R_\mu^\theta(f*g)(z))''}{(1-\lambda)R_\mu^\theta(f*g)(z)+\lambda z(R_\mu^\theta(f*g)(z))'}-\alpha\left|\frac{z(R_\mu^\theta(f*g)(z))'+\lambda z^2(R_\mu^\theta(f*g)(z))''}{(1-\lambda)R_\mu^\theta(f*g)(z)+\lambda z(R_\mu^\theta(f*g)(z))'}-1\right|-(2\beta-1)}\right| \\
\leq\ & \frac{\sum_{k=2}^{\infty} K(k,\mu,\theta)\left[1+\lambda(k-1)\right](k-1)(1-\alpha) b_k |a_k| \left|z^{k-1}\right|}{2(\beta-1)-\sum_{k=2}^{\infty} K(k,\mu,\theta)\left[1+\lambda(k-1)\right]\left[|k-2\beta+1|-\alpha(k-1)\right] b_k |a_k| \left|z^{k-1}\right|} \\
<\ & \frac{\sum_{k=2}^{\infty} K(k,\mu,\theta)\left[1+\lambda(k-1)\right](k-1)(1-\alpha) b_k |a_k|}{2(\beta-1)-\sum_{k=2}^{\infty} K(k,\mu,\theta)\left[1+\lambda(k-1)\right]\left[|k-2\beta+1|-\alpha(k-1)\right] b_k |a_k|}.
\end{aligned}$$

This last expression is bounded above by 1 if

$$\begin{aligned}
& \sum_{k=2}^{\infty} K(k,\mu,\theta)\left[1+\lambda(k-1)\right](k-1)(1-\alpha) b_k |a_k| \\
& + \sum_{k=2}^{\infty} K(k,\mu,\theta)\left[1+\lambda(k-1)\right]\left[|k-2\beta+1|-\alpha(k-1)\right] b_k |a_k| \\
\leq\ & 2(\beta-1).
\end{aligned}$$

Which leads to (2.2), and hence the proof is completed. □

Corollary 5. *Let $f(z) \in M_{\mu}^{\theta}(f, g, \lambda, \beta, \alpha)$. Then*

$$|a_k| \leq \frac{2(\beta-1)}{K(k,\mu,\theta)\left[1+\lambda(k-1)\right]\left[(k-1)(1-2\alpha)+|k-2\beta+1|\right]b_k} \ (k \geq 2). \quad (2.3)$$

The result is sharp for

$$f(z) = z + \frac{2(\beta-1)}{K(k,\mu,\theta)[1+\lambda(k-1)][(k-1)(1-2\alpha)+|k-2\beta+1|]b_k} z^k \ (k \geq 2). \quad (2.4)$$

Putting $g(z) = \frac{z}{1-z} = z + \sum\limits_{k=2}^{\infty} z^k$ (or $b_k = 1$) and $\lambda = 0$ in Theorem 4, we obtain the following corollary:

Corollary 6. *Let $f(z)$ defined by (1.1) satisfy the following condition:*

$$\sum_{k=2}^{\infty} K(k,\mu,\theta)\left[(k-1)(1-2\alpha)+|k-2\beta+1|\right]|a_k| \leq 2(\beta-1).$$

Then $f(z) \in EM_{\mu}^{\theta}(f, \beta, \alpha)$.

Putting $g(z) = \frac{z}{1-z} = z + \sum\limits_{k=2}^{\infty} z^k$ (*or* $b_k = 1$) and $\lambda = 1$ in Theorem 4, we obtain the following corollary:

Corollary 7. *Let $f(z)$ defined by (1.1) satisfy the following condition:*

$$\sum_{k=2}^{\infty} K(k,\mu,\theta)k\left[(k-1)(1-2\alpha)+|k-2\beta+1|\right]|a_k| \leq 2(\beta-1).$$

Then $f(z) \in M_{\mu}^{\theta}(f, \beta, \alpha)$.

Let $M_{\mu}^{\theta *}(f, g, \lambda, \beta, \alpha)$ be the subclass of $M_{\mu}^{\theta}(f, g, \lambda, \beta, \alpha)$ whose coefficients satisfy (2.2). We note that $M_{\mu}^{\theta *}(f, g, \lambda, \beta, \alpha) \subseteq M_{\mu}^{\theta}(f, g, \lambda, \beta, \alpha)$.

Theorem 8. *Let $f(z) \in M_{\mu}^{\theta *}(f, g, \lambda, \beta, \alpha)$, $b_k \geq b_2 > 0$ $(k \geq 2)$. Then for every function $\Psi \in C$, we have*

$$\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]}(f * \Psi)(z) \prec \Psi(z) \quad (2.5)$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{K(2,\mu,\theta)(1+\lambda)\left(2-\alpha-\beta\right)b_2+(\beta-1)}{K(2,\mu,\theta)(1+\lambda)\left(2-\alpha-\beta\right)b_2}. \quad (2.6)$$

The constant $\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]}$ is the best estimate.

Proof. Let $f(z) \in M_{\mu}^{\theta *}(f,g,\lambda,\beta,\alpha)$ and let $\Psi(z) = z + \sum_{k=2}^{\infty} d_k z^k \in C$. Then we have

$$\begin{aligned} & \tfrac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]}(f*\Psi)(z) \\ = & \tfrac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]}\left(z + \sum_{k=2}^{\infty} a_k d_k z^k\right). \end{aligned} \tag{2.7}$$

Thus, by Definition 1, the subordinating result (2.5) will hold true if

$$\left\{\tfrac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} a_k\right\}_{k=1}^{\infty}, \tag{2.8}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 3, this is equivalent to:

$$\operatorname{Re}\left\{1 + \sum_{k=1}^{\infty} \tfrac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2}{[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} a_k z^k\right\} > 0. \tag{2.9}$$

Now, since

$$K(k,\mu,\theta)\left[1+\lambda(k-1)\right]\left[(k-1)(1-2\alpha)+|k-2\beta+1|\right] b_k,$$

is an increasing function of k $(k \geq 2)$, we have

$$\begin{aligned} & \operatorname{Re}\left\{1 + \sum_{k=1}^{\infty} \tfrac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2}{[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} a_k z^k\right\} \\ = & \operatorname{Re}\left\{1 + \tfrac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2}{[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} z\right\} \\ & + \tfrac{1}{[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} \sum_{k=2}^{\infty} K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta) b_2 a_k z^k \\ \geq & 1 - \tfrac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2}{[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} r - \big(\tfrac{1}{[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} \\ & \times \sum_{k=2}^{\infty} \frac{1}{2} K(k,\mu,\theta)\left[1+\lambda(k-1)\right]\left[(k-1)(1-2\alpha)+|k-2\beta+1|\right] b_k |a_k| r^k\big) \\ > & 1 - \tfrac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2}{[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} r - \tfrac{(\beta-1)}{[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} r \\ = & 1 - r > 0 \ (|z| = r < 1), \end{aligned}$$

where we have also made use of (2.2) of Theorem 4. Thus (2.9) holds true in $\mathbb{U}$. This proves (2.5). The inequality (2.6) follows from (2.5) by taking the convex function $\Psi(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \in C$.

To prove the sharpness of the constant $\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]}$, we consider $f_0(z) \in M_{\mu}^{\theta *}(f,g,\lambda,\beta,\alpha)$ given by

$$f_0(z) = z - \frac{(\beta-1)}{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2} z^2. \tag{2.10}$$

Thus from (2.5), we have

$$\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} f_0(z) \prec \frac{z}{1-z}. \tag{2.11}$$

Moreover, it can easily be verified for $f_0(z)$ given by (2.10) that

$$\min_{|z|\le r} \left\{ \operatorname{Re} \frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} f_0(z) \right\} = -\frac{1}{2}. \tag{2.12}$$

This show that $\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]}$ is the best possible. This completes the proof. □

Putting $g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$ (*or* $b_k = 1$) in Theorems 4 and 8, we obtain the following corollary:

Corollary 9. *Let* $f(z) \in M_{\mu}^{\theta *}(f, \frac{z}{1-z}, \lambda, \beta, \alpha)$ *and satisfies the condition*

$$\sum_{k=2}^{\infty} K(k,\mu,\theta)\left[1+\lambda(k-1)\right]\left[(k-1)(1-2\alpha)+|k-2\beta+1|\right]|a_k| \le 2(\beta-1).$$

Then for every $\Psi(z) \in C$, *we have*

$$\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)+(\beta-1)]}(f*\Psi)(z) \prec \Psi(z),$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{K(2,\mu,\theta)(1+\lambda)\,(2-\alpha-\beta)+(\beta-1)}{K(2,\mu,\theta)(1+\lambda)\,(2-\alpha-\beta)}.$$

The constant $\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)+(\beta-1)]}$ *is the best estimate.*

Putting $g(z) = z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k$ (or $b_k = \Gamma_k(\alpha_1)$) in Theorems 4 and 8, we obtain the following corollary:

Corollary 10. *Let* $f(z) \in M_{\mu,q,s}^{\theta *}([\alpha_1];\lambda,\beta,\alpha)$ *and satisfies the condition*

$$\sum_{k=2}^{\infty} K(k,\mu,\theta)\left[1+\lambda(k-1)\right]\left[(k-1)(1-2\alpha)+|k-2\beta+1|\right]\Gamma_k(\alpha_1)\,|a_k| \le 2(\beta-1).$$

Then for every $\Psi(z) \in C$, *we have*

$$\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)\Gamma_2(\alpha_1)}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)\Gamma_2(\alpha_1)+(\beta-1)]}(f*\Psi)(z) \prec \Psi(z),$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)\Gamma_2(\alpha_1)+(\beta-1)}{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)\Gamma_2(\alpha_1)}.$$

The constant $\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)\Gamma_2(\alpha_1)}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)\Gamma_2(\alpha_1)+(\beta-1)]}$ *is the best estimate.*

Putting $g(z) = z + \sum_{k=2}^{\infty}\left[\frac{l+1+\delta(k-1)}{l+1}\right]^m z^k$ (or $b_k = \left[\frac{l+1+\delta(k-1)}{l+1}\right]^m$, $m \in \mathbb{N}_0$; $\delta \geq 0$; $l \geq 0$), in Theorem 4 and Theorem 8, we obtain the following corollary:

Corollary 11. *Let* $f(z) \in N_{\mu}^{\theta *}(l,m,\delta,\lambda,\beta,\alpha)$ *and satisfies the condition*

$$\begin{aligned}&\sum_{k=2}^{\infty} K(k,\mu,\theta)\left[1+\lambda(k-1)\right]\left[(k-1)(1-2\alpha)+|k-2\beta+1|\right]\left[\frac{l+1+\delta(k-1)}{l+1}\right]^m |a_k| \\ \leq\ & 2(\beta-1).\end{aligned}$$

Then for every $\Psi(z) \in C$, *we have*

$$\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)[l+1+\delta)]^m}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)[l+1+\delta]^m+[l+1]^m(\beta-1)]}(f*\Psi)(z) \prec \Psi(z),$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)[l+1+\delta]^m+[l+1]^m(\beta-1)}{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)[l+1+\delta]^m}.$$

The constant $\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)[l+1+\delta]^m}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)[l+1+\delta]^m+[l+1]^m(\beta-1)]}$ *is the best estimate.*

Remark 12. *Putting* $b_k = K(k,\mu,\theta)$ *in Aouf et al. [2], we obtain Corollary 9, above.*

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