SUBORDINATING RESULTS FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY HADAMARD PRODUCT AND ATSHAN AND RAFID OPERATOR

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Abstract. In this paper, we defined a class of analytic functions defined by Hadamard product and Atshan and Rafid operator and obtained some subordinating results for functions in this class.

1 Introduction

Let $S$ be the class of analytic univalent functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \ z \in \mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\} \ (1.1)$$

and $C$ be the subclass of $f(z) \in S$ which are convex in $\mathbb{U}$ and let $M(\beta)$ and $N(\beta)$ denote the subclasses of $S$ consisting of $f(z)$ which satisfying (see [2,3,15,17,18,21])

$$M(\beta) = \text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} < \beta, \ \beta > 1 \ (1.2)$$

and

$$N(\beta) = \text{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \beta, \ \beta > 1. \ (1.3)$$

Then, we note that $f(z) \in N(\beta) \Leftrightarrow z f'(z) \in M(\beta)$.

For $f \in S$ given by (1.1) and $g \in S$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \ (b_k \geq 0), \ (1.4)$$

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then the Hadamard product \( f \ast g \) of \( f \) and \( g \) is defined by

\[
(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g \ast f)(z).
\] (1.5)

Let \( f \) and \( g \) be analytic in \( \mathbb{U} \), then \( f \) is subordinate to \( g \) \((f(z) \prec g(z))\) if there exists an analytic function \( w(z) \), with \( w(0) = 0 \) and \(|w(z)| < 1 \) \((z \in \mathbb{U})\), such that \( f(z) = g(w(z)) \) (see [14]), and if \( g \) is univalent in \( \mathbb{U} \), then

\[
f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).
\]

**Definition 1.** (Subordinating factor sequence) [22] A sequence \( \{d_k\}_{k=1}^{\infty} \) of complex numbers is said to be a subordinating factor sequence if, whenever \( f \) of the form (1.1) is analytic, univalent and convex in \( \mathbb{U} \), we have the subordination given by

\[
\sum_{k=1}^{\infty} a_k d_k z^k \prec f(z) \quad (z \in \mathbb{U}, \; a_1 = 1).
\] (1.6)

For \( f(z) \in S \), Atshan and Rafid [4] defined the operator \( R_{\theta, \mu}^\alpha \) for \( 0 \leq \mu < 1 \), \( 0 \leq \theta \leq 1 \) by

\[
R_{\theta, \mu}^\alpha(f(z)) = \frac{1}{(1 - \mu)^{1+\theta} \Gamma(\theta + 1)} \int_0^{\infty} t^{\theta-1} e^{-\frac{t}{1-\mu}} f(zt) dt = z + \sum_{k=2}^{\infty} K(k, \mu, \theta) a_k z^k,
\] (1.7)

where

\[
K(k, \mu, \theta) = \frac{(1 - \mu)^{k-1} \Gamma(\theta + k)}{\Gamma(\theta + 1)}.
\] (1.8)

**Definition 2.** By using the operator \( R_{\theta, \mu}^\alpha \) and for \( \alpha \leq 0 \), \( \beta > 1 \), \( 0 \leq \lambda \leq 1 \), \( 0 \leq \mu < 1 \), \( 0 \leq \theta \leq 1 \), let \( M_{\mu}^\alpha(f, g, \lambda, \beta, \alpha) \) be the class consisting of functions \( f, g \in S \) satisfying

\[
\text{Re}\left\{ \frac{z(R_{\mu}^\alpha(f \ast g)(z))'}{(1 - \lambda)R_{\mu}^\alpha(f \ast g)(z) + \lambda z R_{\mu}^\alpha(f \ast g)(z)'} - \beta \right\} < \alpha \left| \frac{z(R_{\mu}^\alpha(f \ast g)(z))'}{(1 - \lambda)R_{\mu}^\alpha(f \ast g)(z) + \lambda z R_{\mu}^\alpha(f \ast g)(z)'} - 1 \right|.
\] (1.9)

For suitable choices of \( \lambda \) and \( g \), we obtain the new classes:

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Subordinating results for a class of analytic functions

(i) \[ M_\mu^\theta(f, \frac{z}{1-z}, \lambda, \beta, \alpha) = M_\mu^\theta(f, \lambda, \beta, \alpha) = \]
\[
\left\{ f \in S : \text{Re} \left( \frac{z(R_\mu^\theta f(z))' + \lambda z(R_\mu^\theta f(z))''}{(1 - \lambda)R_\mu^\theta f(z) + \lambda z(R_\mu^\theta f(z))''} - \beta \right) \right\} < \alpha
\]
\[
\left. \left( z(R_\mu^\theta f(z))' + \lambda z(R_\mu^\theta f(z))'' \right) \right|_{(1 - \lambda)R_\mu^\theta f(z) + \lambda z(R_\mu^\theta f(z))''} - 1,
\]
\[
\alpha \leq 0, \beta > 1, 0 \leq \lambda \leq 1, 0 \leq \theta \leq 1, 0 \leq \mu < 1, z \in \mathbb{U}, \}
\]

(ii) \[ M_\mu^\theta(f, \frac{z}{1-z}, 0, \beta, \alpha) = EM_\mu^\theta(f, \beta, \alpha) = \]
\[
\left\{ f \in S : \text{Re} \left( \frac{z(R_\mu^\theta f(z))'}{R_\mu^\theta f(z)} - \beta \right) \right\} < \alpha
\]
\[
\left. \left( z(R_\mu^\theta f(z))' \right) \right|_{R_\mu^\theta f(z)} - 1,
\]
\[
\alpha \leq 0, \beta > 1, 0 \leq \theta \leq 1, 0 \leq \mu < 1, z \in \mathbb{U}, \}
\]

(iii) \[ M_\mu^\theta(f, \frac{z}{1-z}, 1, \beta, \alpha) = M_\mu^\theta(f, \lambda, \beta, \alpha) = \]
\[
\left\{ f \in S : \text{Re} \left( \frac{1 + z(R_\mu^\theta f(z))''}{(R_\mu^\theta f(z))'} - \beta \right) \right\} < \alpha
\]
\[
\left. \left( z(R_\mu^\theta f(z))'' \right) \right|_{(R_\mu^\theta f(z))'}
\]
\[
\alpha \leq 0, \beta > 1, 0 \leq \theta \leq 1, 0 \leq \mu < 1, z \in \mathbb{U}, \}
\]

(iv) \[ M_\mu^\theta(f, g, 0, \beta, \alpha) = M_\mu^\theta(f, g, \beta, \alpha) = \]
\[
\left\{ f \in S : \text{Re} \left( \frac{z(R_\mu^\theta (f * g)(z))'}{R_\mu^\theta (f * g)(z)} - \beta \right) \right\} < \alpha
\]
\[
\left. \left( z(R_\mu^\theta (f * g)(z))' \right) \right|_{R_\mu^\theta (f * g)(z)} - 1,
\]
\[
\alpha \leq 0, \beta > 1, 0 \leq \theta \leq 1, 0 \leq \mu < 1, z \in \mathbb{U}, \}
\]

(v) \[ M_\mu^\theta(f, g, 1, \beta, \alpha) = TM_\mu^\theta(f, g, \beta, \alpha) = \]
\[
\left\{ f \in S : \text{Re} \left( \frac{1 + z(R_\mu^\theta (f * g)(z))''}{(R_\mu^\theta (f * g)(z))'} - \beta \right) \right\} < \alpha
\]
\[
\left. \left( z(R_\mu^\theta (f * g)(z))'' \right) \right|_{(R_\mu^\theta (f * g)(z))'}
\]
\[
\alpha \leq 0, \beta > 1, 0 \leq \theta \leq 1, 0 \leq \mu < 1, z \in \mathbb{U}, \}
\]
Also, we note that:

(1) if \( g(z) = z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1)z^k \) (or \( b_k = \Gamma_k(\alpha_1) \)), where

\[
\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1} \cdots (\alpha_1)_{k-1}}{(\beta_1)_{k-1} \cdots (\beta_s)_{k-1}(k-1)!}
\]

(1.10)

(\( \alpha_i > 0, i = 1, \ldots, q; \beta_j > 0, j = 1, \ldots, s; q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N}, N = \{1, 2, \ldots\} \)) then

\[
R^\theta_k(f * g)(z) = M^\theta_{\mu,q,s}(\alpha_1)f(z) = z + \sum_{k=2}^{\infty} K(k, \mu, \theta)\Gamma_k(\alpha_1)a_kz^k,
\]

where \( K(k, \mu, \theta) \) is given by (1.8) and \( M^\theta_{\mu}(f, g, \lambda, \beta, \alpha) = M^\theta_{\mu,q,s}([\alpha_1], \lambda, \beta, \alpha) \)

\[
f \in S : \text{Re} \left\{ \frac{z(M^\theta_{\mu,q,s}(\alpha_1)f(z))'}{(1 - \lambda)M^\theta_{\mu,q,s}(\alpha_1)f(z) + \lambda z(M^\theta_{\mu,q,s}(\alpha_1)f(z))^\prime} - \beta \right\}
\]

\[
< \alpha \left| \frac{z(M^\theta_{\mu,q,s}(\alpha_1)f(z))'}{z(M^\theta_{\mu,q,s}(\alpha_1)f(z))'} - 1 \right|,
\]

\[
\alpha \leq 0, \beta > 1, 0 < \theta < 1, 0 \leq \mu < 1, \lambda \leq 1, \alpha_i, \beta_j > 0, q, s \in \mathbb{N}_0, z \in \mathcal{U},
\]

where \( g(z) \) is the Dziok-Srivastava operator (see [10, 11]) which contains well known operators (see [5, 6, 8, 9, 12, 13, 16, 19] and [20]):

(2) if \( g(z) = z + \sum_{k=2}^{\infty} \left[ \frac{l+1+\delta(k-1)}{l+1} \right]^m z^k \) (or \( b_k = \left[ \frac{l+1+\delta(k-1)}{l+1} \right]^m \), \( m \in \mathbb{N}_0; \delta \geq 0; l \geq 0 \)), then \( R^\theta_k(f * g)(z) = N^\theta_{\mu,m}(\delta, l)f(z) = z + \sum_{k=2}^{\infty} K(k, \mu, \theta) \left[ \frac{l+1+\delta(k-1)}{l+1} \right]^m a_kz^k \)

and \( M^\theta_{\mu}(f, g, \lambda, \beta, \alpha) = N^\theta_{\mu}(l, m, \delta, \lambda, \beta, \alpha) \)

\[
f \in S : \text{Re} \left\{ \frac{z(N^\theta_{\mu,m}(\delta, l)f(z))'}{(1 - \lambda)N^\theta_{\mu,m}(\delta, l)f(z) + \lambda z(N^\theta_{\mu,m}(\delta, l)f(z))^\prime} - \beta \right\}
\]

\[
< \alpha \left( \frac{z(N^\theta_{\mu,m}(\delta, l)f(z))'}{z(N^\theta_{\mu,m}(\delta, l)f(z))'} - 1 \right),
\]

\[
\alpha \leq 0, \beta > 1, 0 < \theta < 1, 0 \leq \mu < 1, \lambda \leq 1, m \in \mathbb{N}_0; \delta \geq 0; l \geq 0; z \in \mathcal{U},
\]

where \( g(z) \) is the Catas operator introduced and studied by Catas et al. [7].

In this paper the techniques used are similar to those of Aouf et al. [1–3], Nishiwaki and Owa [15], Owa and Nishiwaki [17] and Owa and Srivastava [18].

2 Main results

Unless indicated, we assume that \( \alpha \leq 0, \beta > 1, 0 \leq \lambda \leq 1, 0 \leq \theta \leq 1, 0 \leq \mu < 1, k \geq 2, K(k, \mu, \theta) \) is defined by (1.8), \( b_k \geq 0 \) and \( z \in \mathcal{U} \).

To prove our results we need the following lemma.

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Lemma 3. \[\text{[22]}\] The sequence \(\{d_k\}_{k=1}^{\infty}\) is a subordinating factor sequence if and only if
\[
\text{Re}\left\{1 + 2 \sum_{k=1}^{\infty} d_k z^k\right\} > 0, \quad (z \in \mathbb{U}).
\] (2.1)

Theorem 4. If \(f(z)\) defined by (1.1) satisfies the following condition:
\[
\sum_{k=2}^{\infty} K(k, \mu, \theta) [1 + \lambda(k - 1)] [(k - 1)(1 - 2\alpha) + |k - 2\beta + 1|] b_k |a_k| \leq 2(\beta - 1).
\] (2.2)

Then \(f(z) \in M^\theta_{\mu}(f, g, \lambda, \beta, \alpha)\).

Proof. Assume that (2.2) holds. It suffices to show that
\[
\left|\frac{z(R_0^\theta(f \ast g)(z))' + \lambda z^2 (R_0^\theta(f \ast g)(z))''}{(1 - \lambda) R_0^\theta(f \ast g)(z) + \lambda z (R_0^\theta(f \ast g)(z))'} - 1\right| - 1 - (2\beta - 1) < 1.
\]

We have
\[
\left|\frac{z(R_0^\theta(f \ast g)(z))' + \lambda z^2 (R_0^\theta(f \ast g)(z))''}{(1 - \lambda) R_0^\theta(f \ast g)(z) + \lambda z (R_0^\theta(f \ast g)(z))'} - 1\right| - 1 - (2\beta - 1) \leq \frac{\sum_{k=2}^{\infty} K(k, \mu, \theta) [1 + \lambda(k - 1)] (k - 1)(1 - \alpha) b_k |a_k| |z|^{k-1}}{2(\beta - 1) - \sum_{k=2}^{\infty} K(k, \mu, \theta) [1 + \lambda(k - 1)] [k - 2\beta + 1 - \alpha(k - 1)] b_k |a_k| |z|^{k-1}}.
\]

This last expression is bounded above by 1 if
\[
\sum_{k=2}^{\infty} K(k, \mu, \theta) [1 + \lambda(k - 1)] (k - 1)(1 - \alpha) b_k |a_k| + \sum_{k=2}^{\infty} K(k, \mu, \theta) [1 + \lambda(k - 1)] [k - 2\beta + 1 - \alpha(k - 1)] b_k |a_k| \leq 2(\beta - 1).
\]

Which leads to (2.2), and hence the proof is completed. \(\square\)

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Corollary 5. Let \( f(z) \in M^\theta_\mu(f, g, \lambda, \beta, \alpha) \). Then

\[
|a_k| \leq \frac{2(\beta - 1)}{K(k, \mu, \theta) \left( 1 + \lambda(k-1) \right) \left( (k-1)(1-2\alpha) + |k-2\beta+1| \right) b_k} (k \geq 2). \tag{2.3}
\]

The result is sharp for

\[
f(z) = z + \frac{2(\beta - 1)}{K(k, \mu, \theta) \left( 1 + \lambda(k-1) \right) \left( (k-1)(1-2\alpha) + |k-2\beta+1| \right) b_k} z_k (k \geq 2).	ag{2.4}
\]

Putting \( g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \) (or \( b_k = 1 \)) and \( \lambda = 0 \) in Theorem 4, we obtain the following corollary:

Corollary 6. Let \( f(z) \) defined by (1.1) satisfy the following condition:

\[
\sum_{k=2}^{\infty} K(k, \mu, \theta) \left( (k-1)(1-2\alpha) + |k-2\beta+1| \right) |a_k| \leq 2(\beta - 1).
\]

Then \( f(z) \in EM^\theta_\mu(f, \beta, \alpha) \).

Putting \( g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \) (or \( b_k = 1 \)) and \( \lambda = 1 \) in Theorem 4, we obtain the following corollary:

Corollary 7. Let \( f(z) \) defined by (1.1) satisfy the following condition:

\[
\sum_{k=2}^{\infty} K(k, \mu, \theta) k \left( (k-1)(1-2\alpha) + |k-2\beta+1| \right) |a_k| \leq 2(\beta - 1).
\]

Then \( f(z) \in M^\theta_\mu(f, \beta, \alpha) \).

Let \( M^\theta_\mu(f, g, \lambda, \beta, \alpha) \) be the subclass of \( M^\theta_\mu(f, g, \lambda, \beta, \alpha) \) whose coefficients satisfy (2.2). We note that \( M^\theta_\mu(f, g, \lambda, \beta, \alpha) \subseteq M^\theta_\mu(f, g, \lambda, \beta, \alpha) \).

Theorem 8. Let \( f(z) \in M^\theta_\mu(f, g, \lambda, \beta, \alpha) \), \( b_k \geq b_2 > 0 \) (\( k \geq 2 \)). Then for every function \( \Psi \in C \), we have

\[
\frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{2[K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]}(f * \Psi)(z) < \Psi(z) \tag{2.5}
\]

and

\[
\text{Re} \{ f(z) \} > -\frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)} \frac{(2-\alpha-\beta)b_2}{b_2+(\beta-1)} \tag{2.6}
\]

The constant \( \frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{2[K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} \) is the best estimate.
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Proof. Let \( f(z) \in M^\theta_\mu (f, g, \lambda, \beta, \alpha) \) and let \( \Psi(z) = z + \sum_{k=2}^\infty d_k z^k \in C \). Then we have

\[
\frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{2[K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]}(f \ast \Psi)(z)
= \frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{2[K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]}\left(z + \sum_{k=2}^\infty a_k d_k z^k \right). \tag{2.7}
\]

Thus, by Definition 1, the subordinating result (2.5) will hold true if

\[
\left\{ \frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{2[K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} a_k \xi \right\}_{k=1}^\infty,
\tag{2.8}
\]

is a subordinating factor sequence, with \( a_1 = 1 \). In view of Lemma 3, this is equivalent to:

\[
\text{Re} \left\{ 1 + \sum_{k=1}^\infty \frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{[K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} a_k \xi \right\} > 0. \tag{2.9}
\]

Now, since

\[
K(k, \mu, \theta) [1 + \lambda(k-1)] [(k-1)(1-2\alpha) + k-2\beta + 1] b_k,
\]

is an increasing function of \( k \) \((k \geq 2)\), we have

\[
\text{Re} \left\{ 1 + \sum_{k=1}^\infty \frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{[K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} a_k \xi \right\}
= \text{Re} \left\{ 1 + \frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{[K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} \right\}
+ \frac{1}{[K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} \sum_{k=2}^\infty K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2 a_k \xi
\geq 1 - \frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{[K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} - \frac{1}{[K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} \sum_{k=2}^\infty \frac{1}{2} K(k, \mu, \theta) [1 + \lambda(k-1)] [(k-1)(1-2\alpha) + k-2\beta + 1] b_k |a_k| r^k
\geq 1 - \frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{[K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} - \frac{1}{[K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)]} \sum_{k=2}^\infty \frac{1}{2} K(k, \mu, \theta) [1 + \lambda(k-1)] [(k-1)(1-2\alpha) + k-2\beta + 1] b_k |a_k| r^k
= 1 - r > 0 \quad (|z| = r < 1),
\]

where we have also made use of (2.2) of Theorem 4. Thus (2.9) holds true in \( U \). This proves (2.5). The inequality (2.6) follows from (2.5) by taking the convex function

\[
\Psi(z) = \frac{z}{1-z} = z + \sum_{k=2}^\infty z^k \in C.
\]
To prove the sharpness of the constant $\frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{2K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)}$, we consider $f_0(z) \in M_\mu^{\theta_*}(f, g, \lambda, \beta, \alpha)$ given by

$$f_0(z) = z - \frac{(\beta-1)}{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2} z^2.$$  \hfill (2.10)

Thus from (2.5), we have

$$\frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{2K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)} f_0(z) < \frac{z}{1-z}.$$  \hfill (2.11)

Moreover, it can easily be verified for $f_0(z)$ given by (2.10) that

$$\min_{|z| \leq r} \left\{ \text{Re} \left\{ \frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{2K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)} f_0(z) \right\} \right\} = -\frac{1}{2}.$$  \hfill (2.12)

This shows that $\frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2}{2K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)b_2+(\beta-1)}$ is the best possible. This completes the proof.

Putting $g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$ (or $b_k = 1$) in Theorems 4 and 8, we obtain the following corollary:

**Corollary 9.** Let $f(z) \in M_\mu^{\theta_*}(f, \frac{z}{1-z}, \lambda, \beta, \alpha)$ and satisfies the condition

$$\sum_{k=2}^{\infty} K(k, \mu, \theta) [1 + \lambda(k-1)] [(k-1)(1-2\alpha) + |k-2\beta+1| |a_k|] \leq 2(\beta-1).$$

Then for every $\Psi(z) \in C$, we have

$$\frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)}{2K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)+(\beta-1)} (f * \Psi)(z) < \Psi(z),$$

and

$$\text{Re} \left\{ f(z) \right\} > -\frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta) + (\beta-1)}{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)}. $$

The constant $\frac{K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)}{2K(2, \mu, \theta)(1+\lambda)(2-\alpha-\beta)+(\beta-1)}$ is the best estimate.

Putting $g(z) = z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k$ (or $b_k = \Gamma_k(\alpha_1)$) in Theorems 4 and 8, we obtain the following corollary:

**Corollary 10.** Let $f(z) \in M_\mu^{\theta_*}([\alpha_1]; \lambda, \beta, \alpha)$ and satisfies the condition

$$\sum_{k=2}^{\infty} K(k, \mu, \theta) [1 + \lambda(k-1)] [(k-1)(1-2\alpha) + |k-2\beta+1| \Gamma_k(\alpha_1) |a_k|] \leq 2(\beta-1).$$

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Then for every $\Psi(z) \in C$, we have
\[
\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)\Gamma_2(\alpha_1)}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)\Gamma_2(\alpha_1)+(\beta-1)]} (f * \Psi)(z) < \Psi(z),
\]
and
\[
\Re \{f(z)\} > - \frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)\Gamma_2(\alpha_1)}{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)\Gamma_2(\alpha_1)}.
\]
The constant $\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)\Gamma_2(\alpha_1)}{2[K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)\Gamma_2(\alpha_1)+(\beta-1)]}$ is the best estimate.

Putting $g(z) = z + \sum_{k=2}^{\infty} \left[ \frac{1+\delta(k-1)}{l+1} \right]^m z^k$ (or $b_k = \left[ \frac{1+\delta(k-1)}{l+1} \right]^m$, $m \in \mathbb{N}_0$; $\delta \geq 0$; $l \geq 0$), in Theorem 4 and Theorem 8, we obtain the following corollary:

**Corollary 11.** Let $f(z) \in N^\mu_{\alpha}(l,m,\delta,\lambda,\beta,\alpha)$ and satisfies the condition
\[
\sum_{k=2}^{\infty} K(k,\mu,\theta) [1 + \lambda(k-1)] [(k-1)(1-2\alpha) + |k-2\beta+1|] \left[ \frac{1+\delta(k-1)}{l+1} \right]^m |a_k| \leq 2(\beta-1).
\]
Then for every $\Psi(z) \in C$, we have
\[
\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)[l+1+\delta]^m}{2K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)[l+1+\delta]^m+(\beta-1)}(f * \Psi)(z) < \Psi(z),
\]
and
\[
\Re \{f(z)\} > - \frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)[l+1+\delta]^m+(\beta-1)}{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)[l+1+\delta]^m}.
\]
The constant $\frac{K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)[l+1+\delta]^m}{2K(2,\mu,\theta)(1+\lambda)(2-\alpha-\beta)[l+1+\delta]^m+(\beta-1)}$ is the best estimate.

**Remark 12.** Putting $b_k = K(k,\mu,\theta)$ in Aouf et al. [2], we obtain Corollary 9, above.

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**References**


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