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TWO SHARED SET PROBLEMS IN THE LIGHT OF POWERS OF MEROMORPHIC FUNCTIONS

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Abstract. In the paper, we deal the two shared set problems in view of powers of meromorphic functions and find results in the sense of least cardinality. We have also shown the sharpness of our main results. Moreover, one of our main results improve a result of Yi [25] significantly.

1 Introduction

The uniqueness theory of entire or meromorphic functions via pre-images of it's shared sets is an established and active area of research. The genesis of this theory is the famous "Gross Problem" introduced in 1976 by Fred Gross [9]. So first of all, we recall some basic definitions and the famous "Gross Problem" as follows.

Definition 1.1. *[9] For a non-constant meromorphic function f and $S \subset \mathbb{C} \cup \{\infty\}$, let $E_f(S) = \bigcup_{a \in S}\{(z,p) \in \mathbb{C} \times \mathbb{N} : f(z) = a$ with multiplicity $p\}$ $\left(\overline{E}_f(S) = \bigcup_{a \in S}\{(z,1) \in \mathbb{C} \times \mathbb{N} : f(z) = a\}\right)$. Then we say f, g share the set S CM(IM) if $E_f(S) = E_g(S)$ $\left(\overline{E}_f(S) = \overline{E}_g(S)\right)$.*

If S contains only one element, then we say f and g share the value a CM(IM).

Definition 1.2 ([13],[14])**.** *Let k be a non-negative integer or infinity. For $a \in \overline{\mathbb{C}}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 1.3. *[13] For $S \subset \mathbb{C} \cup \{\infty\}$, we define $E_f(S,k) = \cup_{a \in S} E_k(a; f)$, where k is a non-negative integer $a \in S$ or infinity. Clearly $E_f(S) = E_f(S,\infty)$ and $\overline{E}_f(S) = E_f(S,0)$. Further, if $E_f(S,k) = E_g(S,k)$ for two non-constant*

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meromorphic functions f and g, then we say that f and g share the set S with weight k.

Gross Problem: *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2$ must be identical ? If the answer is affirmative, it would be interesting to know how large both sets would have to be ?*

Regarding "Gross Problem", a lot of investigations [21, 20, 22, 23, 24, 25] have been made by various authors in different time. Finally, Yi [25] settled the problem in 1998 by providing two sets, one containing only one element and the other containing three elements. Below we recall the result.

Theorem A. *[25] Let $S_1 = \{0\}$ and $S_2 = \{z : z^2(z + a) - b = 0\}$, where a and b are two non-zero constants such that the algebraic equation $z^2(z + a) - b = 0$ has no multiple roots. If f and g are any two non-constant entire functions satisfying $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2$, then $f \equiv g$.*

In the same paper [25], Yi also provided examples showing that the cardinality of these two sets are the smallest possible. Now observe that, in view of the notion of weighted sharing one may naturally inquire about the fact whether the sharing conditions of the sets in *Theorem A* are also settled or can be relaxed further. Hence let us formulate this query as follows.

Question 1.1. *Can we have the same result as obtained in* Theorem A *under more relaxed sharing hypothesis?*

To obtain the best possible answer of the above question is one of the motivations of the paper. In fact, in *Theorem 2.2* of this paper we answer this question affirmatively and improve *Theorem A* significantly.

On the other hand, after the initiation of "Gross Problem" researchers also started to study the analogue of "Gross Problem" for meromorphic functions. Below we recall the problem.

Question 1.2. *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2$ must be identical ? If the answer is affirmative it would be interesting to know how large both sets would have to be ?*

With respect to *Question 1.2*, also a number of affirmative answers [17, 8, 27, 2, 28, 5, 21, 4, 19, 6, 7] have been obtained by various authors throughout these years. Naturally like Gross Problem, the research in this direction mainly confined towards obtaining the shared sets with least possible cardinalities. In this connection, the notion of bi-unique range sets introduced by Banerjee in [4] played a vital role to obtain shared sets with smallest possible cardinalities. Below we recall the definition of bi-unique range sets and the result of Banerjee in [4], respectively.

Definition 1.4. *[4] A pair of finite sets S_1 and S_2 in $\mathbb{C}$ is called bi-unique range sets for meromorphic (entire) functions with weights m, k if for any two non-constant meromorphic (entire) functions f and g, $E_f(S_1, m) = E_g(S_1, m)$, $E_f(S_2, k) = E_g(S_2, k)$ imply $f \equiv g$. We write S_i's $i = 1,\ 2$ as BURSMm, k (BURSEm, k) in short. As usual if both $m = k = \infty$, we say S_i's $i = 1,\ 2$ as BURSM (BURSE).*

Theorem B. *[4] Let $S_1 = \{0, 1\}$ and $S_2 = \left\{z : \frac{(n-1)(n-2)}{2}z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2} - d = 0\right\}$, where $n(\geq 5)$ is an integer and $d \neq 0,\ 1,\ \frac{1}{2}$ is a complex number such that $d^2 - d + 1 \neq 0$. Then S_i's $i = 1,\ 2$ are $BURSM1, 3$, $BURSM3, 2$.*

Observe that, in *Theorem B* the least cardinalities of the shared sets are 2 and 5, respectively. Later to obtain the shared sets with lesser cardinalities, Banerjee-Mallick [7] considered the class of derivatives of the meromorphic functions instead of the class of meromorphic functions only. Below we recall the result of Banerjee-Mallick [7].

Theorem C. *[7] Let $S_1 = \{0\}$, $S_2 = \{z : z^n + az^{n-1} + b = 0\}$, where $n(\geq 4)$ be an integer and a, b be two non-zero constants such that $z^n + az^{n-1} + b$ has no multiple zero. If for two non-constant meromorphic functions f and g, with $f^{(k)}$ and $g^{(k)}$ having no simple $-a\frac{(n-1)}{n}$ points; $E_{f^{(k)}}(S_1, 1) = E_{g^{(k)}}(S_1, 1)$ and $E_{f^{(k)}}(S_2, 2) = E_{g^{(k)}}(S_2, 2)$, then $f^{(k)} \equiv g^{(k)}$.*

Clearly S_1 and S_2 in *Theorem C* are bi-unique range sets for the derivatives of meromorphic functions. Also note that in *Theorem C* considering the derivatives of the meromorphic functions instead of the original ones, the authors became successful to obtain smaller sets or in particular smaller bi-unique range sets than those of *Theorem B*. In the same paper, authors also mentioned in their concluding section that using the method adopted to prove Theorem C the lower bound of the degree of the underlying polynomial of a BURSM cannot be reduced further. So natural query arose whether we can obtain further smaller sets for any subclass of meromorphic functions or these are the smallest possible sets in the direction of *Question 1.2*. Pertinent to this, the authors posed the following question in their paper [7] for further investigations.

Question 1.3. *[7] Does there exist any pair of bi-unique range sets, even if for a special class of meromorphic functions, sum of whose cardinalities are less than 5?*

To obtain the answer of *Question 1.3* is the another motivation of the paper. In fact, in *Theorem 2.1* we answer *Question 1.3* affirmatively. We obtain a pair of bi-unique range sets with one and three elements respectively, for a special class of meromorphic functions. This result also provides two shared sets with least possible cardinality in the direction of *Question 1.2*. We have further exhibited two examples to show the sharpness of the result. For this purpose, we consider the class of powers of meromorphic functions. Apropos of that, we define the following notions.

Definition 1.5. *Suppose $M(\mathbb{C})$ denotes the set of all meromorphic functions defined on $\mathbb{C}$. We define $M^d(\mathbb{C})$ to be the collection of all such meromorphic functions which are powers of some meromorphic functions of power at least d, where d is a positive integer. That is, in set theoretic notation, $M^d(\mathbb{C}) = \{f^{d+r} \mid d \in \mathbb{N}, \ r \in \mathbb{N} \cup \{0\} \ \text{and} \ f \in M(\mathbb{C})\}$.*

Clearly, $M^p(\mathbb{C}) \subset M^s(\mathbb{C}) \subset M^1(\mathbb{C}) = M(\mathbb{C})$ whenever $p > s > 1$.

Similar notions can be defined for entire functions and be denoted by $E(\mathbb{C})$ and $E^d(\mathbb{C})$. In that case also we would have $E^p(\mathbb{C}) \subset E^s(\mathbb{C}) \subset E^1(\mathbb{C}) = E(\mathbb{C})$ whenever $p > s > 1$.

Definition 1.6. *A pair of finite sets S_1 and S_2 in $\mathbb{C}$ is called bi-unique range sets for meromorphic (entire) functions of power at least d with weights m, k; if for any two non-constant functions $f, g \in M^d(\mathbb{C})$, $E_f(S_1, m) = E_g(S_1, m)$ and $E_f(S_2, k) = E_g(S_2, k)$ imply $f \equiv g$. We write S_i's $i = 1, 2$ as $BURSP^dMm$, k ($BURSP^dEm$, k) in short.*

If both $m = k = \infty$, then we say S_i's $i = 1, 2$ are bi-unique range sets for meromorphic (entire) functions of power at least d or $BURSP^dM$ ($BURSP^dE$) in short.

As usual, if $m = k = \infty$ and $d = 1$, then we say S_i's $i = 1, 2$ are bi-unique range sets for meromorphic (entire) functions or BURSM (BURSE) in short.

Using this notion of $BURSP^dMm, k(BURSP^dEm, k)$, we answer *Question 1.3* as well as *Question 1.1*. Now we proceed to our main results.

2 Main Results

Let us consider the following polynomial

$$P(z) = z^n + az^{n-1} + b, \tag{2.1}$$

where $n(\geq 2) \in \mathbb{N}$ and $a, b \in \mathbb{C} - \{0\}$ be such that $P(z)$ has only simple zeros.

Theorem 2.1. *Let $P(z)$ be given by (2.1) and $S_1 = \{0\}$, $S_2 = \{z : P(z) = 0\}$. Then S_i's are $BURSP^dM0, 2$ for $n > 2 + \dfrac{13}{2d}$, where $d \geq 2$.*

The following example shows that the condition $d \geq 2$ in *Theorem 2.1* is sharp.

Example 2.1. *Let $S_1 = \{0\}$ and $S_2 = \{z : P(z) = 0\}$, where $P(z)$ is given by (2.1). Now consider*

$$f(z) = -a\frac{e^z + e^{2z} + e^{3z} + \ldots + e^{(n-1)z}}{1 + e^z + e^{2z} + e^{3z} + \ldots + e^{(n-1)z}},$$

$$g(z) = -a\frac{1 + e^z + e^{2z} + e^{3z} + \ldots + e^{(n-2)z}}{1 + e^z + e^{2z} + e^{3z} + \ldots + e^{(n-1)z}}.$$

Then clearly f, g share (S_i, ∞) for $i = 1, 2$; but $f \not\equiv g$.

Corollary 2.1. *Let S_1 and S_2 be defined as in* Theorem 2.1. *Then S_i's are $BURSP^7M0,2$ for $n > 2$.*

Remark 2.1. Corollary 2.1 *answers* Question 1.3 *successfully. Here we would have one set containing only one element and the other set containing only three elements and this pair of sets are bi-unique range sets too. That is, we obtain a pair of bi-unique range sets (for a special class of meromorphic functions) sum of whose cardinalities are less than* 5*. Clearly, this result provides two shared sets with least possible cardinality in the direction of* Question 1.2.

Theorem 2.2. *Let S_1 and S_2 be defined as in* Theorem 2.1. *Then S_i's are $BURSP^dE0,2$ for $n > 2$.*

Remark 2.2. *Observe that the statement of* Theorem 2.2 *itself says the result does not depend upon the values of d; i.e., it is true for all values of d; i.e., S_i's are nothing but $BURSE0,2$; which significantly improves* Theorem A *by relaxing the nature of sharing the sets.*

Now we have the following two examples which show the sharpness of *Corollary 2.1* and *Theorem 2.2* with respect to the cardinalities of the shared sets.

Example 2.2. *Suppose that $S_1 = \{0\}$ and $S_2 = \{a\}$, where $a \in \mathbb{C} - \{0\}$. Consider $f = e^z$ and $g = a^{\frac{2}{d}}e^{-z}$, where $d \in \mathbb{N}$ and by $a^{\frac{2}{d}}$ we mean exactly one of the values of the dth roots of a^2. Then clearly f^d and g^d share S_1 and S_2 CM but $f^d \not\equiv g^d$.*

Example 2.3. *Suppose that $S_1 = \{0\}$ and $S_2 = \{\alpha, \beta\}$, where $\alpha, \beta \in \mathbb{C} - \{0\}$. Consider $f = e^z$ and $g = (\alpha\beta)^{\frac{1}{d}}e^{-z}$, where $d \in \mathbb{N}$ and by $(\alpha\beta)^{\frac{1}{d}}$ we mean exactly one of the values of the dth roots of $\alpha\beta$. Then clearly f^d and g^d share S_1 and S_2 CM but $f^d \neq g^d$.*

Next we exhibit the following two examples in support of *Theorem 2.2* and *Theorem 2.1*.

Example 2.4. *Let $S_1 = \{0\}$ and $S_2 = \{-2, 3, 6\} = \{z : z^3 - 7z^2 + 36 = 0\}$. Then according to* Theorem 2.2 *for any two non-constant entire functions f, g; $E_f(S_1, 0) = E_g(S_1, 0)$ and $E_f(S_2, 2) = E_g(S_2, 2)$ implies $f \equiv g$.*

Example 2.5. *In the above example, if f and g are considered as non-constant meromorphic functions, then $E_{f^{7+r}}(S_1, 0) = E_{g^{7+s}}(S_1, 0)$ and $E_{f^{7+r}}(S_2, 2) = E_{g^{7+s}}(S_2, 2)$ implies $f^{7+r} \equiv g^{7+s}$, where $r, s \in \mathbb{N} \cup \{0\}$.*

3 Lemmas

In this section, we present different lemmas which are required to prove the main results of the paper. Before that, we recall the following definitions of different notations which we use in different lemmas and in the proofs of the main theorems. For standard notations and definitions of Nevanlinna Theory we refer our readers to follow [11, 18].

Definition 3.1. *[12] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid = 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f \mid \leq m)(N(r, a; f \mid \geq m))$ the counting function of those a-points of f whose multiplicities are not greater(less) than m, where each a-point is counted according to its multiplicity. $\overline{N}(r, a; f \mid \leq m)(\overline{N}(r, a; f \mid \geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.*
Also, $N(r, a; f \mid < m), N(r, a; f \mid > m), \overline{N}(r, a; f \mid < m)$ and $\overline{N}(r, a; f \mid > m)$ are defined analogously.

Definition 3.2. *[26] Let f and g be two non-constant meromorphic functions such that f and g share $(a, 0)$, where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ $(\overline{N}_L(r, a; g))$ the reduced counting function of those a-points of f and g where $p > q$ $(q > p)$, by $N_E^{1)}(r, a; f)$ the counting function of those a-points of f and g where $p = q = 1$. Clearly when f and g share (a, m), $m \geq 1$, then $N_E^{1)}(r, a; f) = N(r, a; f \mid = 1)$.*

Definition 3.3. *[13, 14] Let f, g share $(a, 0)$. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Definition 3.4. *[15] Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1, 2, \ldots, q$.*

Consider two arbitrary functions belonging to $M^d(\mathbb{C})$. Then they must be of the form f^{d+r} and g^{d+s}, where f and g be two meromorphic functions with $d \in \mathbb{N}$ and $r, s \in \mathbb{N} \cup \{0\}$. Suppose

$$F = \frac{(f^{d+r})^{n-1}(f^{d+r} + a)}{-b}, \qquad G = \frac{(g^{d+s})^{n-1}(g^{d+s} + a)}{-b}; \tag{3.1}$$

and

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{3.2}$$

$$\Phi = \frac{F'}{F-1} - \frac{G'}{G-1}. \tag{3.3}$$

Lemma 3.1. *[29] If F, G are two non-constant meromorphic functions such that they share $(1,0)$ and $H \not\equiv 0$, then*

$$N_E^{1)}(r,1;F) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 3.2. *[3] Let F and G be two non-constant meromorphic functions sharing $(1,m)$, where $0 \leq m < \infty$. Then*

$$\begin{aligned} & \overline{N}(r,1;F) + \overline{N}(r,1;G) - N_E^{1)}(r,1;F) + \left(m - \frac{1}{2}\right)\overline{N}_*(r,1;F,G) \\ \leq\; & \frac{1}{2}\left[N(r,1;F) + N(r,1;G)\right]. \end{aligned}$$

Lemma 3.3. *Let F and G be given by (3.1) and $H \not\equiv 0$. If f^{d+r}, g^{d+s} share $(0,l)$ and F, G share $(1,m)$ for $0 \leq l < \infty$ and $0 \leq m < \infty$, then*

$$\begin{aligned} N(r,H) \leq\; & \overline{N}(r,-a\frac{(n-1)}{n};f^{d+r}) + \overline{N}(r,\infty;f^{d+r}) + \overline{N}(r,-a\frac{(n-1)}{n};g^{d+s}) \\ & +\overline{N}(r,\infty;g^{d+s}) + \overline{N}_*(r,0;f^{d+r},g^{d+s}) + \overline{N}_*(r,1;F,G) \\ & +\overline{N}_0(r,0;(f^{d+r})') + \overline{N}_0(r,0;(g^{d+s})') + S(r,f^{d+r}) + S(r,g^{d+s}), \end{aligned}$$

where $\overline{N}_0(r,0;(f^{d+r})')$ denotes the reduced counting function corresponding to the zeros of $(f^{d+r})'$ which are not the zeros of $f^{d+r}(f^{d+r}+a\frac{(n-1)}{n})(F-1)$. $\overline{N}_0(r,0;(g^{d+s})')$ is defined similarly.

Proof. Since F, G share $(1,0)$ and H has only simple poles, therefore the result is obvious by some simple calculations. We omit the details. □

Lemma 3.4. *Let F and G be given by (3.1) and $H \not\equiv 0$. If f^{d+r}, g^{d+s} share $(0,l)$ and F, G share $(1,m)$ for $0 \leq l < \infty$ and $0 \leq m < \infty$, then*

$$\begin{aligned} & \left(\frac{n}{2} - 1\right)\left[T(r,f^{d+r}) + T(r,g^{d+s})\right] \\ \leq\; & \overline{N}(r,0;f^{d+r}) + \overline{N}(r,0;g^{d+s}) + 2\left[\overline{N}(r,\infty;f^{d+r}) + \overline{N}(r,\infty;g^{d+s})\right] \\ & +\overline{N}_*(r,0;f^{d+r},g^{d+s}) - (m - \frac{3}{2})\overline{N}_*(r,1;F,G) + S(r,f^{d+r}) + S(r,g^{d+s}). \end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned} (n+1)T(r,f^{d+r}) \leq\; & \overline{N}(r,1;F) + \overline{N}(r,\infty;f^{d+r}) + \overline{N}(r,0;f^{d+r}) \\ & +\overline{N}(r,-a\frac{(n-1)}{n};f^{d+r}) - N_0(r,0;(f^{d+r})') + S(r,f^{d+r}). \end{aligned} \tag{3.4}$$

$$\begin{aligned} (n+1)T(r,g^{d+s}) \leq\; & \overline{N}(r,1;G) + \overline{N}(r,\infty;g^{d+s}) + \overline{N}(r,0;g^{d+s}) \\ & +\overline{N}(r,-a\frac{(n-1)}{n};g^{d+s}) - N_0(r,0;(g^{d+s})') + S(r,g^{d+s}). \end{aligned} \tag{3.5}$$

Now combining (3.4), (3.5) and using *Lemma 3.2*, *Lemma 3.1* and *Lemma 3.3* we get

$$\begin{aligned}
& (n+1)\left[T(r,f^{d+r})+T(r,g^{d+s})\right] \\
\leq\ & \overline{N}(r,1;F)+\overline{N}(r,1;G)+\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s}) \\
& +\overline{N}(r,0;f^{d+r})+\overline{N}(r,0;g^{d+s})+\overline{N}(r,-a\frac{(n-1)}{n};f^{d+r}) \\
& +\overline{N}(r,-a\frac{(n-1)}{n};g^{d+s})-N_0(r,0;(g^{d+s})') \\
& -N_0(r,0;(f^{d+r})')+S(r,f^{d+r})+S(r,g^{d+s}) \\
\leq\ & \frac{n}{2}\left[T(r,f^{d+r})+T(r,g^{d+s})\right]+\left[\overline{N}(r,0;f^{d+r})+\overline{N}(r,0;g^{d+s})\right] \\
& +2\left[\overline{N}(r,-a\frac{(n-1)}{n};f^{d+r})+\overline{N}(r,-a\frac{(n-1)}{n};g^{d+s})\right] \\
& +\overline{N}_*(r,0;f^{d+r},g^{d+s})-(m-\frac{3}{2})\overline{N}_*(r,1;F,G) \\
& +2\left[\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]+S(r,f^{d+r})+S(r,g^{d+s}),
\end{aligned}$$

which proves the lemma. □

Lemma 3.5. *Let S_1 and S_2 be defined as in Theorem 2.1 and F, G be given by (3.1). If $E_{f^{d+r}}(S_1,l)=E_{g^{d+s}}(S_1,l)$ and $E_{f^{d+r}}(S_2,m)=E_{g^{d+s}}(S_2,m)$, where $0\leq l<\infty$, $0\leq m<\infty$ and $H\not\equiv 0$, then*

$$\begin{aligned}
& (2l+1)\left\{\overline{N}\left(r,0;f^{d+r}\mid\geq l+1\right)\right\} \\
\leq\ & \overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})+\overline{N}_*(r,1;F,G)+S(r,f^{d+r})+S(r,g^{d+s}).
\end{aligned}$$

Proof. By the given condition clearly F and G share $(1,m)$. Now we consider two cases as follows.

Case-1 Let $\Phi\equiv 0$. Then from (3.3) we have

$$\begin{aligned}
F-1 &= A(G-1) \\
\implies F' &= AG' \\
\implies F'' &= AG'',
\end{aligned}$$

which in turn implies that $H\equiv 0$, a contradiction.

Case-2 Let $\Phi\not\equiv 0$. Then observe that

$$\Phi=\frac{(f^{d+r})^{n-2}\left(nf^{d+r}+a(n-1)\right)(f^{d+r})'}{-b(F-1)}-\frac{(g^{d+s})^{n-2}\left(ng^{d+s}+a(n-1)\right)(g^{d+s})'}{-b(G-1)}.$$

Let z_0 be a zero of f^{d+r} with multiplicity t. Since $E_{f^{d+r}}(S_1,l) = E_{g^{d+s}}(S_1,l)$, then that would be a zero of Φ of multiplicity $(n-2)t+t-1$ i.e., of multiplicity $(n-1)t-1$ if $t \leq l$ and a zero of multiplicity at least $(n-2)(l+1)+l$ i.e., a zero of multiplicity at least $(n-1)l+(n-2)$ if $t > l$. Since it is clear from the statement of *Theorem 2.1* that $n \not< 3$, so the order of z_0 in Φ is at least $2l+1$ when $t > l$. Hence we can write

$$\begin{aligned}
& \{2l+1\}\left\{\overline{N}(r,0;f^{d+r} \mid\geq l+1)\right\} \\
\leq\ & N(r,0;\Phi) \\
\leq\ & T(r,\Phi) \\
\leq\ & N(r,\infty;\Phi)+S(r,F)+S(r,G) \\
\leq\ & \overline{N}_*(r,1;F,G)+\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})+S(r,f^{d+r})+S(r,g^{d+s}).
\end{aligned}$$

□

Lemma 3.6. *Let S_1, S_2 be defined as in Theorem 2.1 and F, G be given by (3.1). Further suppose that $\omega_1,\omega_2\ldots\omega_n$ are the members of the set S_2. If $E_{f^{d+r}}(S_1,l) = E_{g^{d+s}}(S_1,l)$ and $E_{f^{d+r}}(S_2,m) = E_{g^{d+s}}(S_2,m)$, where $0 \leq l < \infty$, $2 \leq m < \infty$ and $H \not\equiv 0$, then*

$$\overline{N}_*(r,1;F,G) \leq \frac{3}{2(m-1)}\left[\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]+S(r,f^{d+r})+S(r,g^{d+s}).$$

Proof. First we note that '0' is not a member of S_2. Therefore proceeding as follows with the help of *Lemma 3.5* for $l=0$ we get,

$$\begin{aligned}
& \overline{N}_*(r,1;F,G) \\
\leq\ & \overline{N}(r,1;F \mid\geq m+1) \\
\leq\ & \frac{1}{m}\ \left(N(r,1;F)-\overline{N}(r,1;F)\right) \\
\leq\ & \frac{1}{m}\left[\sum_{j=1}^{n}\left(N(r,\omega_j;f^{d+r})-\overline{N}(r,\omega_j;f^{d+r})\right)\right] \\
\leq\ & \frac{1}{m}\left[N\left(r,0;(f^{d+r})' \mid f^{d+r}\neq 0\right)\right] \\
\leq\ & \frac{1}{m}\left[N\left(r,\infty;\frac{f^{d+r}}{(f^{d+r})'}\right)\right] \\
\leq\ & \frac{1}{m}\left[N\left(r,\infty;\frac{(f^{d+r})'}{f^{d+r}}\right)\right]+S(r,f^{d+r}) \\
\leq\ & \frac{1}{m}\left[\overline{N}(r,0;f^{d+r})+\overline{N}(r,\infty;f^{d+r})\right]+S(r,f^{d+r}) \\
\leq\ & \frac{1}{m}\left[2\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})+\overline{N}_*(r,1;F,G)\right]+S(r,f^{d+r})+S(r,g^{d+s}),
\end{aligned}$$

which clearly implies

$$\overline{N}_*(r,1;F,G) \leq \frac{1}{m-1}\left[2\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right] +S(r,f^{d+r})+S(r,g^{d+s}). \tag{3.6}$$

Similarly, applying the above method for G instead of F we can obtain

$$\overline{N}_*(r,1;F,G) \leq \frac{1}{m-1}\left[2\overline{N}(r,\infty;g^{d+s})+\overline{N}(r,\infty;f^{d+r})\right] +S(r,f^{d+r})+S(r,g^{d+s}). \tag{3.7}$$

Now adding (3.6) and (3.7) we get the desired result. □

Lemma 3.7. *Let S_1, S_2 be defined as in Theorem 2.1 and F, G be given by (3.1) with $d \geq 2$. Further suppose that $\omega_1,\omega_2 \ldots \omega_n$ are the members of the set S_2. If $E_{f^{d+r}}(S_1,l)=E_{g^{d+s}}(S_1,l)$ and $E_{f^{d+r}}(S_2,m)=E_{g^{d+s}}(S_2,m)$, where $0 \leq l < \infty$, $1 \leq m < \infty$ and $H \not\equiv 0$, then*

$$\overline{N}_*(r,1;F,G) \leq \frac{5}{2(3m-1)}\left[\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]+S(r,f^{d+r})+S(r,g^{d+s}).$$

Proof. Since $d \geq 2$, so $\overline{N}(r,0;f^{d+r}) = \overline{N}(r,0;f^{d+r} \mid\geq 2)$ and $\overline{N}(r,0;g^{d+s}) = \overline{N}(r,0;g^{d+s} \mid\geq 2)$. Now proceeding similarly like *Lemma 3.6*, in view of *Lemma 3.5* we get,

$$\begin{aligned} &\overline{N}_*(r,1;F,G)\\ \leq\ & \frac{1}{m}\left[\overline{N}(r,0;f^{d+r})+\overline{N}(r,\infty;f^{d+r})\right]+S(r,f^{d+r})\\ \leq\ & \frac{1}{m}\left[\overline{N}(r,0;f^{d+r}\mid\geq 2)+\overline{N}(r,\infty;f^{d+r})\right]+S(r,f^{d+r})\\ \leq\ & \frac{1}{m}\left[\frac{1}{3}\left(\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})+\overline{N}_*(r,1;F,G)\right)+\overline{N}(r,\infty;f^{d+r})\right]\\ &+S(r,f^{d+r})\\ \leq\ & \frac{1}{3m}\left[4\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})+\overline{N}_*(r,1;F,G)\right]+S(r,f^{d+r}), \end{aligned}$$

which implies

$$\overline{N}_*(r,1;F,G) \leq \frac{1}{3m-1}\left[4\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]+S(r,f^{d+r}). \tag{3.8}$$

Similar approach for g^{d+s} will provide

$$\overline{N}_*(r,1;F,G) \leq \frac{1}{3m-1}\left[4\overline{N}(r,\infty;g^{d+s})+\overline{N}(r,\infty;f^{d+r})\right]+S(r,g^{d+s}). \tag{3.9}$$

Combining (3.8) and (3.9) we get

$$N_*(r,1;F,G) \leq \frac{5}{2(3m-1)}\left[\overline{N}(r,\infty;g^{d+s})+\overline{N}(r,\infty;f^{d+r})\right]+S(r,f^{d+r})+S(r,g^{d+s}).$$

□

Lemma 3.8. *Let F and G be defined by (3.1). Then $FG \neq 1$ for $n \geq 3$.*

Proof. If possible suppose that $FG=1$. That is

$$(f^{d+r})^{n-1}(f^{d+r}+a)(g^{d+s})^{n-1}(g^{d+s}+a)=b^2. \tag{3.10}$$

Clearly applying the first fundamental theorem on (3.10) we would get

$$T(r,f^{d+r})=T(r,g^{d+s})+O(1). \tag{3.11}$$

Since f^{d+r}, g^{d+s} share $(0,0)$, so (3.10) clearly implies that f^{d+r} and g^{d+s} both omit the value 0. Also note that $\overline{N}(r,\infty;f^{d+r})=\overline{N}(r,-a;g^{d+s})$ and $\overline{N}(r,\infty;g^{d+s})=\overline{N}(r,-a;f^{d+r})$, so each $-a$ point of g^{d+s} or f^{d+r} is of multiplicity at least n. Hence by the second fundamental theorem in view of (3.11) we get

$$\begin{aligned} T(r,f^{d+r}) &\leq \overline{N}(r,0;f^{d+r})+\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,-a;f^{d+r})+S(r,f^{d+r}) \\ &\leq \overline{N}(r,-a;g^{d+s})+\overline{N}(r,-a;f^{d+r})+S(r,f^{d+r}) \\ &\leq \frac{2}{n}T(r,f^{d+r})+S(r,f^{d+r}), \end{aligned}$$

which is a contradiction for $n \geq 3$. □

Lemma 3.9. *Let F and G be defined by (3.1). Then $F \equiv G$ implies $f^{d+r} \equiv g^{d+s}$ for $n \geq 2$, where $d \geq 2$.*

Proof. Since $F \equiv G$. Therefore we have

$$(f^{d+r})^{n-1}(f^{d+r}+a)=(g^{d+s})^{n-1}(g^{d+s}+a) \tag{3.12}$$

By substituting $h=\dfrac{g^{d+s}}{f^{d+r}}$ in the above equation we get

$$(f^{d+r})^n(1-h^n)+a(f^{d+r})^{n-1}(1-h^{n-1})=0. \tag{3.13}$$

If h is non-constant, then from (3.13) we have

$$f^{d+r}=-a\frac{h^{n-1}-1}{h^n-1}=-a\frac{\prod_{i=1}^{n-2}(h-\alpha_i)}{\prod_{i=1}^{n-1}(h-\beta_i)}, \tag{3.14}$$

where α_i's are distinct $(n-1)$th roots of unity with $\alpha_i \neq 1$ and β_i's are distinct nth roots of unity with $\beta_i \neq 1$. Clearly $\alpha_i \neq \beta_j$. Note that from (3.14), it is obvious that each α_i and β_i point of h is of multiplicity at least d. Further, from (3.12) we get that f^{d+r}, g^{d+s} share $(0,\infty)$ and (∞,∞). Hence h does not have any zero or pole. So, by the second fundamental theorem we get

$$\begin{aligned}
&(2n-3)T(r,h)\\
\leq\ & \sum_{i=1}^{n-2}\overline{N}(r,\alpha_i;h)+\sum_{i=1}^{n-1}\overline{N}(r,\beta_i;h)+\overline{N}(r,0;h)+\overline{N}(r,\infty;h)+S(r,h)\\
\leq\ & \frac{(2n-3)}{d}T(r,h)+S(r,h),
\end{aligned}$$

which is a contradiction for $n \geq 2$ as $d \geq 2$.
Thus h is a constant, which implies $h^n = h^{n-1} = 1$; i.e., $h = 1$ and hence $f^{d+r} \equiv g^{d+s}$. □

Remark 3.1. *Note that if f^{d+r} and g^{d+s} are entire functions in* Lemma 3.9, *then from (3.14) one can easily conclude that h omits β_i points for $i = 1, 2, \ldots, n-1$, where $n \geq 2$ at the same time h omits zeros and poles as discussed above, which contradicts the fact that h is non-constant. Hence we would have $f^{d+r} \equiv g^{d+s}$ for $n \geq 2$ even if $d = 1$.*

4 Proof of the Theorems

Proof of Theorem 2.1. Let F and G be defined by (3.1). Then F, G share $(1,2)$.

Case-1 Suppose $H \not\equiv 0$. Since f^{d+r}, g^{d+s} share $(0,0)$, so we have $\overline{N}_*(r,0;f^{d+r},g^{d+s}) \leq \overline{N}(r,0;f^{d+r}) = \overline{N}(r,0;g^{d+s})$. Also we have $\overline{N}(r,0;f^{d+r}) = \overline{N}(r,0;f \mid\geq 2)$ as $d \geq 2$. Now, using *Lemma 3.4* for $m = 2$ and $l = 0$, *Lemma 3.5* for $l = 1$ and then *Lemma 3.7* for $m = 2$, we get

$$\begin{aligned}
&(\frac{n}{2}-1)\left[T(r,f^{d+r})+T(r,g^{d+s})\right]\\
\leq\ & \left[\overline{N}(r,0;f^{d+r})+\overline{N}(r,0;g^{d+s})\right]+\overline{N}(r,0;f^{d+r})-\frac{1}{2}\overline{N}_*(r,1;F,G)\\
&+2\left[\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]+S(r,f^{d+r})+S(r,g^{d+s})\\
\leq\ & 3\overline{N}(r,0;f^{d+r})+2\left[\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]\\
&-\frac{1}{2}\overline{N}_*(r,1;F,G)+S(r,f^{d+r})+S(r,g^{d+s})\\
\leq\ & 3\overline{N}(r,0;f^{d+r}\mid\geq 2)+2\left[\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]\\
&-\frac{1}{2}\overline{N}_*(r,1;F,G)+S(r,f^{d+r})+S(r,g^{d+s})
\end{aligned}$$

$$
\begin{aligned}
&\leq 3\left[\frac{1}{3}\left(\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})+\overline{N}_*(r,1;F,G)\right)\right]\\
&\quad +2\left[\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]-\frac{1}{2}\overline{N}_*(r,1;F,G)+S(r,f^{d+r})+S(r,g^{d+s})\\
&\leq 3\left[\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]+\frac{1}{2}\overline{N}_*(r,1;F,G)+S(r,f^{d+r})+S(r,g^{d+s})\\
&\leq 3\left[\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]+\frac{1}{2}\left[\frac{1}{2}\left(\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right)\right]\\
&\quad +S(r,f^{d+r})+S(r,g^{d+s})\\
&\leq \left[3+\frac{1}{4}\right]\left[\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]+S(r,f^{d+r})+S(r,g^{d+s})\\
&\leq \frac{13}{4d}\left[N(r,\infty;f^{d+r})+N(r,\infty;g^{d+s})\right]+S(r,f^{d+r})+S(r,g^{d+s})\\
&\leq \left(\frac{13}{4d}\right)\left[T(r,f^{d+r})+T(r,g^{d+s})\right]+S(r,f^{d+r})+S(r,g^{d+s}),
\end{aligned}
$$

which is a contradiction for $n > 2 + \dfrac{13}{2d}$.

Case-2 Suppose $H \equiv 0$. Then on integration we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{4.1}$$

where $A(\neq 0), B$ are complex constants. From (4.1), clearly we have

$$T(r,f^{d+r}) = T(r,g^{d+s}) + S(r,g^{d+s}). \tag{4.2}$$

Now we can write (4.1) as

$$F = \frac{(B+1)G + A - B - 1}{BG + A - B}. \tag{4.3}$$

Hence let us consider the following subcases.

Subcase-2.1 Let $B \neq 0$.

Subcase-2.1.1 Let $B \neq -1$. Obviously $\frac{A-B-1}{B+1} \neq \frac{A-B}{B}$. For if $\frac{A-B-1}{B+1} = \frac{A-B}{B}$, then $A = 0$, which is absurd. Therefore

$$\overline{N}(r, \frac{B-A}{B}; G) = \overline{N}(r,\infty;F). \tag{4.4}$$

Now using the second fundamental theorem in view of (4.2) we have

$$
\begin{aligned}
T(r,G) &\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,\frac{B-A}{B};G)+S(r,G)\\
&\leq \overline{N}(r,0;g^{d+s})+\overline{N}(r,-a;g^{d+s})+\overline{N}(r,\infty;g^{d+s})+\overline{N}(r,\infty;f^{d+r})+S(r,G)\\
&\leq \left(\frac{1+\frac{3}{d}}{n}\right)T(r,G)+S(r,G),
\end{aligned}
$$

which is a contradiction for $n > 1 + \frac{3}{d}$.

Subcase-2.1.2 Let $B = -1$. Then from (4.3) we get

$$F = \frac{A}{-G + A + 1}. \tag{4.5}$$

Subcase-2.1.2.1 Let $A + 1 \neq 0$. Then $\overline{N}(r, A + 1; G) = \overline{N}(r, \infty; F)$ and $\overline{N}(r, \infty; G) = \overline{N}(r, 0; F)$. Now using the second fundamental theorem in view of (4.2) we have

$$\begin{aligned} T(r, G) &\leq \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + \overline{N}(r, A + 1; G) + S(r, G) \\ &\leq \overline{N}(r, 0; g^{d+s}) + \overline{N}(r, -a; g^{d+s}) + \overline{N}(r, \infty; g^{d+s}) + \overline{N}(r, \infty; f^{d+r}) + S(r, G) \\ &\leq \left(\frac{1 + \frac{3}{d}}{n} \right) T(r, G) + S(r, G), \end{aligned}$$

which is a contradiction for $n > 1 + \frac{3}{d}$.

Subcase-2.1.2.2 Let $A + 1 = 0$. Then $FG = 1$. Since $n > 2 + \frac{13}{2d}$, so in view of *Lemma 3.8*, this case is invalid.

Subcase-2.2 Suppose $B = 0$ then from (4.3) we get

$$AF = G + A - 1. \tag{4.6}$$

Subcase-2.2.1 Let $A \neq 1$. Therefore (4.6) implies $\overline{N}(r, 0; F) = \overline{N}(r, 1 - A; G)$. Now using the second fundamental theorem in view of (4.2), we get

$$\begin{aligned} & T(r, G) \\ \leq\ & \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + \overline{N}(r, 1 - A; G) + S(r, G) \\ \leq\ & \overline{N}(r, 0; g^{d+s}) + \overline{N}(r, -a; g^{d+s}) + \overline{N}(r, \infty; g^{d+s}) + \overline{N}(r, 0; f^{d+r}) + \overline{N}(r, -a; f^{d+r}) \\ & + S(r, G) \\ \leq\ & \left(\frac{2 + \frac{3}{d}}{n} \right) T(r, G) + S(r, G), \end{aligned}$$

which is a contradiction for $n > 2 + \frac{3}{d}$.

Subcase-2.2.2 Let $A = 1$ i.e., $F \equiv G$. So in view of *Lemma 3.9*, we get $f^{d+r} \equiv g^{d+s}$ as $n > 2 + \frac{13}{2d}$. □

Proof of Theorem 2.2. Let F and G be defined by (3.1). Then F, G share $(1, 2)$.

Case-1 Suppose $H \not\equiv 0$. Then using $l = 0$ and $m = 2$ in *Lemma 3.4*, *Lemma 3.5* and *Lemma 3.6* respectively, we get

$$\begin{aligned}
& (\frac{n}{2}-1)\left[T(r,f^{d+r})+T(r,g^{d+s})\right] \\
\leq & \left[\overline{N}(r,0;f^{d+r})+\overline{N}(r,0;g^{d+s})\right]+\overline{N}(r,0;f^{d+r})-\frac{1}{2}\overline{N}_*(r,1;F,G) \\
& +2\left[\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]+S(r,f^{d+r})+S(r,g^{d+s}) \\
\leq & 3\overline{N}(r,0;f^{d+r})+2\left[\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]+S(r,f^{d+r})+S(r,g^{d+s}) \\
\leq & 3\overline{N}_*(r,1;F,G)+5\left[\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]+S(r,f^{d+r})+S(r,g^{d+s}) \\
\leq & \left[\frac{9}{2}+5\right]\left[\overline{N}(r,\infty;f^{d+r})+\overline{N}(r,\infty;g^{d+s})\right]+S(r,f^{d+r})+S(r,g^{d+s}) \\
\leq & S(r,f^{d+r})+S(r,g^{d+s}),
\end{aligned}$$

which is a contradiction for $n \geq 3$.

Case-2 Suppose $H \equiv 0$. Then using $\overline{N}(r,\infty;f^{d+r}) = S(r,f^{d+r})$, $\overline{N}(r,\infty;g^{d+s}) = S(r,g^{d+s})$, *Remark 3.1* and proceeding similarly like Case-2 of *Theorem 2.1*, we obtain $f^{d+r} \equiv g^{d+s}$ for $n \geq 3$. In this process, we just need to deal the Subcase-2.2.1 in a slight detail as follows.

Here we would again have $AF = G + A - 1$ with $A \neq 1$. Since f^{d+r}, g^{d+s} share $(0,0)$, so for a $z_0 \in \mathbb{C}$ if $f^{d+r}(z_0) = 0$, then $g^{d+s}(z_0) = 0$. For that z_0, we must have $AF(z_0) = G(z_0) + A - 1$. But this contradicts the fact that $A \neq 1$. So, f^{d+r} and g^{d+s} must omit the value 0; i.e., $\overline{N}(r,0;f^{d+r}) = S(r,f^{d+r})$ and $\overline{N}(r,0;g^{d+s}) = S(r,g^{d+s})$. Now applying this fact in Subcase-2.2.1 of *Theorem 2.1*, we would have a contradiction for $n \geq 3$. □

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