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COUPLED PETTIS HADAMARD FRACTIONAL DIFFERENTIAL SYSTEMS WITH RETARDATION AND ANTICIPATION

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Abstract. In this article, we study some existence results concerning the weak solutions for some coupled systems of Hadamard fractional differential equations with the mixed arguments of anticipations and retardation. By utilizing a fixed point theorem of Mönch and the technique of measure of weak noncompactness, we obtain our existence results. Finally, we present an example illustrating the applicability of the imposed conditions.

1 Introduction

The study of fractional differential equations has received great attention from many researchers, both in theory and in applications; we refer the reader to the monographs of Abbas *et al.* [1, 2], Kilbas *et al.* [18], Samko *et al.* [23], and the recent papers [30, 31], and the references therein.

The measure of weak noncompactness is introduced by De Blasi [14]. The strong measure of noncompactness was considered by Banaś and Goebel [9] and in many papers; see for example, Akhmerov *et al.* [7], Alvàrez [8], Benchohra *et al.* [12], Guo *et al.* [16], and the references therein. In [12, 21] the author considered some existence results by applying measure of noncompactness techniques. Recently, several authors used the technique of measure of weak noncompactness for other results (existence, stability,...); see [2, 10, 11], and the references therein.

In [3, 4], the authors studied the existence of weak solutions for some classes of coupled systems of Hadamard fractional differential and integral equations. Integer order differential equations with retardation and anticipation have been considered by many authors, see for example [6, 15, 24, 25, 26, 27, 28, 29].

In the present paper, we consider the following coupled system of Pettis–Hadamard

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$$\begin{cases}(u(t),v(t)) = (\phi_1(t),\phi_2(t));\ t\in[1-h_1,1],\\ ((^HD_1^{r_1}u)(t),(^HD_1^{r_2}v)(t)) = (f_1(t,u_t,v_t,u^t,v^t),f_2(t,u_t,v_t,u^t,v^t));\ t\in I:=[1,e],\\ (u(t),v(t)) = (\psi_1(t),\psi_2(t));\ t\in[e,e+h_2],\end{cases} \tag{1.1}$$

where $r_i\in(1,2]$, $f_i : I\times C_1\times C_1\times C_2\times C_2\to E$; $i=1,2$ are given continuous functions, $C_1 := C[-h_1,0]$, $C_2 := C[0,h_2]$, $h_1,h_2>0$, $\phi_i\in C[1-h_1,1]$; with $\phi_i(1)=0$, and $\psi_i\in C[e,e+h_2]$; with $\psi_i(e)=0$. Furthermore, $u_t : [-h_1,0]\to E$ such that $u_t(s)=u(t+s)$; $s\in[-h_1,0]$, and $u^t : [0,h_2]\to E$ such that $u^t(\sigma)=u(t+\sigma)$; $\sigma\in[0,h_2]$. $(E,\|\cdot\|_E)$ is a real (or complex) Banach space with dual E^*, such that E is the dual of a weakly compactly generated Banach space X and $^HD_1^{r_i}$ is the Pettis–Hadamard fractional derivative of order r_i; $i=1,2$.

In this paper, we establish some existence results for coupled systems of Hadamard fractional differential equations with the mixed arguments of anticipations and retardation.

2 Preliminaries

By $C(I)$ we denote the Banach space of all continuous functions w from I into E with the supremum norm

$$\|w\|_\infty := \sup_{t\in I}\|w(t)\|_E.$$

Also $C([1-h_1,e+h_2])$ denotes the Banach space of all continuous functions from $[1-h_1,e+h_2]$ into E with the supremum norm.
As usual, by $AC(I)$ we denote the space of absolutely continuous functions from I into E. Also, $C^2([1-h_1,e+h_2]) := C([1-h_1,e+h_2])\times C([1-h_1,e+h_2])$, denotes the product Banach space with the norm

$$\|(u,v)\|_{C^2([1-h_1,e+h_2])} = \|u\|_{C([1-h_1,e+h_2])} + \|v\|_{C([1-h_1,e+h_2])}.$$

Let $(E,w)=(E,\sigma(E,E^*))$ be the Banach space E with its weak topology.

Definition 1. *A Banach space X is called weakly compactly generated (WCG, in short) if it contains a weakly compact set whose linear span is dense in X.*

Definition 2. *A function $h:E\to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any (u_n) in E with $u_n\to u$ in (E,w) then $h(u_n)\to h(u)$ in (E,w)).*

Definition 3. *[22] The function $u : I\to E$ is said to be Pettis integrable on I if and only if there is an element $u_J\in E$ corresponding to each $J\subset I$ such that $\phi(u_J)=\int_J\phi(u(s))ds$ for all $\phi\in E^*$, where the integral on the right hand side is assumed to exist in the sense of Lebesgue, (by definition, $u_J=\int_J u(s)ds$).*

Let $P(I,E)$ be the space of all $E-$valued Pettis integrable functions on I, and $L^1(I,E)$, be the Banach space of measurable functions $u: I \to E$ which are Bochner integrable. Define the class $P_1(I,E)$ by

$$P_1(I,E) = \{u \in P(I,E) : \varphi(u) \in L^1(I,\mathbb{R});\ for\ every\ \varphi \in E^*\}.$$

The space $P_1(I,E)$ is normed by

$$\|u\|_{P_1} = \sup_{\varphi \in E^*,\ \|\varphi\| \le 1} \int_1^e |\varphi(u(x))| d\lambda x,$$

where λ stands for a Lebesgue measure on I.

The following result is due to Pettis (see [[22], Theorem 3.4 and Corollary 3.41]).

Proposition 4. *[22] If $u \in P_1(I,E)$ and h is a measurable and essentially bounded real-valued function, then $uh \in P_1(J,E)$.*

For all what follows, the sign " $\int$ " denotes the Pettis integral.
Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [18] for a more detailed analysis.

Definition 5. *[18] The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1(I,E)$, is defined as*

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

where $\Gamma(\cdot)$ is the Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt;\ \xi > 0,$$

provided the integral exists.
Let $g \in P_1(I,E)$. For every $\varphi \in E^*$, we have

$$\varphi({}^H I_1^q g)(t) = ({}^H I_1^q \varphi g)(t); for\ a.e.\ t \in I.$$

Analogously to the Riemann-Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way. Set

$$\delta = x\frac{d}{dx},\ q > 0,\ n = [q] + 1,$$

where $[q]$ is the integer part of q, and

$$AC_\delta^n := \{u : I \to E : \delta^{n-1}[u(x)] \in AC(I)\}.$$

Definition 6. *[17, 18] The Hadamard fractional derivative of order q applied to the function $w \in AC^n_\delta$ is defined as*

$$({}^H D^q_1 w)(x) = \delta^n ({}^H I^{n-q}_1 w)(x) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-q-1} \frac{w(s)}{s} ds.$$

Corollary 7. *[18] Let $q > 0$ and $n = [q] + 1$. The equality $D^q h(t) = 0$ is valid if and only if*

$$h(t) = \sum_{j=1}^{n} c_j (\log t)^{q-j} \ \text{for each } t \in I,$$

where $c_j \in \mathbb{R}$ $(j = 1, \dots, n)$ are arbitrary constants.

From Lemma 2.3 in [5], we concluded the following Lemma:

Lemma 8. *Let $h_1, h_2 > 0,\ 1 < \alpha \leq 2,\ \phi \in C([-h_1, 0], E)$ with $\phi(0) = 0,\ \psi \in C([0, h_2], E)$ with $\psi(0) = 0$ and $\sigma : I \to E$ be a continuous function. The linear problem*

$$\begin{cases} u_1 = \phi, \\ ({}^H D^\alpha u)(t) = \sigma(t); \quad t \in I, \\ u^e = \psi, \end{cases}$$

has a following unique solution

$$u(t) = -\int_1^e G(t,s) \frac{\sigma(s)}{s} ds, \ \text{if} \quad t \in I$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\log t)^{\alpha-1}(1 - \log s)^{\alpha-1} - (\log t - \log s)^{\alpha-1}; \ 1 \leq s \leq t \leq e, \\ (\log t)^{\alpha-1}(1 - \log s)^{\alpha-1}; \ 1 \leq t \leq s \leq e. \end{cases} \tag{2.1}$$

Definition 9. *[14] Let E be a Banach space, Ω_E the bounded subsets of E and B_1 the unit ball of E. The De Blasi measure of weak noncompactness is the map $\mu : \Omega_E \to [0, \infty)$ defined by*
$\mu(X) = \inf\{\epsilon > 0 :$ there exists a weakly compact subset Ω of $E : X \subset \epsilon B_1 + \Omega\}$.

The De Blasi measure of weak noncompactness satisfies the following properties:

(a) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$,

(b) $\mu(A) = 0 \Leftrightarrow$ A is weakly relatively compact,

(c) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$,

(d) $\mu(\overline{A}^{\omega}) = \mu(A)$, ($\overline{A}^{\omega}$ denotes the weak closure of A),

(e) $\mu(A+B) \leq \mu(A) + \mu(B)$,

(f) $\mu(\lambda A) = |\lambda|\mu(A)$,

(g) $\mu(conv(A)) = \mu(A)$,

(h) $\mu(\cup_{|\lambda|\leq h}\lambda A) = h\mu(A)$.

The next result follows directly from the Hahn-Banach theorem.

Proposition 10. *Let E be a normed space, and $x_0 \in E$ with $x_0 \neq 0$. Then, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : I \to E$ let us denote by

$$V(t) = \{v(t) : v \in V\};\ t \in I,$$

and

$$V(I) = \{v(t) : v \in V,\ t \in I\}.$$

Lemma 11. *[16] Let $H \subset C$ be a bounded and equicontinuous. Then the function $t \to \mu(H(t))$ is continuous on I, and*

$$\mu_C(H) = \max_{t\in I} \mu(H(t)),$$

and

$$\mu\left(\int_I u(s)ds\right) \leq \int_I \mu(H(s))ds,$$

where $H(s) = \{u(s) : u \in H,\ s \in I\}$, and μ_C is the De Blasi measure of weak noncompactness defined on the bounded sets of C.

For our purpose we will need the following fixed point theorem:

Theorem 12. *[20] Let Q be a nonempty, closed, convex and equicontinuous subset of a metrizable locally convex vector space $C(I)$ such that $0 \in Q$. Suppose $T : Q \to Q$ is weakly-sequentially continuous. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \Rightarrow V \quad \textit{is relatively weakly compact}, \tag{2.2}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

3 Existence of weak solutions

Let us start by defining what we mean by a weak solution of the coupled system (1.1).

Definition 13. *A coupled functions $(u,v) \in C^2([1-h_1, e+h_2])$ is said to be a weak solution of the coupled system (1.1) if (u,v) satisfies the equations $({}^H D_1^{r_1} u)(t) = f_1(t, u_t, v_t, u^t, v^t)$ and $({}^H D_1^{r_2} v)(t) = f_2(t, u_t, v_t, u^t, v^t)$ on I, and the conditions $(u(t), v(t)) = (\phi_1(t), \phi_2(t))$; $t \in [1-h_1, 1]$, with $\phi_i(1) = 0$; $i = 1,2$ and $(u(t), v(t)) = (\psi_1(t), \psi_2(t))$; $t \in [e, e+h_2]$, with $\psi_i(e) = 0$; $i = 1,2$ hold.*

The following hypotheses will be used in the sequel.

(H_1) For a.e. $t \in I$, the functions $u \to f_i(t,u,\cdot,\cdot,\cdot)$, $v \to f_i(t,\cdot,v,\cdot,\cdot)$, $w \to f_i(t,\cdot,\cdot,w,\cdot)$ and $z \to f_i(t,\cdot,\cdot,\cdot,z)$; $i = 1,2$ are weakly sequentially continuous,

(H_2) For a.e. $u, v \in C_1$, and $w, z \in C_2$, the functions $t \to f_i(t,u,v,w,z)$ are Pettis integrable a.e. on I,

(H_3) There exist $p_i \in C(I, [0,\infty))$ such that for all $\varphi \in E^*$, we have

$$|\varphi(f_i(t,u,v,w,z))| \leq \frac{p_i(t)(\|u\|_{C_1} + \|v\|_{C_1} + \|w\|_{C_2} + \|z\|_{C_2})}{1 + \|\varphi\| + \max\{\|u\|_{C_1}, \|v\|_{C_1}, \|w\|_{C_2}, \|z\|_{C_2}\}};$$

for a.e. $t \in I$, and each $u, v \in C_1$, and $w, z \in C_2$,

(H_4) For each bounded and measurable sets $B_i \subset C_i^2$; $i = 1,2$ and for each $t \in I$, we have

$$\mu(f_1(t, B_1, B_2), 0) \leq p_1(t)\mu(B), \text{ and } \mu(0, f_2(t, B_1, B_2), 0) \leq p_2(t)\mu(B),$$

where

$$(f_1(t, B_1, B_2), 0) = \{(f_1(t, u_t, v_t, u^t, v^t), 0) : (u_t, v_t) \in B_1,\ (u^t, v^t) \in B_2\},$$

and

$$(0, f_2(t, B_1, B_2)) = \{(0, f_2(t, u_t, v_t, u^t, v^t)) : (u_t, v_t) \in B_1,\ (u^t, v^t) \in B_2\}.$$

Set

$$p_i^* = \sup_{t \in I} p_i(t);\ i = 1,2,$$

Theorem 14. *Assume that the hypotheses $(H_1) - (H_4)$ hold. If*

$$L := \frac{2p_1^*}{\Gamma(1+r_1)} + \frac{2p_2^*}{\Gamma(1+r_2)} < 1, \tag{3.1}$$

then the coupled system (1.1) has at least one weak solution defined on $[1-h_1, e+h_2]$.

Proof. Define the operators $N_1, N_2 : C[1-h_1, e+h_2] \to C[1-h_1, e+h_2]$ by

$$(N_1u)(t) = \begin{cases} \phi_1(t);\ t \in [1-h_1, 1], \\ -\displaystyle\int_1^e G_1(t,s)\frac{f_1(s,u_s,v_s,u^s,v^s)}{s}ds;\ t \in I, \\ \psi_1(t);\ t \in [e, e+h_2], \end{cases} \tag{3.2}$$

and

$$(N_2v)(t) = \begin{cases} \phi_2(t);\ t \in [1-h_1, 1], \\ -\displaystyle\int_1^e G_2(t,s)\frac{f_2(s,u_s,v_s,u^s,v^s)}{s}ds;\ t \in I, \\ \psi_2(t);\ t \in [e, e+h_2], \end{cases} \tag{3.3}$$

where

$$G_i(t,s) = \frac{1}{\Gamma(r_i)} \begin{cases} (\log t)^{r_i-1}(1-\log s)^{r_i-1} - (\log t - \log s)^{r_i-1};\ 1 \le s \le t \le e, \\ (\log t)^{r_i-1}(1-\log s)^{r_i-1};\ 1 \le t \le s \le e, \end{cases} \quad ;\ i = 1,2.$$

Consider the continuous operator $N : C^2[1-h_1, e+h_2] \to C^2[1-h_1, e+h_2]$ defined by

$$(N(u,v))(t) = ((N_1u)(t), (N_2v)(t))$$

$$= \left(-\int_1^e G_1(t,s)\frac{f_1(s,u_s,v_s,u^s,v^s)}{s}ds, -\int_1^e G_2(t,s)\frac{f_2(s,u_s,v_s,u^s,v^s)}{s}ds\right);\ t \in I. \tag{3.4}$$

We can show that our operator N is well defined. Indeed; the hypotheses imply that for all $t \in I$, the functions $t \mapsto G(\cdot,t)$, and $t \mapsto f_i(s,u_s,v_s,u^s,v^s)$ are Pettis integrable, over I.

In all what follows, we denote $\|w\|_{C[1-h_1,e+h_2]}$ by $\|w\|_C$.
Set $R = \max\{R_1, R_2\}$, with

$$R_i > \max\left\{\frac{8p_i^*}{\Gamma(1+r_i)}, \|\phi_i\|_{C[1-h_1,1]}, \|\psi_i\|_{C[e,e+h_2]}\right\};\ i = 1,2,$$

and consider the closed, convex end equicontinuous set

$$Q = \Big\{(u,v) \in C^2[1-h_1, e+h_2] : \|(u,v)\|_{C^2[1-h_1,e+h_2]} \le R,\ \|u(t_2)-u(t_1)\|_E$$
$$\le p_1^* \int_1^e |G_1(t_2,s) - G_1(t_1,s)|\frac{ds}{s},\ \textit{and}\ \|v(t_2)-v(t_1)\|_E \le p_2^* \int_1^e |G_2(t_2,s)-G_2(t_1,s)|\frac{ds}{s}\Big\}.$$

We shall show that the operator N satisfies all the assumptions of Theorem 12. The proof will be given in three steps.

Step 1. *N maps Q into itself.*
Let $(u,v)\in Q$; $t\in I$ and assume that $(N(u,v))(t)\neq(0,0)$. Then there exists $\varphi_i\in E^*$; $i=1,2$ such that for each $t\in I$, we have $\|(N(u,v))(t)\|_E=(\varphi_1(|(N_1u)(t)),\varphi_2(|(N_2v)(t)))$. Thus

$$\|(N_1u)(t)\|_E=\varphi_1\left(\int_1^e G_1(t,s)f_1(s,u_s,v_s)\frac{ds}{s}\right).$$

For each $t\in I$ and any $i=1,2$, we have

$$\begin{aligned}\int_1^e |G_i(t,s)|\frac{ds}{s} &\leq \frac{1}{\Gamma(r_i)}\left[\int_1^t\left(\log\frac{t}{s}\right)^{r_i-1}\frac{ds}{s}+(\log t)^{r_i-1}\int_1^e\left(\log\frac{e}{s}\right)^{r_i-1}\frac{ds}{s}\right]\\ &\leq \tfrac{2}{\Gamma(r_i)}\int_1^e\left(\log\frac{e}{s}\right)^{r_i-1}\frac{ds}{s}=\frac{2}{\Gamma(1+r_i)}.\end{aligned}\tag{3.5}$$

Next, from (H_3), we have

$$|f_i(t,u,v,w,z)|\leq\frac{4p_i\max\{\|u\|_{C_1},\|v\|_{C_1},\|w\|_{C_2},\|z\|_{C_2}\}}{\max\{\|u\|_{C_1},\|v\|_{C_1},\|w\|_{C_2},\|z\|_{C_2}\}r}=4p_1^*.$$

Thus, for each $t\in I$, we have

$$\begin{aligned}\|(N_1u)(t)\|_E &\leq \int_1^e|G_1(t,s)|\frac{|\varphi(f_1(s,u_s,v_s,u^s,v^s))|}{s}ds\\ &\leq \frac{8p_1^*}{\Gamma(1+r_1)}\\ &\leq R_1,\end{aligned}$$

Hence, for each $t\in[1-h_1,e+h_2]$, we have

$$\|(N_1u)(t)\|_E\leq R_1.$$

Also, for each $t\in[1-h_1,e+h_2]$, we obtain

$$\|(N_2v)(t)\|_E\leq R_2.$$

Therefore, for each $t\in[1-h_1,e+h_2]$ and all $(u,v)\in Q$, we get

$$\|(N(u,v)(t)\|_E\leq R.$$

Next, let $t_1,t_2\in I$ such that $t_1<t_2$ and let $(u,v)\in Q$, with

$$(N(u,v))(t_2)-(N(u,v))(t_1)\neq(0,0).$$

Then there exists $\varphi_i\in E^*$, $i=1,2$ with $\|\varphi_1\|=1$ such that

$$\|(N_1u)(t_2)-(N_1u)(t_1)\|_E=\varphi_1((N_1u)(t_2)-(N_1u)(t_1)),$$

and

$$\|(N_2v)(t_2)-(N_2v)(t_1)\|_E=\varphi_2((N_2v)(t_2)-(N_2v)(t_1)).$$

Thus

$$\begin{aligned}&\|(N_1u)(t_2)-(N_1u)(t_1)\|_E=\varphi_1((N_1u)(t_2)-(N_1u)(t_1))\\&\leq\varphi_1\left(\int_1^e(G_2(t_2,s)f_2(s,u_s,v_s,u^s,v^s)-G_1(t_1,s)f_1(s,u_s,v_s,u^s,v^s))\frac{ds}{s}\right),\end{aligned}$$

This gives

$$\begin{aligned}\|(N_1u)(t_2)-(N_1u)(t_1)\|_E&\leq\int_1^e|G_1(t_2,s)-G_1(t_1,s)||f_1(s,u_s,v_s,u^s,v^s)|\frac{ds}{s}\\&\leq p_1^*\int_1^e|G_1(t_2,s)-G_1(t_1,s)|\frac{ds}{s}.\end{aligned}$$

Also, we can obtain

$$\begin{aligned}\|(N_2v)(t_2)-(N_2v)(t_1)\|_E&\leq\int_1^e|G_2(t_2,s)-G_2(t_1,s)||f_2(s,u_s,v_s,u^s,v^s)|\frac{ds}{s}\\&\leq p_2^*\int_1^e|G_2(t_2,s)-G_2(t_1,s)|\frac{ds}{s}.\end{aligned}$$

Hence $N(Q)\subset Q$.

Step 2. *N is weakly-sequentially continuous.*
Let (u_n,v_n) be a sequence in Q and let $(u_n(t),v_n(t))\to(u(t),v(t))$ in $(E,\omega)\times(E,\omega)$ for each $t\in[1-h_1,e+h_2]$. Fix $t\in[1-h_1,e+h_2]$, since the functions $f_i;\ i=1,2$ satisfy the assumption (H_1), we have $f_i(t,u_{nt},v_{nt},u^{nt},v^{nt})$ converge weakly uniformly to $f_i(t,u_t,v_t,u^t,v^t)$. Hence the Lebesgue dominated convergence theorem for Pettis integral implies $(N(u_n,v_n))(t)$ converges weakly uniformly to $(N(u,v))(t)$ in $(E,\omega)\times(E,\omega)$, for each $t\in[1-h_1,e+h_2]$. Thus, $N(u_n,v_n)\to N(u,v)$. Hence, $N:Q\to Q$ is weakly-sequentially continuous.

Step 3. *The implication (2.2) holds.*
Let V be a subset of Q such that $\overline{V}=\overline{conv}(N(V)\cup\{(0,0)\})$. Obviously

$$V(t)\subset\overline{conv}(NV)(t))\cup\{(0)\}),\ for\ all\ t\in[1-h_1,e+h_2].$$

Further, as V is bounded and equicontinuous, by Lemma 3 in [13], the function $t\to(v_1(t),v_2(t))=\mu(V(t))$ is continuous on $[1-h_1,e+h_2]$. From (H_3), (H_4), Lemma 11 and the properties of the measure μ, for any $t \in [1-h_1, e+h_2]$, we have

$$
\begin{aligned}
& \mu(V(t)) \leq \mu((NV)(t) \cup \{0\}) \\
\leq\ & \mu((NV)(t)) \\
=\ & \mu(\{(N_1v_1)(t)))(t), (N_2v_2)(t));\ (v_1,v_2) \in V\}) \\
\leq\ & \int_1^e \mu(\{(G_1(t,s)f_1(s,v_1(s),v_2(s),v_1(s),v_2(s)), \\
& G_2(t,s)f_2(s,v_1(s),v_2(s),v_1(s),v_2(s)));\ (v_1,v_2) \in V\})\frac{ds}{s} \\
\leq\ & \int_1^e |G_1(t,s)|\mu(\{(f_1(s,v_1(s),v_2(s),v_1(s),v_2(s)),0);\ (v_1,v_2) \in V\})\frac{ds}{s} \\
+\ & \int_1^e |G_2(t,s)|\mu(\{(0,f_2(s,v_1(s),v_2(s),v_1(s),v_2(s)));\ (v_1,v_2) \in V\})\frac{ds}{s} \\
\leq\ & \int_1^e |G_1(t,s)|p_1(s)\mu(\{(v_1(s),v_2(s));\ (v_1,v_2) \in V\})\frac{ds}{s} \\
+\ & \int_1^e |G_2(t,s)|p_2(s)\mu(\{(v_1(s),v_2(s));\ (v_1,v_2) \in V\})\frac{ds}{s} \\
\leq\ & \int_1^e |G_1(t,s)|p_1(s)\mu(V(s))\frac{ds}{s} + \int_1^e |G_2(t,s)|p_2(s)\mu(V(s))\frac{ds}{s} \\
\leq\ & \left(p_1^* \int_1^e |G_1(t,s)|\frac{ds}{s} + p_2^* \int_1^e |G_2(t,s)|\frac{ds}{s}\right) \sup_{t \in I} \mu(V(t)) \\
\leq\ & \left(\frac{2p_1^*}{\Gamma(1+r_1)} + \frac{2p_2^*}{\Gamma(1+r_2)}\right) \sup_{t \in I} \mu(V(t)) \\
=\ & L \sup_{t \in I} \mu(V(t)).
\end{aligned}
$$

Thus

$$\sup_{t \in I} \mu(V(t)) \leq L \sup_{t \in I} \mu(V(t)).$$

Hence, the inequality (3.1) gives $\sup_{t \in I} \mu(V(t)) = 0$, that is $\mu(V(t)) = 0$; for each $t \in [1-h_1, e+h_2]$, and then Theorem 2 in [19] implies that V is weakly relatively compact in $C^2[1-h_1, e+h_2]$. Consequently; Theorem 12 implies that N has at least a fixed point which is a solution of the coupled system (1.1).

4 An Example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

We consider the following coupled system of Hadamard fractional differential equations

$$\begin{cases} (u(t), v(t)) = (\phi_1(t), \phi_2(t)); \ t \in [-1, 1], \\ (({}^H D_1^{\frac{3}{2}} u_n)(t), ({}^H D_1^{\frac{3}{2}} v_n)(t)) = (f_n(t, u_t, v_t, u^t, v^t), g_n(t, u_t, v_t, u^t, v^t)); \ t \in [1, e], \\ (u(t), v(t)) = (\psi_1(t), \psi_2(t)); \ t \in [e, e+3], \end{cases} \tag{4.1}$$

where $\phi_1(x) = 1 - e^x, \ \phi_2(x) = 0; \ x \in [-1, 1], \ \psi_1(x) = -1 + \sin x, \ \psi_2(x) = 0; \ x \in [e, e+3]$, $\phi_i \in C[-1, 1]$ with $\phi_i(1) = 0$, and $\psi_i \in C[e, e+3]$ with $\psi_i(e) = 0$,

$$f_n(t, u, v, w, z) = \frac{ct^2(e^{-7} + e^{-5-t})}{1 + \|u\|_{C[-1,1]} + \|v\|_{C[-1,1]} + \|w\|_{C[e,e+3]} + \|z\|_{C[e,e+3]}} u_n(t); \ t \in [1, e],$$

and

$$g_n(t, u_t, v_t) = \frac{ct^2 e^{-6}}{1 + \|u\|_{C[-1,1]} + \|v\|_{C[-1,1]} + \|u\|_{C[e,e+3]} + \|v\|_{C[e,e+3]}}; \ t \in [1, e],$$

with

$$u = (u_1, u_2, \ldots, u_n, \ldots), \ v = (v_1, v_2, \ldots, v_n, \ldots), \ and \ c := \frac{e^4}{24} \Gamma\left(\frac{1}{2}\right).$$

Set

$$f = (f_1, f_2, \ldots, f_n, \ldots), \ g = (g_1, g_2, \ldots, g_n, \ldots).$$

Clearly, the functions f and g are continuous.
For each $u, v \in C[-1, 1], \ w, z \in C[e, e+3]$, and $t \in [1, e]$, we have

$$\|f(t, u, v, w, z)\|_E \leq \frac{ct^2(e^{-7} + e^{-5-t})}{1 + \|u\|_{C[-1,1]} + \|v\|_{C[-1,1]} + \|u\|_{C[e,e+3]} + \|v\|_{C[e,e+3]}},$$

and

$$\|g(t, u, v, w, z)\|_E \leq \frac{ct^2 e^{-6}}{1 + \|u\|_{C[-1,1]} + \|v\|_{C[-1,1]} + \|u\|_{C[e,e+3]} + \|v\|_{C[e,e+3]}}.$$

Hence, the hypothesis (H_3) is satisfied with $p_1^* = p_2^* = ce^{-4}$.
We shall show that condition (3.1) holds. Indeed,

$$\frac{2p_1^*}{\Gamma(1 + r_1)} + \frac{2p_2^*}{\Gamma(1 + r_2)} = \frac{c}{e^4 \Gamma(\frac{5}{2})} = \frac{1}{2} < 1.$$

A simple computations show that all conditions of Theorem 14 are satisfied. It follows that the coupled system (4.1) has at least one solution on $[-1, 3 + e]$.

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