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FURTHER RESULTS OF CERTAIN BIHARI-GAMIDOV TYPE INTEGRAL INEQUALITIES IN TWO INDEPENDENT VARIABLES ON TIME SCALES AND APPLICATIONS

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Abstract. In this paper, we investigate some new nonlinear Bihar-Gamidov type integral inequalities in two independent variables on time scale. Also two illustrative examples to highlight the utility of our results are given.

1 Introduction

Integral inequalities play an important role in the qualitative analysis of the solutions of differential and integral equations see [8, 11, 14, 15]. Recently, many versions of Gamidov inequalities can be found in [4, 7, 9, 10, 12, 13] and the references therein.

In particular, Sh. G.Gamidov [10] , while studying the boundary value problem for higher order differential equations, initiated the study of obtaining explicit upper bounds on the integral inequalities of the forms

$$u(t) \leq c + \int_{\alpha}^{t} f(s)u(s)ds + \int_{\alpha}^{\beta} g(s)u(s)ds, \tag{1.1}$$

for $t \in [\alpha, \beta]$, under some suitable conditions on the functions involved in (1.1).

In [13], Pachpatte established more general Gamidov inequalities as follows :

$$u(t) \leq a(t) + \int_{\alpha}^{t} b(t,s)u(s)ds + \int_{\alpha}^{\beta} c(s)u(s)ds. \tag{1.2}$$

In [7], K. Cheng, C. Guo have studied the general version in two independent variables as follows:

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$$u(x,y) \leq$$
$$a(x,y) + b(x,y)\int_0^x \int_0^y f(t,s)u(t,s)dsdt + c(x,y)\int_0^M \int_0^N g(t,s)u(t,s)dsdt. \quad (1.3)$$

in [4], K. Boukerrioua et al, have studied the general version in two independent variables as follows:

$$u(x,y) \leq$$
$$a(x,y) + b(x,y)\int_0^x \int_0^y f(t,s)u(t,s)dsdt + c(x,y)\int_0^M \int_0^N g(t,s)n(u(t,s))dsdt. \quad (1.4)$$

In the present paper, we shall consider the problem of obtaining new explicit upper bounds on the general versions of (1.4) on time scale which can be used as tools in the study of qualitative behavior of solutions of certain classes of integral equations.

2 Preliminaries on time scales theory

In this section, we begin by giving some necessary materials for our study.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The forward jump operator σ on $\mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \in \mathbb{T}$ for all $t \in \mathbb{T}$,C_{rd} denotes the set of rd-continuous functions and the set $\mathbb{T}^k$ which is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$. The graininess function $\mu : \mathbb{T} \to [0,\infty[$ is defined by $\mu(t) := \sigma(t) - t$.A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu p \neq 0$ on $\mathbb{T}^k$.$\Re(\mathbb{T},\mathbb{R})$ denotes the set of regressive and rd-continuous functions and we define the set of all positively regressive functions by

$$\mathfrak{R}^+(\mathbb{T},\mathbb{R}) = \{p \in \mathfrak{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$$

Throughout this paper,we always assume that $\mathbb{T}_1$and $\mathbb{T}_2$ are time scales, $\Omega = [\alpha,\beta]_{\mathbb{T}_1} \times [\gamma,\theta]_{\mathbb{T}_2}$ and we write $x^{\Delta_t}(t,s)$ for the partial delta derivatives of $x(t,s)$ with respect to t.

We define the time scales interval by

$$[\alpha,\beta]_{\mathbb{T}} = \{t \in \mathbb{T} : \alpha \leq t \leq \beta\}, \quad (2.1)$$

Note that $[\alpha,\beta]^k_{\mathbb{T}} = \begin{cases} [\alpha,\beta]_{\mathbb{T}} \text{ if } \beta \text{ is left- dense,} \\ [\alpha,\beta[_{\mathbb{T}} = [\alpha,\rho(\beta)]_{\mathbb{T}} = [\alpha,\beta[\text{ if } \beta \text{ is left- scattered.} \end{cases}$

Also CC_{rd} denote the set of functions $f(t,s)$ on Ω with the following properties :

1-f is rd-continuous in t,

2-f is rd-continuous in s,

3-if $(t_1,s_1) \in \Omega$,with t_1 right-dense or maximal and s_1 right-dense or maximal, then f is continuous at (t_1,s_1),

4-if t_1 and $s_1$are both left-dense, then the limit of $f(t,s)$ exists as (t,s) approaches (t_1,s_1) along any path in the region

$$R_{LL}(t_1,s_1) = \left\{(t,s) : t \in [a,t_1] \cap [\alpha,\beta]_{\mathbb{T}_1}\; s \in [c,s_1] \cap [\gamma,\theta]_{\mathbb{T}_2}\right\}.$$

for more details about properties of the set CC_{rd}, one can see [1] .

The following Lemmas are very useful in our main results.

Lemma 1. *[8] Assume that $a \geq 0, p \geq q \geq 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} + \frac{p-q}{p} K^{\frac{q}{p}}, \tag{2.2}$$

for any $K > 0$.

Lemma 2. *[2] Assume that $u, b \in C_{rd}, a \in \mathfrak{R}^+$. And if*

$$u^{\Delta}(t) \leq a(t)u(t) + b(t),\ t \geq t_0, t \in \mathbb{T}^k. \tag{2.3}$$

then

$$u(t) \leq u(t_0)e_a(t,t_0) + \int_{t_0}^{t} e_a(t,\sigma(\tau))b(\tau)\Delta\tau, t \geq t_0,\ \ t \in \mathbb{T}^k. \tag{2.4}$$

For more discussion on time scales, we refer the reader to [2, 3] .

Now we state the main results of this work.

3 Main result

In this section, some Bihari- Gamidov type integral inequalities in two independent variables on time scales are investigated. For convenience, it is always assumed that p, q, r are real constants such that $0 < q, r \leq p$, $1 \leq p$ and $\alpha, \beta \in \mathbb{T}_1, \gamma, \theta \in \mathbb{T}_2$.For a given function a defined in a domain Ω with two variables, we say a is decreasing function if, for all $(p,q),(P,Q) \in \Omega$ with $p \leq P, q \leq Q$,one always has $a(P,Q) \leq a(p,q)$.

Lemma 3. *Assume that $u(x,y)$, $a(x,y)$, $c(x,y)$, $g(x,y) \in CCrd(\Omega, \mathbb{R}_+)$. Let*

$$n : \mathbb{R}_+ \to \mathbb{R}_+$$

be a differentiable increasing function on $]0,+\infty[$ *with continuous nonincreasing first derivative* n' *on* $]0,+\infty[$*. If*

$$u(x,y) \le a(x,y) + c(x,y)\int_{\alpha}^{\beta}\int_{\gamma}^{\theta} g(x,y)n(u(x,y))\Delta y\Delta x, \tag{3.1}$$

for $(x,y)\in\Omega$*. And if*

$$\int_{\alpha}^{\beta}\int_{\gamma}^{\theta} g(x,y)c(x,y)n'(a(x,y))\Delta y\Delta x < 1, \tag{3.2}$$

then the following explicit estimates

$$u(x,y) \le a(x,y) + \frac{c(x,y)\int_{\alpha}^{\beta}\int_{\gamma}^{\theta} g(x,y)n(a(x,y))\Delta y\Delta x}{1-\int_{\alpha}^{\beta}\int_{\gamma}^{\theta} g(x,y)c(x,y)n'(a(x,y))\Delta y\Delta x}, \tag{3.3}$$

holds for $(x,y)\in\Omega$.

Proof. Obviously,$\int_{\alpha}^{\beta}\int_{\gamma}^{\theta} g(x,y)n(u(x,y))\Delta y\Delta x$ is a constant.

Letting

$$\int_{\alpha}^{\beta}\int_{\gamma}^{\theta} g(x,y)n(u(x,y))\Delta y\Delta x = \Gamma$$

from (3.1), we have

$$u(x,y) \le a(x,y) + c(x,y)\Gamma \tag{3.4}$$

Since n is increasing on $]0,+\infty[$, then

$$n(u(x,y)) \le n(a(x,y) + c(x,y)\Gamma). \tag{3.5}$$

Applying the mean value Theorem for the function n, then for every $x_1 \ge y_1 > 0$ there exists $c \in]y_1, x_1[$ such that

$$n(x_1) - n(y_1) = n'(c)(x_1 - y_1) \le n'(y_1)(x_1 - y_1). \tag{3.6}$$

Which gives

$$n(u(x,y)) \le n'(a(x,y))c(x,y)\Gamma + n(a(x,y), \tag{3.7}$$

Taking into account that $g(x,y)$ is positive, then

$$g(x,y)n(u(x,y)) \le g(x,y)n'(a(x,y))c(x,y)\Gamma + g(x,y)n(a(x,y)). \tag{3.8}$$

Integrating for both sides of (3.8) on Ω, we have

$$\begin{aligned}\Gamma &= \int_{\alpha}^{\beta}\int_{\gamma}^{\theta} g(x,y)n(u(x,y))\Delta y\Delta x \qquad (3.9)\\ &\le \int_{\alpha}^{\beta}\int_{\gamma}^{\theta} g(x,y)n(a(x,y))\Delta y\Delta x + \Gamma\int_{\alpha}^{\beta}\int_{\gamma}^{\theta} g(x,y)c(x,y)n'(a(x,y))\Delta y\Delta x.\end{aligned}$$

Consequently, it follows from inequality (3.9) that

$$\Gamma \leq \frac{\int_\alpha^\beta \int_\gamma^\theta g(x,y)n(a(x,y))\Delta y \Delta x}{1-\int_\alpha^\beta \int_\gamma^\theta g(x,y)c(x,y)n'(a(x,y))\Delta y \Delta x}. \tag{3.10}$$

Substituting the inequality above into (3.4), we get the explicit estimate (3.3) for $u(x,y)$. □

Theorem 4. *Let* $u(t,s)$, $a(t,s), f(t,s), g(t,s) \in CC_{rd}(\Omega, \mathbb{R}_+)$. *Further,* $n: \mathbb{R}_+ \to \mathbb{R}_+$*is a differentiable increasing function on* $]0,+\infty[$ *with continuous nonincreasing first derivative* n' *on* $]0,+\infty[$. *If*

$$u(t,s) \leq a(t,s) + \int_\alpha^t \int_\gamma^s f(\tau,\eta)\, u(\tau,\eta)\Delta\eta\Delta\tau + \int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta)\ n(u(\tau,\eta))\ \Delta\eta\Delta\tau, \tag{3.11}$$

then

$$u(t,s) \leq A^*(t,s) + \frac{C^*(t,s)\int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta)n(a(\tau,\eta))\Delta\eta\Delta\tau}{1-\int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta)C^*(\tau,\eta)n'(a(\tau,\eta))\Delta\eta\Delta\tau}, \tag{3.12}$$

where

$$A^*(t,s) = a(t,s) + \int_\alpha^t e_{A(.,s)}(t,\sigma(\tau))B(\tau,s)\Delta\tau, \tag{3.13}$$

$$C^*(t,s) = e_{A(.,s)}(t,\alpha), \tag{3.14}$$

$$A(t,s) = \int_\gamma^s f(t,\eta)\,\Delta\eta, \tag{3.15}$$

$$B(t,s) = \int_\gamma^s f(t,\eta)\, a(t,\eta)\Delta\eta. \tag{3.16}$$

Proof. Define a function $z(t,s)$ by

$$z(t,s) = \int_\alpha^t \int_\gamma^s f(\tau,\eta)\, u(\tau,\eta)\Delta\eta\Delta\tau + \int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta)n(u(\tau,\eta))\Delta\eta\Delta\tau, \tag{3.17}$$

then

$$u(t,s) \leq a(t,s) + z(t,s) = w(t,s). \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$\begin{aligned} z^{\Delta_t}(t,s) &\leq \int_\gamma^s f(t,\eta)\, a(t,\eta)\Delta\eta + \int_\gamma^s f(t,\eta)\, z(t,\eta)\Delta\eta \\ &\leq \int_\gamma^s f(t,\eta)\, a(t,\eta)\Delta\eta + (\int_\gamma^s f(t,\eta)\,\Delta\eta) z(t,s), \end{aligned} \tag{3.19}$$

Since $z(t,s)$ is nondecreasing in s for each fixed t, then, the inequality (3.19) can be written as

$$z^{\Delta_t}(t,s) \le B(t,s) + A(t,s)z(t,s), \tag{3.20}$$

where $A(t,s)$ and $B(t,s)$ are defined by (3.15) and (3.16).

Noting that

$$z(\alpha,s) = \int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta)n(u(\tau,\eta))\Delta\eta\Delta\tau \le \int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta)n(a(\tau,\eta)+z(\tau,\eta))\Delta\eta\Delta\tau. \tag{3.21}$$

Using Lemma 2 to (3.20), we have

$$z(t,s) \le z(\alpha,s)e_{A(.,s)}(t,\alpha) + \int_\alpha^t e_{A(.,s)}(t,\sigma(\tau))B(\tau,s)\Delta\tau, \tag{3.22}$$

From (3.22) and (3.21), we get

$$\begin{aligned} z(t,s) \quad \le \quad & e_{A(.,s)}(t,\alpha)\int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta)n(a(\tau,\eta)+z(\tau,\eta))\Delta\eta\Delta\tau \\ & + \int_\alpha^t e_{A(.,s)}(t,\sigma(\tau))B(\tau,s)\Delta\tau, \end{aligned} \tag{3.23}$$

The inequality (3.23) can be reformulated as

$$w(t,s) \le A^*(t,s) + C^*(t,s)\int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta)n(w(\tau,\eta))\Delta\eta\Delta\tau, \tag{3.24}$$

where $A^*(t,s)$,$C^*(t,s)$ are defined as in (3.13) and (3.14).

Using Lemma 3 to (3.24) and from (3.18), we obtain the desired inequality in (3.12). □

Corollary 5. *Assume that the hypotheses of Theorem 4 hold. Then*

$$u(t,s) \le a(t,s) + \int_\alpha^t \int_\gamma^s f(\tau,\eta)\, u(\tau,\eta)\Delta\eta\Delta\tau + \int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta)\ \arctan(u(\tau,\eta))\ \Delta\eta\Delta\tau,$$

then

$$u(t,s) \le A^*(t,s) + \frac{C^*(t,s)\int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta)\arctan(a(\tau,\eta))\Delta\eta\Delta\tau}{1 - \int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta)\frac{C^*(\tau,\eta)}{1+a^2(\tau,\eta)}\Delta\eta\Delta\tau},$$

where $A^*(t,s), C^*(t,s)$ are defined as in Theorem 4, provided that

$$\int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta)\frac{C^*(\tau,\eta)}{1+a^2(\tau,\eta)}\Delta\eta\Delta\tau < 1.$$

Corollary 6. *Assume that the hypotheses of Theorem 4 hold. Then*

$$u(t,s) \leq a(t,s) + \int_\alpha^t \int_\gamma^s f(\tau,\eta)\, u(\tau,\eta) \Delta\eta\Delta\tau + \int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta) \ln(u(\tau,\eta)+1)\ \Delta\eta\Delta\tau,$$

implies

$$u(t,s) \leq A^*(t,s) + \frac{C^*(t,s) \int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta) \ln(a(\tau,\eta)+1)\ \Delta\eta\Delta\tau}{1 - \int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta) \frac{C^*(\tau,\eta)}{1+a(\tau,\eta)} \Delta\eta\Delta\tau},$$

where $A^*(t,s), C^*(t,s)$ are defined as in Theorem 4 , provided that

$$\int_\alpha^\beta \int_\gamma^\theta g(\tau,\eta) \frac{C^*(\tau,\eta)}{1+a(\tau,\eta)} \Delta\eta\Delta\tau < 1.$$

Theorem 7. *Let $u(t,s)$, $a(t,s), b(t,s), c(t,s)$, $f(t,s), g(t,s), h(t,s) \in CC_{rd}(\Omega, \mathbb{R}_+)$ and $b(t,s)+c(t,s)$ is decreasing, $p,q,r \in \mathbb{R}_+$ such that $p \geq q > 0$, $p \geq r > 0$, $p \geq 1$. Let $n : \mathbb{R}_+ \to \mathbb{R}_+$is a differentiable increasing function n' on $]0,+\infty[$ with continuous nonincreasing first derivative n' on $]0,+\infty[$. If $u(t,s)$ satisfies*

$$\begin{aligned} u^p(t,s) \quad &\leq \quad a(t,s) + b(t,s) \int_\alpha^t \int_\gamma^s \ [f(\tau,\eta)\, u^q(\tau,\eta) + g(\tau,\eta)\, u(\tau,\eta)]\, \Delta\eta\Delta\tau \\ &\quad + c(t,s) \int_\alpha^\beta \int_\gamma^\theta h(\tau,\eta) n(\ u^r(\tau,\eta))\ \Delta\eta\Delta\tau. \end{aligned} \tag{3.25}$$

Then

$$\begin{aligned} u(t,s) \quad &\leq \\ &\left[A_1^*(t,s) + \frac{C_1^*(t,s) \int_\alpha^\beta \int_\gamma^\theta g_1(\tau,\eta) n(a_1(\tau,\eta)) \Delta\eta\Delta\tau}{1 - \int_\alpha^\beta \int_\gamma^\theta g_1(\tau,\eta) C_1^*(\tau,\eta) n'(a_1(\tau,\eta)) \Delta\eta\Delta\tau} \right]^{\frac{1}{r}}, \end{aligned} \tag{3.26}$$

holds for $(t,s) \in [\alpha,\beta]_{\mathbb{T}_1}^k \times [\gamma,\theta]_{\mathbb{T}_2}$, where

$$A_1^*(t,s) \quad = \quad a_1(t,s) + \int_\alpha^t e_{A_1(.,s)}(t,\sigma(\tau)) B_1(\tau,s) \Delta\tau, \tag{3.27}$$

$$C_1^*(t,s) \quad = \quad e_{A_1(.,s)}(t,\alpha), \tag{3.28}$$

and

$$\begin{aligned} a_1(t,s) \quad &= \quad \frac{r}{p} K^{\frac{r-p}{p}} a(t,s) + \frac{p-r}{p} K^{\frac{r}{p}} \\ &\quad + \frac{r}{p} K^{\frac{r-p}{p}} b^*(\alpha,\gamma) \int_\alpha^t \int_\gamma^s (f(\tau,\eta) (\frac{q}{p} K^{\frac{q-p}{p}} a(\tau,\eta) + \frac{p-q}{p} K^{\frac{q}{p}}) \\ &\quad + g(\tau,\eta) (\frac{1}{p} K^{\frac{1-p}{p}} a(\tau,\eta) + \frac{p-1}{p} K^{\frac{1}{p}})) \Delta\eta\Delta\tau, \end{aligned} \tag{3.29}$$

$$f_1(t,s) = \left[\frac{q}{p}K^{\frac{q-p}{p}} f(t,s) + \frac{1}{p}K^{\frac{1-p}{p}} g(t,s)\right] b^*(\alpha,\gamma), \tag{3.30}$$

$$g_1(t,s) = \frac{r}{p}k^{\frac{r-p}{p}} b^*(\alpha,\gamma)h(t,s), \tag{3.31}$$

and

$$\begin{aligned} A_1(t,s) &= \int_\gamma^s f_1(t,\eta)\,\Delta\eta,, \\ B_1(t,s) &= \int_\gamma^s f_1(t,\eta)\,a_1(t,\eta)\Delta\eta, \end{aligned}$$

with

$$\int_\alpha^\beta \int_\gamma^\theta g_1(\tau,\eta)C_1^*(\tau,\eta)n'(a_1(\tau,\eta))\Delta\eta\Delta\tau < 1.$$

Proof. The inequality (3.25) can be reformulated as

$$\begin{aligned} u^p(t,s) \leq\ & a(t,s) + b^*(t,s)(\int_\alpha^t \int_\gamma^s [f(\tau,\eta)\,u^q(\tau,\eta) + g(\tau,\eta)\,u(\tau,\eta)]\,\Delta\eta\Delta\tau \\ & + \int_\alpha^\beta \int_\gamma^\theta h(\tau,\eta)n(u^r(\tau,\eta))\Delta\eta\Delta\tau), \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} b^*(t,s) &= b(t,s) + c(t,s), \\ b^*(t,s) &\leq b^*(\alpha,\gamma). \end{aligned}$$

Define a function $z(t,s)$ by

$$\begin{aligned} z(t,s) =\ & \int_\alpha^t \int_\gamma^s (f(\tau,\eta)\,u^q(\tau,\eta) + g(\tau,\eta)\,u(\tau,\eta))\Delta\eta\Delta\tau, \\ & + \int_\alpha^\beta \int_\gamma^\theta h(\tau,\eta)\ n(u^r(\tau,\eta))\Delta\eta\Delta\tau. \end{aligned} \tag{3.33}$$

By lemma 1, we have

$$\begin{aligned} u(t,s) &\leq (a(t,s) + b^*(\alpha,\gamma)z(t,s))^{\frac{1}{p}} \leq \frac{1}{p}K^{\frac{1-p}{p}}(a(\tau,\eta) + b^*(\alpha,\gamma)z(\tau,\eta)) + \frac{p-1}{p}K^{\frac{1}{p}}, \\ u^q(t,s) &\leq (a(t,s) + b^*(\alpha,\gamma)z(t,s))^{\frac{q}{p}} \\ &\leq \frac{q}{p}K^{\frac{q-p}{p}}(a(t,s) + b^*(\alpha,\gamma)z(t,s)) + \frac{p-q}{p}K^{\frac{q}{p}} \end{aligned} \tag{3.34}$$

$$\begin{aligned} u^r(t,s) &\le (a(t,s)+b^*(\alpha,\gamma)z(t,s))^{\frac{r}{p}} \\ &\le \frac{r}{p}K^{\frac{r-p}{p}}(a(t,s)+b^*(\alpha,\gamma)z(t,s))+\frac{p-r}{p}K^{\frac{r}{p}}=w(t,s). \end{aligned} \tag{3.35}$$

It follows from (3.33),(3.34) and (3.35) that

$$\begin{aligned} z(t,s) \le & \int_\alpha^t\int_\gamma^s\Big[f(\tau,\eta)\,(\frac{q}{p}K^{\frac{q-p}{p}}(a(\tau,\eta)+b^*(\alpha,\gamma)z(\tau,\eta))+\frac{p-q}{p}K^{\frac{q}{p}}) \\ &+g(\tau,\eta)\,(\frac{1}{p}K^{\frac{1-p}{p}}(a(\tau,\eta)+b^*(\alpha,\gamma)z(\tau,\eta))+\frac{p-1}{p}K^{\frac{1}{p}})\Big]\,\Delta\eta\Delta\tau \\ &+\int_\alpha^\beta\int_\gamma^\theta h(\tau,\eta)\; n\left(\frac{r}{p}K^{\frac{r-p}{p}}(a(\tau,\eta)+b^*(\alpha,\gamma)z(\tau,\eta))+\frac{p-r}{p}K^{\frac{r}{p}}\right)\;\Delta\eta\Delta\tau. \end{aligned} \tag{3.36}$$

It gives ,

$$\begin{aligned} & z(t,s)\le \\ & \int_\alpha^t\int_\gamma^s\Big[f(\tau,\eta)\,(\frac{q}{p}K^{\frac{q-p}{p}}a(\tau,\eta)+\frac{p-q}{p}K^{\frac{q}{p}})+g(\tau,\eta)\,(\frac{1}{p}K^{\frac{1-p}{p}}a(\tau,\eta)+\frac{p-1}{p}K^{\frac{1}{p}})\Big]\,\Delta\eta\Delta\tau \\ &\quad +\int_\alpha^t\int_\gamma^s\left[\frac{q}{p}K^{\frac{q-p}{p}}f(\tau,\eta)+\frac{1}{p}K^{\frac{1-p}{p}}g(\tau,\eta)\right]b^*(\alpha,\gamma)z(\tau,\eta)\Delta\eta\Delta\tau \\ &\quad\quad +\int_\alpha^\beta\int_\gamma^\theta h(\tau,\eta)\; n(w(\tau,\eta))\;\Delta\eta\Delta\tau, \end{aligned} \tag{3.37}$$

where $w(t,s)$ is defined as in (3.35).

Multiplying both sides of the inequality (3.37) by $\frac{r}{p}K^{\frac{r-p}{p}}\; b^*(\alpha,\gamma)$ and adding $\frac{r}{p}K^{\frac{r-p}{p}}a(t,s)+\frac{p-r}{p}K^{\frac{r}{p}}$ to the resultant inequality, we obtain the following inequality

$$w(t,s)\le a_1(t,s)+\int_\alpha^t\int_\gamma^s f_1(\tau,\eta)\,w(\tau,\eta)\,\Delta\eta\Delta\tau+\int_\alpha^\beta\int_\gamma^\theta g_1(\tau,\eta)n(w(\tau,\eta))\Delta\eta\Delta\tau, \tag{3.38}$$

where $a_1(t,s)$ is defined as in (3.29) and $f_1(t,s)$ and $g_1(t,s)$ are defined as in (3.30) and (3.31).

Applying Theorem 4 to inequality (3.38), we obtain

$$w(t,s)\le A_1^*(t,s)+\frac{C_1^*(t,s)\int_\alpha^\beta\int_\gamma^\theta g_1(\tau,\eta)n(a_1(\tau,\eta))\Delta\eta\Delta\tau}{1-\int_\alpha^\beta\int_\gamma^\theta g_1(\tau,\eta)C_1^*(\tau,\eta)n'(a_1(\tau,\eta))\Delta\eta\Delta\tau}, \tag{3.39}$$

then from (3.35) and (3.39), we obtain the desired inequality in (3.26). □

Theorem 8. *Assume that* $u(t,s)$,$a(t,s)$,$b(t,s)$, $c(t,s)$, $g(t,s) \in CC_{rd}(\Omega, \mathbb{R}_+)$ *and* $b(t,s)+c(t,s)$ *is decreasing,* $p \geq 1$ *is a real constant. Let* $n : \mathbb{R}_+ \to \mathbb{R}_+$ *be a differentiable increasing function on* $]0,+\infty[$ *with continuous nonincreasing first derivative* n' *on* $]0,+\infty[$. *Let* $f : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *is* CC_{rd} *on* Ω *and continuous on* $\mathbb{R}_+$ *such that*

$$0 \leq f(t,s,x) - f(t,s,y) \leq h(t,s,y)(x-y)$$

for $(t,s) \in \Omega$, $x \geq y \geq 0$ *where* $h : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *is* CC_{rd} *on* Ω *and continuous on* $\mathbb{R}_+$, *then*

$$u^p(t,s) \leq a(t,s)+b(t,s)\int_\alpha^t\int_\gamma^s f(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau+c(t,s)\int_\alpha^\beta\int_\gamma^\theta g(\tau,\eta)n(\ u^r\,(\tau,\eta))\ \Delta\eta\Delta\tau, \tag{3.40}$$

implies

$$u(t,s) \leq \left[A_2^*(t,s) + \frac{C_2^*(t,s)\int_\alpha^\beta\int_\gamma^\theta g_2(\tau,\eta)n(a_2(\tau,\eta))\Delta\eta\Delta\tau}{1-\int_\alpha^\beta\int_\gamma^\theta g_2(\tau,\eta)C_2^*(\tau,\eta)n'(a_2(\tau,\eta))\Delta\eta\Delta\tau}\right]^{\frac{1}{r}}, \tag{3.41}$$

for $(t,s) \in [\alpha,\beta]_{\mathbb{T}_1}^k \times [\gamma,\theta]_{\mathbb{T}_2}$, *provided that*

$$\int_\alpha^\beta\int_\gamma^\theta g_2(\tau,\eta)C_2^*(\tau,\eta)n'(a_2(\tau,\eta))\Delta\eta\Delta\tau < 1,$$

where

$$A_2^*(t,s) = a_2(t,s) + \int_\alpha^t e_{A_2(.,s)}(t,\sigma(\tau))B_2(\tau,s)\Delta\tau, \tag{3.42}$$

$$C_2^*(t,s) = e_{A_2(.,s)}(t,\alpha). \tag{3.43}$$

$$\begin{aligned} a_2(t,s) &= \frac{r}{p}K^{\frac{r-p}{p}}b^*(\alpha,\gamma)\int_\alpha^t\int_\gamma^s f(\tau,\eta,\frac{1}{p}K^{\frac{1-p}{p}}a(\tau,\eta)+\frac{p-1}{p}K^{\frac{1}{p}})\Delta\eta\Delta\tau \\ &\quad +\frac{r}{p}K^{\frac{r-p}{p}}a(t,s)+\frac{p-r}{p}K^{\frac{r}{p}}, \qquad (3.44) \\ B_2(t,s) &= \int_\gamma^s f_2\,(t,\eta)\,a_2(t,\eta)\Delta\eta, \\ A_2(t,s) &= \int_\gamma^s f_2(t,\eta)\Delta\eta, \end{aligned}$$

$$f_2(t,s) = h(t,s,\frac{1}{p}K^{\frac{1-p}{p}}a(t,s)+\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}b^*(\alpha,\gamma), \tag{3.45}$$

$$g_2(t,s) = \frac{r}{p}K^{\frac{r-p}{p}}b^*(\alpha,\gamma)g(t,s)\ . \tag{3.46}$$

Proof. The inequality (3.40) can be reformulated as

$$u^p(t,s) \leq$$

$$a(t,s)+b^*(t,s)(\int_\alpha^t\int_\gamma^s f(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau+\int_\alpha^\beta\int_\gamma^\theta g(\tau,\eta)n(u^r(\tau,\eta))\Delta\eta\Delta\tau). \tag{3.47}$$

Define a function $z(t,s)$ by

$$z(t,s)=\int_\alpha^t\int_\gamma^s f(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau\ +\int_\alpha^\beta\int_\gamma^\theta g(\tau,\eta)n(u^r(\tau,\eta))\Delta\eta\Delta\tau. \tag{3.48}$$

From (3.47) and (3.48), we have

$$u(t,s)\leq (a(t,s)+b^*(\alpha,\gamma)z(t,s))^{\frac{1}{p}}, \tag{3.49}$$

Using Lemma 1 to (3.49), we obtain

$$\begin{aligned} z(t,s) \ \leq\ & \int_\alpha^t\int_\gamma^s \left[\begin{array}{c} f(\tau,\eta,\frac{1}{p}K^{\frac{1-p}{p}}a(\tau,\eta)+\frac{p-1}{p}K^{\frac{1}{p}}+\frac{1}{p}K^{\frac{1-p}{p}}b^*(\alpha,\gamma)z(\tau,\eta)) \\ -f(\tau,\eta,\frac{1}{p}K^{\frac{1-p}{p}}a(\tau,\eta)+\frac{p-1}{p}K^{\frac{1}{p}}) \\ +f(\tau,\eta,\frac{1}{p}K^{\frac{1-p}{p}}a(\tau,\eta)+\frac{p-1}{p}K^{\frac{1}{p}}) \end{array}\right]\Delta\eta\Delta\tau \\ & +\int_\alpha^\beta\int_\gamma^\theta g(\tau,\eta)n\left(\begin{array}{c} \frac{r}{p}K^{\frac{r-p}{p}}(a(\tau,\eta) \\ +b^*(\alpha,\gamma)z(\tau,\eta))+\frac{p-r}{p}K^{\frac{r}{p}} \end{array}\right)\Delta\eta\Delta\tau, \end{aligned} \tag{3.50}$$

Noting the assumptions on f , we have

$$\begin{aligned} z(t,s) \ \leq\ & \int_\alpha^t\int_\gamma^s f(\tau,\eta,\frac{1}{p}K^{\frac{1-p}{p}}a(\tau,\eta)+\frac{p-1}{p}K^{\frac{1}{p}})\Delta\eta\Delta\tau \\ & +\int_\alpha^t\int_\gamma^s h(\tau,\eta,\frac{1}{p}K^{\frac{1-p}{p}}a(\tau,\eta)+\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}b^*(\alpha,\gamma)z(\tau,\eta)\Delta\eta\Delta\tau \\ & +\int_\alpha^\beta\int_\gamma^\theta g(\tau,\eta)n(w(\tau,\eta))\Delta\eta\Delta\tau, \end{aligned} \tag{3.51}$$

where

$$w(t,s)=\frac{r}{p}K^{\frac{r-p}{p}}(a(t,s)+b^*(\alpha,\gamma)z(t,s))+\frac{p-r}{p}K^{\frac{r}{p}}.$$

Multiplying both sides of (3.51) by $\frac{r}{p}K^{\frac{r-p}{p}}\ b^{*}(\alpha,\gamma)$ and adding $\frac{r}{p}K^{\frac{r-p}{p}}a(t,s)+\frac{p-r}{p}K^{\frac{r}{p}}$to both sides of the resultant inequality, we obtain

$$w(t,s)\leq$$
$$a_2(t,s)+\int_{\alpha}^{t}\int_{\gamma}^{s}h(\tau,\eta,\frac{1}{p}K^{\frac{1-p}{p}}a(\tau,\eta)+\frac{p-1}{p}K^{\frac{1}{p}})\frac{1}{p}K^{\frac{1-p}{p}}b^{*}(\alpha,\gamma)w(\tau,\eta)\Delta\eta\Delta\tau$$
$$+\int_{\alpha}^{\beta}\int_{\gamma}^{\theta}\frac{r}{p}K^{\frac{r-p}{p}}b^{*}(\alpha,\gamma)g(\tau,\eta)n(w(\tau,\eta))\Delta\eta\Delta\tau. \quad (3.52)$$

Then the inequality (3.52) can be expressed as

$$\begin{aligned} w(t,s) \quad &\leq \quad a_2(t,s)+\int_{\alpha}^{t}\int_{\gamma}^{s}f_2(\tau,\eta)w(\tau,\eta)\Delta\eta\Delta\tau \\ &+\int_{\alpha}^{\beta}\int_{\gamma}^{\theta}g_2(\tau,\eta)n(w(\tau,\eta))\Delta\eta\Delta\tau, \end{aligned} \quad (3.53)$$

where $a_2(t,s)$ is defined by (3.44) and $f_2(t,s)$ and $g_2(t,s)$ are defined by (3.45) and (3.46).

Using Theorem 4 to inequality (3.53) and from (3.35), we obtain the desired inequality in (3.41). □

4 Applications:

In this section, we present some applications of our results to study boundedness and uniqueness of solutions of a Volterra-Fredholm integral equation in two independent variables on time scales.

Example 9. *Consider the following intergral equation on time scales*

$$\begin{aligned} z^{p}(t,s) \quad &= \quad a(t,s)+b(t,s)\int_{\alpha}^{t}\int_{\gamma}^{s}F(\tau,\eta,z(\tau,\eta))\Delta\eta\Delta\tau + \\ & \quad\quad c(t,s)\int_{\alpha}^{\beta}\int_{\gamma}^{\theta}G(\tau,\eta,z(\tau,\eta))\Delta\eta\Delta\tau, \end{aligned} \quad (4.1)$$

for $(t,s)\in\Omega$, *where* $z(t,s)\in CCrd(\Omega,\mathbb{R})$,$a(t,s),b(t,s),c(t,s)\in CCrd(\Omega,\mathbb{R}_{+})$, $F(t,s,z),G(t,s,z)\in CCrd(\Omega\times\mathbb{R},\mathbb{R})$.

Theorem 10. *Assume that the functions F and G in (4.1) satisfy the conditions*

$$\begin{aligned} |F(t,s,z)| &\leq f(t,s)\,|z|^q + g(t,s)\,|z| \\ |G(t,s,z)| &\leq h(t,s)n(|z|^r), \end{aligned} \tag{4.2}$$

where $f(t,s), h(t,s)$ and n are defined as in Theorem 7, with $n'0) = 0$, p, q, r are constants such that $p \geq q > 0$, $p \geq r > 0$, $p \geq 1$. If $z(t,s)$ is a solution of (4.1)-(4.2), then $z(t,s)$ satisfies

$$\begin{aligned} z(t,s) &\leq \\ &\left[A_1^*(t,s) + \frac{C_1^*(t,s)\int_\alpha^\beta\int_\gamma^\theta g_1(\tau,\eta)n(a_1(\tau,\eta))\Delta\eta\Delta\tau}{1-\int_\alpha^\beta\int_\gamma^\theta g_1(\tau,\eta)C_1^*(\tau,\eta)n'(a_1(\tau,\eta))\Delta\eta\Delta\tau}\right]^{\frac{1}{r}}, \end{aligned} \tag{4.3}$$

hold for $(t,s) \in [\alpha,\beta]_{\mathbb{T}_1}^k \times [\gamma,\theta]_{\mathbb{T}_2}$,where $A_1^*(t,s), C_1^*(t,s), g_1(t,s), a_1(t,s)$ are defined as in Theorem 7.

Proof. Assume that $z(t,s)$ is the unique solution of (4.1), from (4.2), we have

$$\begin{aligned} |z^p(t,s)| &\leq a(t,s) + b(t,s)\int_\alpha^t\int_\gamma^s f(\tau,\eta)\,|z(\tau,\eta)|^q + g(\tau,\eta)\,|z(\tau,\eta)|\,\Delta\eta\,\Delta\tau \\ &\quad + c(t,s)\int_\alpha^\beta\int_\gamma^\theta h(\tau,\eta)n(|z(\tau,\eta)|^r)\Delta\eta\,\Delta\tau. \end{aligned} \tag{4.4}$$

Now by applying Theorem 7 to (4.4), we obtained the required inequality (4.3). ☐

Proposition 11. *Assume that the functions F and G in (4.1) satisfy the conditions*

$$\begin{aligned} |F(t,s,z) - F(t,s,\overline{z})| &\leq f(t,s)\,|z-\overline{z}|\,, \\ |G(t,s,z) - G(t,s,\overline{z})| &\leq h(t,s)n(|z-\overline{z}|), \end{aligned} \tag{4.5}$$

where $f(t,s)$, $h(t,s)$ and n are defined as in Theorem 7 and $z(t,s)$ is a solution of (4.1) (in the case $p = 1$). Then (4.1) has at most one solution.

Proof. Let $z(t,s)$ and $\overline{z}(t,s)$ be two solutions of (4.1), that is,

$$\begin{aligned} \overline{z}(t,s) &= a(t,s) + b(t,s)\int_\alpha^t\int_\gamma^s F(\tau,\eta,\overline{z})\Delta\eta\,\Delta\tau \\ &\quad + c(t,s)\int_\alpha^\beta\int_\gamma^\theta G(\tau,\eta,\overline{z})\Delta\eta\,\Delta\tau. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we have

$$
\begin{aligned}
|z(t,s)-\overline{z}(t,s)| &\leq b(t,s)\int_{\alpha}^{t}\int_{\gamma}^{s}|F(\tau,\eta,z)-F(\tau,\eta,\overline{z})|\,\Delta\eta\,\Delta\tau \qquad (4.7)\\
&\quad +c(t,s)\int_{\alpha}^{\beta}\int_{\gamma}^{\theta}|G(\tau,\eta,z)-G(\tau,\eta,\overline{z})|\,\Delta\eta\,\Delta\tau\\
&\leq b(t,s)\int_{\alpha}^{t}\int_{\gamma}^{s}f(\tau,\eta)\,|z-\overline{z}|\,\Delta\eta\,\Delta\tau\\
&\quad +c(t,s)\int_{\alpha}^{\beta}\int_{\gamma}^{\theta}h(\tau,\eta)n(|z-\overline{z}|)\Delta\eta\,\Delta\tau.
\end{aligned}
$$

According to Theorem 7, we obtain that $|z(t,s)-\overline{z}(t,s)|\leq 0$, which implies $z(t,s)=\overline{z}(t,s)$ for $(t,s)\in\Omega$. □

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