FURTHER RESULTS OF CERTAIN BIHARI-GAMIDOV TYPE INTEGRAL INEQUALITIES IN TWO INDEPENDENT VARIABLES ON TIME SCALES AND APPLICATIONS

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Abstract. In this paper, we investigate some new nonlinear Bihar-Gamidov type integral inequalities in two independent variables on time scale. Also two illustrative examples to highlight the utility of our results are given.

1 Introduction

Integral inequalities play an important role in the qualitative analysis of the solutions of differential and integral equations see [8, 11, 14, 15]. Recently, many versions of Gamidov inequalities can be found in [4, 7, 9, 10, 12, 13] and the references therein.

In particular, Sh. G.Gamidov [10], while studying the boundary value problem for higher order differential equations, initiated the study of obtaining explicit upper bounds on the integral inequalities of the forms

$$ u(t) \leq c + \int_{\alpha}^{t} f(s)u(s)ds + \int_{\alpha}^{\beta} g(s)u(s)ds, \quad (1.1) $$

for $t \in [\alpha, \beta]$, under some suitable conditions on the functions involved in (1.1).

In [13], Pachpatte established more general Gamidov inequalities as follows:

$$ u(t) \leq a(t) + \int_{\alpha}^{t} b(t, s)u(s)ds + \int_{\alpha}^{\beta} c(s)u(s)ds. \quad (1.2) $$

In [7], K. Cheng, C. Guo have studied the general version in two independent variables as follows:

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\[ u(x, y) \leq a(x, y) + b(x, y) \int_0^x \int_0^y f(t, s) u(t, s) ds dt + c(x, y) \int_0^M \int_0^N g(t, s) u(t, s) ds dt. \quad (1.3) \]

In [4], K. Boukerrioua et al, have studied the general version in two independent variables as follows:

\[ u(x, y) \leq a(x, y) + b(x, y) \int_0^x \int_0^y f(t, s) u(t, s) ds dt + c(x, y) \int_0^M \int_0^N g(t, s) u(t, s) ds dt + n(u(t, s)) ds dt. \quad (1.4) \]

In the present paper, we shall consider the problem of obtaining new explicit upper bounds on the general versions of (1.4) on time scale which can be used as tools in the study of qualitative behavior of solutions of certain classes of integral equations.

2 Preliminaries on time scales theory

In this section, we begin by giving some necessary materials for our study.

A time scale \( T \) is an arbitrary nonempty closed subset of \( \mathbb{R} \). The forward jump operator \( \sigma \) on \( T \) is defined by \( \sigma(t) := \inf \{ s \in T : s > t \} \in T \) for all \( t \in T \). \( C_{rd} \) denotes the set of rd-continuous functions and the set \( T^k \) which is derived from the time scale \( T \) as follows: If \( T \) has a left-scattered maximum \( m \), then \( T^k = T - \{ m \} \), otherwise, \( T^k = T \). The graininess function \( \mu : T \rightarrow [0, \infty[ \) is defined by \( \mu(t) := \sigma(t) - t \). A function \( p : T \rightarrow \mathbb{R} \) is called regressive if \( 1 + \mu p \neq 0 \) on \( T^k \). \( \mathcal{R}(T, \mathbb{R}) \) denotes the set of regressive and rd-continuous functions and we define the set of all positively regressive functions by

\[ \mathcal{R}^+(T, \mathbb{R}) = \{ p \in \mathcal{R} : 1 + \mu p(t) > 0 \text{ for all } t \in T \} \]

Throughout this paper, we always assume that \( T_1 \) and \( T_2 \) are time scales, \( \Omega = [\alpha, \beta]_{T_1} \times [\gamma, \theta]_{T_2} \) and we write \( x^{\Delta t}(t, s) \) for the partial delta derivatives of \( x(t, s) \) with respect to \( t \).

We define the time scales interval by

\[ [\alpha, \beta]_T = \{ t \in T : \alpha \leq t \leq \beta \}, \quad (2.1) \]

Note that \( [a, \beta]_{kT} = \begin{cases} [a, \beta]_T & \text{if } \beta \text{ is left- dense}, \\ [a, \rho(\beta)]_T = [a, \beta]_T & \text{if } \beta \text{ is left- scattered.} \end{cases} \)
Also \( CC_{rd} \) denote the set of functions \( f(t, s) \) on \( \Omega \) with the following properties:

1. \( f \) is rd-continuous in \( t \),
2. \( f \) is rd-continuous in \( s \),
3. if \((t_1, s_1) \in \Omega, t_1 \) right-dense or maximal and \( s_1 \) right-dense or maximal,

then \( f \) is continuous at \((t_1, s_1)\),
4. if \( t_1 \) and \( s_1 \) are both left-dense, then the limit of \( f(t, s) \) exists as \((t, s)\) approaches \((t_1, s_1)\) along any path in the region

\[
R_{LL}(t_1, s_1) = \{(t, s) : t \in [a, t_1] \cap [\alpha, \beta]_{T_1} s \in [c, s_1] \cap [\gamma, \theta]_{T_2}\}.
\]

for more details about properties of the set \( CC_{rd} \), one can see [1].

The following Lemmas are very useful in our main results.

**Lemma 1.** [8] Assume that \( a \geq 0, p \geq q \geq 0 \) and \( p \neq 0 \), then

\[
a^\frac{q}{p} \leq \frac{q}{p} K^{\frac{q-p}{p}} + \frac{p-q}{p} K^{\frac{q}{p}},
\]

(2.2)

for any \( K > 0 \).

**Lemma 2.** [2] Assume that \( u, b \in C_{rd}, a \in \mathbb{R}^+ \). And if

\[
u^\Delta(t) \leq a(t)u(t) + b(t), \ t \geq t_0, t \in T^k.
\]

(2.3)

then

\[
u(t) \leq u(t_0)e_a(t, t_0) + \int_{t_0}^{t} e_a(t, \sigma(\tau))b(\tau)\Delta \tau, \ t \geq t_0, t \in T^k.
\]

(2.4)

For more discussion on time scales, we refer the reader to [2, 3].

Now we state the main results of this work.

**3 Main result**

In this section, some Bihari- Gamidov type integral inequalities in two independent variables on time scales are investigated. For convenience, it is always assumed that \( p, q, r \) are real constants such that \( 0 < q, r \leq p, 1 \leq p \) and \( \alpha, \beta \in T_1, \gamma, \theta \in T_2 \). For a given function \( a \) defined in a domain \( \Omega \) with two variables, we say \( a \) is decreasing function if, for all \((p, q), (P, Q) \in \Omega \) with \( p \leq P, q \leq Q \), one always has \( a(P, Q) \leq a(p, q) \).

**Lemma 3.** Assume that \( u(x, y), a(x, y), c(x, y), g(x, y) \in CC_{rd}(\Omega, \mathbb{R}^+) \). Let

\[
n : \mathbb{R}^+ \rightarrow \mathbb{R}^+
\]
be a differentiable increasing function on $]0, +\infty[$ with continuous nonincreasing first derivative $n'$ on $]0, +\infty[$. If
\[
u(x, y) \leq a(x, y) + c(x, y) \int_\alpha^\beta \int_\gamma^\theta g(x, y)n(u(x, y))\Delta y\Delta x, \tag{3.1}\]
for $(x, y) \in \Omega$. And if
\[
\int_\alpha^\beta \int_\gamma^\theta g(x, y)c(x, y)n'(a(x, y))\Delta y\Delta x < 1, \tag{3.2}\]
then the following explicit estimates
\[
u(x, y) \leq a(x, y) + c(x, y)\Gamma + n(a(x, y)) \tag{3.3}\]
holds for $(x, y) \in \Omega$.

Proof. Obviously, $\int_\alpha^\beta \int_\gamma^\theta g(x, y)n(u(x, y))\Delta y\Delta x$ is a constant.

Letting
\[
\int_\alpha^\beta \int_\gamma^\theta g(x, y)n(u(x, y))\Delta y\Delta x = \Gamma
\]
from (3.1), we have
\[
u(x, y) \leq a(x, y) + c(x, y)\Gamma \tag{3.4}
\]
Since $n$ is increasing on $]0, +\infty[$, then
\[
n(u(x, y)) \leq n(a(x, y) + c(x, y)\Gamma). \tag{3.5}
\]
Applying the mean value Theorem for the function $n$, then for every $x_1 \geq y_1 > 0$ there exists $c \in ]y_1, x_1[$ such that
\[
n(x_1) - n(y_1) = n'(c)(x_1 - y_1) \leq n'(y_1)(x_1 - y_1). \tag{3.6}
\]
Which gives
\[
n(u(x, y)) \leq n'(a(x, y))c(x, y)\Gamma + n(a(x, y)). \tag{3.7}
\]
Taking into account that $g(x, y)$ is positive, then
\[
g(x, y)n(u(x, y)) \leq g(x, y)n'(a(x, y))c(x, y)\Gamma + g(x, y)n(a(x, y)). \tag{3.8}
\]
Integrating for both sides of (3.8) on $\Omega$, we have
\[
\Gamma = \int_\alpha^\beta \int_\gamma^\theta g(x, y)n(u(x, y))\Delta y\Delta x \tag{3.9}
\]
\[
\leq \int_\alpha^\beta \int_\gamma^\theta g(x, y)n(a(x, y))\Delta y\Delta x + \Gamma \int_\alpha^\beta \int_\gamma^\theta g(x, y)c(x, y)n'(a(x, y))\Delta y\Delta x.
\]
Consequently, it follows from inequality (3.9) that
\[
\Gamma \leq \frac{\int_\alpha^\beta \int_\gamma^\theta g(x, y)n(a(x, y))\Delta y\Delta x}{1 - \int_\alpha^\beta \int_\gamma^\theta g(x, y)c(x, y)n'(a(x, y))\Delta y\Delta x}.
\] (3.10)

Substituting the inequality above into (3.4), we get the explicit estimate (3.3) for \( u(x, y) \).

**Theorem 4.** Let \( u(t, s), a(t, s), f(t, s), g(t, s) \in C_{rd}(\Omega, \mathbb{R}^+) \). Further, \( n : \mathbb{R}^+ \to \mathbb{R}^+ \) is a differentiable increasing function on \([0, +\infty[\) with continuous nonincreasing first derivative \( n' \) on \([0, +\infty[\). If
\[
u(t, s) \leq a(t, s) + \int_\alpha^t \int_\gamma^s f(\tau, \eta)u(\tau, \eta)\Delta \eta \Delta \tau + \int_\alpha^\beta \int_\gamma^\theta g(\tau, \eta)n(u(\tau, \eta))\Delta \eta \Delta \tau,
\] (3.11)
then
\[
u(t, s) \leq A^*(t, s) + \frac{C^*(t, s)\int_\alpha^\beta \int_\gamma^\theta g(\tau, \eta)n(a(\tau, \eta))\Delta \eta \Delta \tau}{1 - \int_\alpha^\beta \int_\gamma^\theta g(\tau, \eta)C^*(\tau, \eta)n'(a(\tau, \eta))\Delta \eta \Delta \tau},
\] (3.12)
where
\[
A^*(t, s) = a(t, s) + \int_\alpha^t e_{A(\cdot, s)}(t, \sigma(t))B(\tau, s)\Delta \tau,
\] (3.13)
\[
C^*(t, s) = e_{A(\cdot, s)}(t, \alpha),
\] (3.14)
\[
A(t, s) = \int_\gamma^s f(\tau, \eta)\Delta \eta,
\] (3.15)
\[
B(t, s) = \int_\gamma^s f(\tau, \eta)a(t, \eta)\Delta \eta.
\] (3.16)

**Proof.** Define a function \( z(t, s) \) by
\[

z(t, s) = \int_\alpha^t \int_\gamma^s f(\tau, \eta)u(\tau, \eta)\Delta \eta \Delta \tau + \int_\alpha^\beta \int_\gamma^\theta g(\tau, \eta)n(u(\tau, \eta))\Delta \eta \Delta \tau,
\] (3.17)
then
\[
u(t, s) \leq a(t, s) + z(t, s) = w(t, s).
\] (3.18)

It follows from (3.17) and (3.18) that
\[
z^\Delta(t, s) \leq \int_\gamma^s f(\tau, \eta)a(t, \eta)\Delta \eta + \int_\gamma^s f(\tau, \eta)z(\tau, \eta)\Delta \eta
\] (3.19)
\[
\leq \int_\gamma^s f(\tau, \eta)a(t, \eta)\Delta \eta + (\int_\gamma^s f(\tau, \eta)\Delta \eta)z(t, s).
\]
Since \( z(t, s) \) is nondecreasing in \( s \) for each fixed \( t \), then, the inequality (3.19) can be written as
\[
\begin{align*}
\Delta_t z(t, s) & \leq B(t, s) + A(t, s) z(t, s),
\end{align*}
\]
(3.20)
where \( A(t, s) \) and \( B(t, s) \) are defined by (3.15) and (3.16).

Noting that
\[
\begin{align*}
z(\alpha, s) &= \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g(\tau, \eta) n(u(\tau, \eta)) \Delta \eta \Delta \tau \\
&\leq \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g(\tau, \eta) n(a(\tau, \eta) + z(\tau, \eta)) \Delta \eta \Delta \tau.
\end{align*}
\]
(3.21)

Using Lemma 2 to (3.20), we have
\[
\begin{align*}
z(t, s) &\leq z(\alpha, s) e^{A(\cdot, s)(t, \alpha)} + \int_{\alpha}^{t} e^{A(\cdot, s)(t, \sigma(\tau))} B(\tau, s) \Delta \tau,
\end{align*}
\]
(3.22)
From (3.22) and (3.21), we get
\[
\begin{align*}
z(t, s) &\leq e^{A(\cdot, s)(t, \alpha)} \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g(\tau, \eta) n(w(\tau, \eta)) \Delta \eta \Delta \tau \\
&+ \int_{\alpha}^{t} e^{A(\cdot, s)(t, \sigma(\tau))} B(\tau, s) \Delta \tau,
\end{align*}
\]
(3.23)
The inequality (3.23) can be reformulated as
\[
\begin{align*}
w(t, s) &\leq A^{*}(t, s) + C^{*}(t, s) \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g(\tau, \eta) n(w(\tau, \eta)) \Delta \eta \Delta \tau,
\end{align*}
\]
(3.24)
where \( A^{*}(t, s), C^{*}(t, s) \) are defined as in (3.13) and (3.14).

Using Lemma 3 to (3.24) and from (3.18), we obtain the desired inequality in (3.12).

**Corollary 5.** Assume that the hypotheses of Theorem 4 hold. Then
\[
\begin{align*}
u(t, s) &\leq a(t, s) + \int_{\alpha}^{t} \int_{\gamma}^{s} f(\tau, \eta) u(\tau, \eta) \Delta \eta \Delta \tau + \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g(\tau, \eta) \arctan(u(\tau, \eta)) \Delta \eta \Delta \tau,
\end{align*}
\]
then
\[
\begin{align*}
u(t, s) &\leq A^{*}(t, s) + \frac{C^{*}(t, s) \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g(\tau, \eta) \arctan(a(\tau, \eta)) \Delta \eta \Delta \tau}{1 - \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g(\tau, \eta) \frac{C^{*}(\tau, \eta)}{1 + a^2(\tau, \eta)} \Delta \eta \Delta \tau},
\end{align*}
\]
where \( A^{*}(t, s), C^{*}(t, s) \) are defined as in Theorem 4, provided that
\[
\int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g(\tau, \eta) \frac{C^{*}(\tau, \eta)}{1 + a^2(\tau, \eta)} \Delta \eta \Delta \tau < 1.
\]
Corollary 6. Assume that the hypotheses of Theorem 4 hold. Then
\[ u(t, s) \leq a(t, s) + \int_a^{\gamma} \int_{\gamma}^t f(\tau, \eta) u(\tau, \eta) \Delta\eta\Delta\tau + \int_{\alpha}^{\beta} \int_{\gamma}^\theta g(\tau, \eta) \ln(u(\tau, \eta) + 1) \Delta\eta\Delta\tau, \]
implies
\[ u(t, s) \leq A^*(t, s) + C^*(t, s) \int_{\alpha}^{\beta} \int_{\gamma}^\theta g(\tau, \eta) \ln(a(\tau, \eta) + 1) \Delta\eta\Delta\tau \]
where \( A^*(t, s), C^*(t, s) \) are defined as in Theorem 4, provided that
\[ \int_{\alpha}^{\beta} \int_{\gamma}^\theta g(\tau, \eta) C^*(\tau, \eta) \Delta\eta\Delta\tau < 1. \]

Theorem 7. Let \( u(t, s), a(t, s), b(t, s), c(t, s), f(t, s), g(t, s), h(t, s) \in CC_{rd}(\Omega, \mathbb{R}_+) \) and \( b(t, s) + c(t, s) \) is decreasing, \( p, q, r \in \mathbb{R}_+ \) such that \( p \geq q > 0, p \geq r > 0, p \geq 1. \) Let \( n: \mathbb{R}_+ \to \mathbb{R}_+ \) is a differentiable increasing function \( n' \) on \([0, +\infty[\) with continuous nonincreasing first derivative \( n' \) on \([0, +\infty[\). If \( u(t, s) \) satisfies
\[ u^p(t, s) \leq a(t, s) + b(t, s) \int_a^{\gamma} \int_{\gamma}^t \bigl[ f(\tau, \eta) u^q(\tau, \eta) + g(\tau, \eta) u(\tau, \eta) \bigr] \Delta\eta\Delta\tau \]
\[ + c(t, s) \int_{\alpha}^{\beta} \int_{\gamma}^\theta h(\tau, \eta) n( u^r(\tau, \eta) ) \Delta\eta\Delta\tau. \]  

Then
\[ u(t, s) \leq \left[ A_1^*(t, s) + C_1^*(t, s) \int_{\alpha}^{\beta} \int_{\gamma}^\theta g(\tau, \eta) n(a(\tau, \eta)) \Delta\eta\Delta\tau \right]^{\frac{1}{p}}, \]
holds for \((t, s) \in [\alpha, \beta]_{\tau_1} \times [\gamma, \theta]_{\tau_2}, \) where
\[ A_1^*(t, s) = a_1(t, s) + \int_{\alpha}^t e_{A_1(., s)}(t, \sigma(\tau)) B_1(\tau, s) \Delta\tau, \]
\[ C_1^*(t, s) = e_{A_1(., s)}(t, \alpha), \]
and
\[ a_1(t, s) = \frac{r}{p} K^{\frac{q-p}{p}} a(t, s) + \frac{p-r}{p} K^{\frac{q}{p}} \]
\[ + \frac{r}{p} K^{\frac{q-p}{p}} b^*(\alpha, \gamma) \int_{\alpha}^t \int_{\gamma}^s (f(\tau, \eta) \left( \frac{q}{p} K^{\frac{q-p}{p}} a(\tau, \eta) + \frac{p-q}{p} K^{\frac{q}{p}} \right) 
\[ + g(\tau, \eta) \left( \frac{1}{p} K^{\frac{1}{p}} a(\tau, \eta) + \frac{p-1}{p} K^{\frac{1}{p}} \right) \Delta\eta\Delta\tau, \]

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\[ f_1(t, s) = \left[ \frac{q}{p} K^{\frac{2}{r}} f(t, s) + \frac{1}{p} K^{\frac{1}{r}} g(t, s) \right] b^*(\alpha, \gamma), \]  
(3.30)

\[ g_1(t, s) = \frac{r}{p} K^{\frac{2}{r}} b^*(\alpha, \gamma) h(t, s), \]  
(3.31)

and

\[ A_1(t, s) = \int_{\gamma}^{s} f_1(t, \eta) \Delta \eta, \]

\[ B_1(t, s) = \int_{\gamma}^{s} f_1(t, \eta) a_1(t, \eta) \Delta \eta, \]

with

\[ \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g_1(\tau, \eta) C^*_1(\tau, \eta) n'(a_1(\tau, \eta)) \Delta \eta \Delta \tau < 1. \]

**Proof.** The inequality (3.25) can be reformulated as

\[
\begin{align*}
u^p(t, s) & \leq a(t, s) + b^*(t, s) \left( \int_{\alpha}^{t} \int_{\gamma}^{s} \left[ f(\tau, \eta) u^q(\tau, \eta) + g(\tau, \eta) u(\tau, \eta) \right] \Delta \eta \Delta \tau \\
& \quad + \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} h(\tau, \eta) n(u^r(\tau, \eta)) \Delta \eta \Delta \tau \right),
\end{align*}
\]  
(3.32)

where

\[ b^*(t, s) = b(t, s) + c(t, s), \]

\[ b^*(t, s) \leq b^*(\alpha, \gamma). \]

Define a function \( z(t, s) \) by

\[ z(t, s) = \int_{\alpha}^{t} \int_{\gamma}^{s} \left( f(\tau, \eta) u^q(\tau, \eta) + g(\tau, \eta) u(\tau, \eta) \right) \Delta \eta \Delta \tau, \]  
(3.33)

\[ + \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} h(\tau, \eta) n(u^r(\tau, \eta)) \Delta \eta \Delta \tau. \]

By lemma 1, we have

\[ u(t, s) \leq (a(t, s) + b^*(\alpha, \gamma) z(t, s))^\frac{1}{p} \leq \frac{1}{p} K^{\frac{1}{r}} (a(t, \eta) + b^*(\alpha, \gamma) z(t, \eta)) + \frac{p}{p-1} K^\frac{1}{p}, \]

\[ u^q(t, s) \leq (a(t, s) + b^*(\alpha, \gamma) z(t, s))^\frac{q}{p} \]

\[ \leq \frac{q}{p} K^{\frac{2}{r}} (a(t, s) + b^*(\alpha, \gamma) z(t, s)) \]  
(3.34)

\[
\begin{align*}
&\leq \frac{q}{p} K^{\frac{2}{r}} (a(t, s) + b^*(\alpha, \gamma) z(t, s)) + \frac{p}{p} K^\frac{q}{p} K^\frac{2}{r}.
\end{align*}
\]

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Further results of certain Bihari-Gamidov type integral inequalities

\[ u^*(t, s) \leq (a(t, s) + b^*(\alpha, \gamma)z(t, s))^{\frac{1}{\gamma}} \]
\[ \leq \frac{r}{p} \frac{z^{\frac{p}{1-r}}}{p} (a(t, s) + b^*(\alpha, \gamma)z(t, s))^{\frac{1}{p}} + \frac{p-r}{p} K^{\frac{q}{p}} = w(t, s). \]

It follows from (3.33),(3.34) and (3.35) that
\[ z(t, s) \leq \int_{\alpha}^{t} \int_{\gamma}^{s} \left[ f(\tau, \eta) \left( \frac{q}{p} K^{\frac{q-r}{p}} (a(\tau, \eta) + b^*(\alpha, \gamma)z(\tau, \eta)) + \frac{p-q}{p} K^{\frac{q}{p}} \right) 
+ g(\tau, \eta) \left( \frac{1}{p} K^{\frac{1-q}{p}} (a(\tau, \eta) + b^*(\alpha, \gamma)z(\tau, \eta)) + \frac{1-p}{p} K^{\frac{1}{p}} \right) \Delta \eta \Delta \tau 
+ \int_{\alpha}^{\beta} h(\tau, \eta) n \left[ \frac{r}{p} K^{\frac{q}{p}} (a(\tau, \eta) + b^*(\alpha, \gamma)z(\tau, \eta)) + \frac{p-r}{p} K^{\frac{q}{p}} \right] \Delta \eta \Delta \tau. \]

It gives,
\[ z(t, s) \leq \int_{\alpha}^{t} \int_{\gamma}^{s} \left[ f(\tau, \eta) \left( \frac{q}{p} K^{\frac{q-r}{p}} a(\tau, \eta) + \frac{p-q}{p} K^{\frac{q}{p}} \right) + g(\tau, \eta) \left( \frac{1}{p} K^{\frac{1-q}{p}} a(\tau, \eta) + \frac{1-p}{p} K^{\frac{1}{p}} \right) \Delta \eta \Delta \tau 
+ \int_{\alpha}^{t} \int_{\gamma}^{s} \left[ \frac{q}{p} K^{\frac{q-r}{p}} f(\tau, \eta) + \frac{1}{p} K^{\frac{1-q}{p}} g(\tau, \eta) \right] b^*(\alpha, \gamma)z(\tau, \eta) \Delta \eta \Delta \tau 
+ \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} h(\tau, \eta) n(w(\tau, \eta)) \Delta \eta \Delta \tau, \]

where \( w(t, s) \) is defined as in (3.35).

Multiplying both sides of the inequality (3.37) by \( \frac{z^{\frac{p}{1-r}}}{p} K^{\frac{q}{p}} b^*(\alpha, \gamma) \) and adding \( \frac{z^{\frac{p}{1-r}}}{p} K^{\frac{q}{p}} a(t, s) + \frac{p-r}{p} K^{\frac{q}{p}} \) to the resultant inequality, we obtain the following inequality
\[ w(t, s) \leq a_1(t, s) + \int_{\alpha}^{t} \int_{\gamma}^{s} f_1(\tau, \eta) w(\tau, \eta) \Delta \eta \Delta \tau + \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g_1(\tau, \eta) n(w(\tau, \eta)) \Delta \eta \Delta \tau, \]

(3.38)

where \( a_1(t, s) \) is defined as in (3.29) and \( f_1(t, s) \) and \( g_1(t, s) \) are defined as in (3.30) and (3.31).

Applying Theorem 4 to inequality (3.38), we obtain
\[ w(t, s) \leq A_1(t, s) + \frac{C_1^*(t, s) \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g_1(\tau, \eta) n(a_1(\tau, \eta)) \Delta \eta \Delta \tau}{1 - \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g_1(\tau, \eta) C_1^*(\tau, \eta) n(a_1(\tau, \eta)) \Delta \eta \Delta \tau}, \]

(3.39)
than from (3.35) and (3.39), we obtain the desired inequality in (3.26).  

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Theorem 8. Assume that $u(t,s), a(t,s), b(t,s), c(t,s), g(t,s) \in C_{rd}(\Omega, \mathbb{R}^+)$ and $b(t,s) + c(t,s)$ is decreasing, $p \geq 1$ is a real constant. Let $n : \mathbb{R}^+ \to \mathbb{R}^+$ be a differentiable increasing function on $[0, +\infty[$ with continuous nonincreasing first derivative $n'$ on $[0, +\infty[$. Let $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is $C_{rd}$ on $\Omega$ and continuous on $\mathbb{R}^+$ such that

$$0 \leq f(t, s, x) - f(t, s, y) \leq h(t, s, y)(x - y)$$

for $(t, s) \in \Omega$, $x \geq y \geq 0$ where $h : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is $C_{rd}$ on $\Omega$ and continuous on $\mathbb{R}^+$, then

$$u^p(t,s) \leq a(t,s) + b(t,s) \int_{\alpha}^{t} \int_{\gamma}^{s} f(\tau, \eta, u(\tau, \eta)) \Delta \eta \Delta \tau + c(t,s) \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g(\tau, \eta)n(u^r(\tau, \eta)) \Delta \eta \Delta \tau,$$

implies

$$u(t,s) \leq \left[ A^*_2(t,s) + \frac{C^*_2(t,s) \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g_2(\tau, \eta)n(a_2(\tau, \eta)) \Delta \eta \Delta \tau}{1 - \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g_2(\tau, \eta)C^*_2(\tau, \eta)n'(a_2(\tau, \eta)) \Delta \eta \Delta \tau} \right]^\frac{1}{p},$$

for $(t, s) \in [\alpha, \beta] \times [\gamma, \theta]$, provided that

$$\int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g_2(\tau, \eta)C^*_2(\tau, \eta)n'(a_2(\tau, \eta)) \Delta \eta \Delta \tau < 1,$$

where

$$A^*_2(t, s) = \int_{\alpha}^{t} \int_{\gamma}^{s} f(\tau, \eta, u(\tau, \eta)) \Delta \eta \Delta \tau,$$

$$C^*_2(t, s) = e_{A^*_2(\cdot, s)}(t, \alpha).$$

$$a_2(t, s) = \frac{r}{p} K^{\frac{r-p}{p-1}} b^*(\alpha, \gamma) \int_{\alpha}^{t} \int_{\gamma}^{s} f(\tau, \eta, \frac{1}{p} K^{\frac{r-p}{p-1}} a(\tau, \eta) + \frac{p-1}{p} K^{\frac{1}{p}}) \Delta \eta \Delta \tau + \frac{r}{p} K^{\frac{r-p}{p-1}} a(t, s) + \frac{p-r}{p} K^{\frac{1}{p}},$$

$$B^*_2(t, s) = \int_{\gamma}^{s} f_2(t, \eta) a_2(t, \eta) \Delta \eta,$$

$$A^*_2(t, s) = \int_{\gamma}^{s} f_2(t, \eta) a_2(t, \eta) \Delta \eta.$$
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\[ f_2(t, s) = h(t, s, \frac{1}{p} K^{\frac{1}{p}} a(t, s) + \frac{p-1}{p} K^{\frac{1}{p^{*}}}) \frac{1}{p} K^{\frac{1}{p^{*}}} b^*(\alpha, \gamma), \quad (3.45) \]

\[ g_2(t, s) = \frac{r}{p} K^{\frac{1}{p^{*}}} b^*(\alpha, \gamma) g(t, s). \quad (3.46) \]

**Proof.** The inequality (3.40) can be reformulated as

\[ u^p(t, s) \leq a(t, s) + b^*(t, s) \left( \int_a^t \int_{\gamma}^s f(\tau, \eta, u(\tau, \eta)) \Delta \eta \Delta \tau + \int_a^\beta \int_{\gamma}^\theta g(\tau, \eta)n(\tilde{u}^r(\tau, \eta)) \Delta \eta \Delta \tau \right). \quad (3.47) \]

Define a function \( z(t, s) \) by

\[ z(t, s) = \int_a^t \int_{\gamma}^s f(\tau, \eta, u(\tau, \eta)) \Delta \eta \Delta \tau + \int_a^\beta \int_{\gamma}^\theta g(\tau, \eta)n(\tilde{u}^r(\tau, \eta)) \Delta \eta \Delta \tau. \quad (3.48) \]

From (3.47) and (3.48), we have

\[ u(t, s) \leq (a(t, s) + b^*(t, s)z(t, s))^{\frac{1}{p}}, \quad (3.49) \]

Using Lemma 1 to (3.49), we obtain

\[ z(t, s) \leq \int_a^t \int_{\gamma}^s \left[ f(\tau, \eta, \frac{1}{p} K^{\frac{1}{p}} a(\tau, \eta) + \frac{p-1}{p} K^{\frac{1}{p^{*}}} b^*(\alpha, \gamma) z(\tau, \eta)) \right] \Delta \eta \Delta \tau + \int_a^\beta \int_{\gamma}^\theta g(\tau, \eta)n(\tilde{u}^{r^*}(\tau, \eta)) \Delta \eta \Delta \tau, \quad (3.50) \]

Noting the assumptions on \( f \), we have

\[ z(t, s) \leq \int_a^t \int_{\gamma}^s f(\tau, \eta, \frac{1}{p} K^{\frac{1}{p}} a(\tau, \eta) + \frac{p-1}{p} K^{\frac{1}{p^{*}}} b^*(\alpha, \gamma) z(\tau, \eta)) \Delta \eta \Delta \tau + \int_a^\beta \int_{\gamma}^\theta g(\tau, \eta)n(\tilde{u}^{r^*}(\tau, \eta)) \Delta \eta \Delta \tau, \quad (3.51) \]

where

\[ w(t, s) = \frac{r}{p} K^{\frac{1}{p^{*}}} (a(t, s) + b^*(\alpha, \gamma) z(t, s)) + \frac{p-1}{p} K^{\frac{1}{p^{*}}} \]

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Multiplying both sides of (3.51) by \( \frac{r}{p} K^{\frac{1-p}{p}} \) and adding \( \frac{r}{p} K^{\frac{1-p}{p}} a(t, s) + \frac{r}{p} K^{\frac{1-p}{p}} \) to both sides of the resultant inequality, we obtain

\[
w(t, s) \leq a_2(t, s) + \int_{\alpha}^{t} \int_{\gamma}^{s} \left[ h(\tau, \eta, \frac{1}{p} K^{\frac{1-p}{p}} a(\tau, \eta) + \frac{p-1}{p} K^{\frac{1-p}{p}} b^*(\alpha, \gamma) w(\tau, \eta) \Delta \eta \Delta \tau \right]
\]

Then the inequality (3.52) can be expressed as

\[
w(t, s) \leq a_2(t, s) + \int_{\alpha}^{t} \int_{\gamma}^{s} f_2(\tau, \eta) w(\tau, \eta) \Delta \eta \Delta \tau \]

where \( a_2(t, s) \) is defined by (3.44) and \( f_2(t, s) \) and \( g_2(t, s) \) are defined by (3.45) and (3.46).

Using Theorem 4 to inequality (3.53) and from (3.35), we obtain the desired inequality in (3.41).

4 Applications:

In this section, we present some applications of our results to study boundedness and uniqueness of solutions of a Volterra-Fredholm integral equation in two independent variables on time scales.

**Example 9.** Consider the following integral equation on time scales

\[
z^p(t, s) = a(t, s) + b(t, s) \int_{\alpha}^{t} \int_{\gamma}^{s} F(\tau, \eta, z(\tau, \eta)) \Delta \eta \Delta \tau + c(t, s) \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} G(\tau, \eta, z(\tau, \eta)) \Delta \eta \Delta \tau,
\]

for \((t, s) \in \Omega\), where \( z(t, s) \in CCrd(\Omega, \mathbb{R}), a(t, s), b(t, s), c(t, s) \in CCrd(\Omega, \mathbb{R}_+), F(t, s, z), G(t, s, z) \in CCrd(\Omega \times \mathbb{R}, \mathbb{R}). \)

**Theorem 10.** Assume that the functions \( F \) and \( G \) in (4.1) satisfy the conditions
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\[ |F(t, s, z)| \leq f(t, s) |z|^q + g(t, s) |z| \] \hspace{1cm} (4.2)

\[ |G(t, s, z)| \leq h(t, s) n(|z|^r), \]

where \( f(t, s), h(t, s) \) and \( n \) are defined as in Theorem 7, with \( n'(0) = 0 \), \( p, q, r \) are constants such that \( p \geq q > 0, p \geq r > 0, p \geq 1 \). If \( z(t, s) \) is a solution of (4.1)-(4.2), then \( z(t, s) \) satisfies

\[ z(t, s) \leq \left[ A^*_1(t, s) + \frac{C^*_1(t, s) \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g_1(\tau, \eta) n(a_1(\tau, \eta)) \Delta \eta \Delta \tau}{1 - \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} g_1(\tau, \eta) C^*_1(\tau, \eta) n'(a_1(\tau, \eta)) \Delta \eta \Delta \tau} \right]^{\frac{1}{r}}, \]

hold for \( (t, s) \in [\alpha, \beta]_T \times [\gamma, \theta]_T \), where \( A^*_1(t, s), C^*_1(t, s), g_1(t, s), a_1(t, s) \) are defined as in Theorem 7.

Proof. Assume that \( z(t, s) \) is the unique solution of (4.1), from (4.2), we have

\[ |z^p(t, s)| \leq a(t, s) + b(t, s) \int_{a}^{b} \int_{\gamma}^{\theta} f(\tau, \eta) |z(\tau, \eta)|^q + g(\tau, \eta) |z(\tau, \eta)| \Delta \eta \Delta \tau \]

\[ + c(t, s) \int_{a}^{b} \int_{\gamma}^{\theta} h(\tau, \eta) n(|z(\tau, \eta)|^r) \Delta \eta \Delta \tau. \] \hspace{1cm} (4.4)

Now by applying Theorem 7 to (4.4), we obtained the required inequality (4.3).

**Proposition 11.** Assume that the functions \( F \) and \( G \) in (4.1) satisfy the conditions

\[ |F(t, s, z) - F(t, s, \bar{z})| \leq f(t, s) |z - \bar{z}|, \] \hspace{1cm} (4.5)

\[ |G(t, s, z) - G(t, s, \bar{z})| \leq h(t, s) n(|z - \bar{z}|), \]

where \( f(t, s), h(t, s) \) and \( n \) are defined as in Theorem 7 and \( z(t, s) \) is a solution of (4.1) (in the case \( p = 1 \)). Then (4.1) has at most one solution.

Proof. Let \( z(t, s) \) and \( \bar{z}(t, s) \) be two solutions of (4.1), that is,

\[ \bar{z}(t, s) = a(t, s) + b(t, s) \int_{a}^{b} \int_{\gamma}^{\theta} F(\tau, \eta, \bar{z}) \Delta \eta \Delta \tau \]

\[ + c(t, s) \int_{a}^{b} \int_{\gamma}^{\theta} G(\tau, \eta, \bar{z}) \Delta \eta \Delta \tau. \] \hspace{1cm} (4.6)

From (4.5) and (4.6), we have

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\begin{equation}
\begin{aligned}
|z(t, s) - \overline{z}(t, s)| & \leq b(t, s) \int_{\alpha}^{t} \int_{\gamma}^{s} |F(\tau, \eta, z) - F(\tau, \eta, \overline{z})| \Delta \eta \Delta \tau \\
& + c(t, s) \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} |G(\tau, \eta, z) - G(\tau, \eta, \overline{z})| \Delta \eta \Delta \tau \\
& \leq b(t, s) \int_{\alpha}^{t} \int_{\gamma}^{s} f(\tau, \eta) |z - \overline{z}| \Delta \eta \Delta \tau \\
& + c(t, s) \int_{\alpha}^{\beta} \int_{\gamma}^{\theta} h(\tau, \eta) n(|z - \overline{z}|) \Delta \eta \Delta \tau.
\end{aligned}
\end{equation}

According to Theorem 7, we obtain that $|z(t, s) - \overline{z}(t, s)| \leq 0$, which implies $z(t, s) = \overline{z}(t, s)$ for $(t, s) \in \Omega$. □

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