Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 16 (2021), 127 – 135

FIXED POINT THEOREMS FOR SOME GENERALIZED CONTRACTIVE MAPPINGS OVER A LOCALLY CONVEX TOPOLOGICAL VECTOR SPACE

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Abstract. In this paper we prove some useful fixed point theorems and common fixed point theorems for a class of non-linear mappings acting on locally convex topological vector space with supporting examples.

1 Introduction

Rapid stride of fixed point theory owes much to Stefan Banach [1], a Polish mathematician for his historic findings of fixed point of contraction mapping over a complete metric space, in 1922. Since then enormous literatures on fixed points have grown by extension of contraction principle either by relaxing spatial structure or by inviting even discontinuous non-linear mappings to come to play their respective roles to derive fixed points of these mappings or by both. Several renowned Mathematicians like Kannan [4], Chatterjea [2], Reich [6] have generalized Banach contraction mapping in their own way and obtained various fixed point theorems.
Researchers have been trying to develope fixed point theory in a Topological Vector Spaces. In 1976, Sehgal et. al. [10] proved a fixed point theorem in a locally convex topological vector space using $\mathcal{U}$-contraction from which one can derive the famous the Banach Contraction Principle in such context. In 2016, Tang et.al. [11] had proved the fixed point theorem for (ψ, ϕ) contractive mapping in a locally convex topological vector space using Minkowski functionals.
Very recently in 2019, Roy & Saha [7] established fixed point theorem of Kannan type contractive mapping in a locally convex topological vector space without using Minkowski functional.

In this article we establish some fixed point theorems using analougue of Chatterjea type contractive mapping in a complete locally convex topological vector space.

2020 Mathematics Subject Classification: Primary 47H10; Secondary 54H25.
Keywords: fixed point; Chatterjea type contractive mapping; locally convex topological vector space.

2 Preliminaries

In this section we skip elementary definitions and properties of vector spaces together with some basic results related to Topological Vector Space (for these we may refer [3], [8], [9], [5], [7], [9]).

Definition 1. *A vector space X over a field $\mathbb{F}(\mathbb{R}$ or $\mathbb{C})$ equipped with a T_1 topology τ is said to be a Topological Vector Space(TVS) if the following conditions hold:*
(i) The map $(x, y) \mapsto x + y$, for all $x, y \in X$ is continuous, i.e., for every neighborhood W of $x + y$ there exists neighborhoods U of x and V of y such that $U + V \subset W$.
(ii) The map $(\lambda, x) \mapsto \lambda x$ for all $x \in X$ and $\lambda \in \mathbb{F}$ is continuous, i.e., for any neighborhood W of λx there exists a neighborhood of λ,say $N_\delta(\lambda)$ and a neighborhood V of x such that $\alpha V \subset W$, where $\alpha \in N_\delta(\lambda)$.

Definition 2. *A neighborhood base $\mathcal{B}$ of $\theta \in X$ of a TVS (X, τ) is called a local base, i.e., for every neighborhood V of θ there exists a member $B \in \mathcal{B}$ such that $\theta \in B \subset V$.*

Definition 3. *A TVS X is said to be locally convex if X has a local base whose numbers are all convex sets.*

There are so many locally convex topological vector spaces. Followings are two examples which are locally convex topological vector spaces without being metrizable.

Example 4. *(I)[7] Consider the space $C_c^\infty(K_r)$ of infinitely differentiable functions on $\mathbb{R}^n$ with compact support contained in K_r, where $K_r = B[\theta, r]$, closed ball centered at θ with radius $r > 0$. Then $C_c^\infty(K_r)$ is a Frechet space, where the topology τ_r is built with the family of seminorms given by: for every $p \in \mathbb{N}$, $\|f\|_p^{(r)} = \sup_{x\in K_r} |f^{(p)}(x)|$ for all $f \in C_c^\infty(K_r)$. Then from the family of topological spaces $\{(C_c^\infty(K_r), \tau_r) : r \in \mathbb{N}\}$ we have the natural LF-space structure on $C_c^\infty(\mathbb{R}^n)$, which is a complete locally convex TVS but not a Frechet space.*

(II) The space $\mathbb{R}^{[0,1]}$ of all functions from $[0, 1]$ to $\mathbb{R}$ forms a complete locally convex TVS which is not metrizable.

Definition 5. *Let X be TVS. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for every neighborhood V of $\theta \in X$ there exists a natural number n_0 such that $x_m - x_n \in V$ whenever $n, m \geq n_0$.*

Definition 6. *A sequence $\{x_n\}$ in X is said to be convergent to an element $x \in X$ if for any neighborhood V of $\theta \in X$, there exists a $n_0 \in \mathbb{N}$ such that $x_n - x \in V$ whenever $n \geq n_0$ and we can write it $x_n \to x$ as $n \to \infty$. Also we can say that x is the limit of $\{x_n\}$.*

Definition 7. *A TVS X is said to be sequentially complete if every Cauchy sequence in X is convergent to an element in X.*

Definition 8. *Let (X,τ) be a locally convex TVS and $\{T_n\}$ be a sequence of self maps on X. Then $\{T_n\}$ converges uniformly to a self map T on X if for each every neighborhood V of $\theta \in X$ there exists $n_0 \in \mathbb{N}$ such that $T_n x - Tx \in V$ whenever $n \geq n_0$, for all $x \in X$.*

Lemma 9. *[7] Let X be a locally convex TVS and $\{x_n\}$ be a sequence in X. If for any neighborhood V of $\theta \in X$ there exist $k > 0$ such that for all $n \in \mathbb{N}$, $x_{n+1} - x_n \in \mu^n kV$ for some $\mu \in (0,1)$ then $\{x_n\}$ is a Cauchy sequence in X.*

Definition 10. *[7] Let X be a locally convex topological vector space. A mapping $T : X \to X$ is said to be a Kannan-type contractive mapping if for every neighborhood V of $\theta \in X$ there exists $0 < \alpha < \frac{1}{2}$ such that $Tx - Ty - \alpha(y - Ty) \in \alpha V$ whenever $x - Tx \in V$ for all $x, y \in X$.*

Theorem 11. *[7] Let X be a complete locally convex topological vector space and let $T : X \to X$ be a Kannan-type contractive mapping with the constant α. Then T has a unique fixed point in X.*

Example 12. *[7] Take the LF-space $X = C_c^\infty(\mathbb{R}^n)$ and the map $T : X \to X$ by $Tx = -\frac{1}{2}x$ for all $x \in X$. Then T is a Kannan-type contractive mapping with the constant $\alpha = \frac{1}{3}$ and an element $f \in X$ defined by $f(t) = 0$ for all $t \in \mathbb{R}^n$, is the unique fixed point of T in X.*

3 Main Result

In this section first we define an analogue of Chatterjea type contractive mapping on a locally convex topological vector space and then we extablish some fixed point theorems on it with some supporting examples.

Definition 13. *Let X be a locally convex topological vector space. A mapping $T : X \to X$ is said to be a Chatterjea type contractive mapping if there exists a constant $\alpha \in (0, \frac{1}{2})$ such that for every neighborhood V of $\theta \in X$, $Tx - Ty - \alpha(x - Ty) \in \alpha V$, whenever $y - Tx \in V$ for all $x, y \in X$.*

Theorem 14. *Let X be a sequentially complete locally convex topological vector space and $T : X \to X$ be a Chatterjea type contractive mapping. Then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$. Construct the sequence of iteration $x_n = T(x_{n-1})$, $n = 1, 2, 3, \cdots$

Let V be a neighborhood of $\theta \in X$. As V is absorbing, there exists $\lambda > 0$ such that

$x_0 - x_1 \in \lambda V$. Now $x_1 - Tx_0 = x_1 - x_1 = \theta \in \frac{1}{\alpha}V$. So $Tx_0 - Tx_1 - \alpha(x_0 - Tx_1) \in V$. Then $Tx_0 - Tx_1 - \alpha(x_0 - Tx_1) \in \bigcap_{V \in \mathcal{N}\{\theta\}} V = \{\theta\}$, where $\mathcal{N}\{\theta\}$ is the collection of all neighborhoods of θ. Hence $(1-\alpha)(x_1 - x_2) - \alpha(x_0 - x_1) = \theta$, that is, $x_1 - x_2 \in \frac{\alpha}{1-\alpha}\lambda V$. In a similar way we can get $x_2 - x_3 \in \left(\frac{\alpha}{1-\alpha}\right)^2 \lambda V$.
Proceeding in this way, at the n-th stage we get, $x_n - x_{n+1} \in \left(\frac{\alpha}{1-\alpha}\right)^n \lambda V$.
Since $0 < \alpha < 1/2$, we have $\frac{\alpha}{1-\alpha} < 1$ and so by Lemma 9, $\{x_n\}$ is a Cauchy sequence in X.
Since X is sequentially complete, so $\{x_n\}$ is convergent and let $\lim_{n\to\infty} x_n = u \in X$. So for every neighborhood U of θ, there exists a positive integer N such that $x_{n+1} - u \in U$, whenever $n \geq N$. Thus $u - Tx_n \in U$, whenever $n \geq N$. Hence by hypothesis, $Tx_n - Tu - \alpha(x_n - Tu) \in \alpha U$, whenever $n \geq N$. i.e. $x_{n+1} - Tu - \alpha x_n + \alpha Tu \in \alpha U$, whenever $n \geq N$.
Letting $n \to \infty$ we get, $u - Tu - \alpha u + \alpha Tu \in \alpha U$, which in turn implies that $u - Tu \in \frac{\alpha}{1-\alpha}U$. Since U is an arbitrary neighborhood of θ, we get $u = Tu$, implies that is u is a fixed point of T.
For uniqueness let $v \in X$ be another fixed point of T. i.e., $Tv = v$.
Let W be a neighborhood of $\theta \in X$. Since W is absorbing there exist $\gamma > 0$ such that $u - v \in \gamma W$. Then by hypothesis, $Tu - Tv - \alpha(u - Tv) \in \alpha\gamma W$, that is $Tu - Tv \in \frac{\alpha}{1-\alpha}\gamma W$, which implies $u - v \in \frac{\alpha}{1-\alpha}\gamma W$. Applying same argument we can get $u - v \in \left(\frac{\alpha}{1-\alpha}\right)^2 \gamma W$. Continuing in this way at the n-th stage we get $u - v \in \left(\frac{\alpha}{1-\alpha}\right)^n \gamma W$. Since $\frac{\alpha}{1-\alpha)} < 1$ so letting $n \to \infty$ we get $u = v$. Hence fixed point of T is unique.

□

Example 15. *Consider the space $X = \mathbb{R}^{[0,1]}$. Then X is a sequentially complete locally convex topological vector space which is not metrizable. Fix $F \in X$. Define $T_F : X \to X$ by $T_F(f) = \frac{f}{5} + F$ for all $f \in X$. Then T_F is a Chatterjea type contractive mapping with $\alpha = \frac{1}{6} < \frac{1}{2}$. T_F has a unique fixed point $\frac{5}{4}F \in X$.*

Theorem 16. *Let X be a sequentially complete locally convex topological vector space. Let $T_1, T_2 : X \to X$ be two maps satisfying*

$$T_1x - T_2y - \alpha(x - T_2y) \in \alpha V \text{ whenever } y - T_1x \in V \quad (3.1)$$

and

$$T_2x - T_1y - \alpha(x - T_1y) \in \alpha V \text{ whenever } y - T_2x \in V \quad (3.2)$$

for $0 < \alpha < \frac{1}{2}$ then T_1 and T_2 have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X by $x_{2n+1} = T_1(x_{2n})$ and $x_{2n} = T_2(x_{2n-1})$ for all $n = 0, 1, 2, \cdots$. Now, $x_0 - x_1 \in X$. Let V be a neighborhood of $\theta \in X$. Since V is absorbing there exists $\lambda > 0$ such that $x_0 - x_1 \in \lambda V$.
Now $x_1 - T_1 x_0 = x_1 - x_1 = \theta \in \frac{1}{\alpha} V$. Then by (3.1) we have, $T_1 x_0 - T_2 x_1 - \alpha(x_0 - T_2 x_1) \in V$. So $T_1 x_0 - T_2 x_1 - \alpha(x_0 - T_2 x_1) \in \bigcap_{V \in \mathcal{N}\{\theta\}} V = \{\theta\}$, where $\mathcal{N}\{\theta\}$ is the collection of all neighborhoods of $\theta \in X$. Thus $x_1 - x_2 = \frac{\alpha}{1-\alpha}(x_0 - x_1) \in \frac{\alpha}{1-\alpha} \lambda V$.
Again $x_2 - T_2 x_1 = x_2 - x_2 = \theta \in \frac{1}{\alpha} V$. Then by using equation (3.2), $T_2 x_1 - T_1 x_2 - \alpha(x_1 - T_1 x_2) \in V$. In a similar way we can show that $x_2 - x_3 \in \left(\frac{\alpha}{1-\alpha}\right)^2 \lambda V$.
Continuing in this way we get at the n-th stage we get, $x_n - x_{n+1} \in \left(\frac{\alpha}{1-\alpha}\right)^n \lambda V$.
Since $\frac{\alpha}{1-\alpha} < 1$, by Lemma 9 we have $\{x_n\}$ is a Cauchy sequence in X. As X is sequentially complete so $\{x_n\}$ is convergent in X and let $\lim_{n\to\infty} x_n = u \in X$.
Then for every neighborhood W, there exists $n \in \mathbb{N}$ such that $x_{2n+1} - u \in W$ for all $n \geq N$. Without loss of generality we can assume that W is symmetric. Thus $u - T_1 x_{2n} \in W$ for all $n \geq N$. So using equation (3.1), $T_1 x_{2n} - T_2 u - \alpha(x_{2n} - T_2 u) \in \alpha W$, for all $n \geq N$, which implies $x_{2n+1} - T_2 u - \alpha(x_{2n} - T_2 u) \in \alpha W$ for all $n \geq N$. Letting $n \to \infty$ we get, $u - T_2 u \in \frac{\alpha}{1-\alpha} W$. Since W is an arbitrary neighborhood of $\theta \in X$ we get $u = T_2 u$, showing that u is a fixed point of T_2.
In a similar way we can show that $u = T_1 u$. That is u is a fixed point of T_1.
Hence u is a common fixed point of T_1 and T_2. Uniqueness of u is also clear. □

Theorem 17. *Let X be a sequentially complete locally convex topological vector space. Let $\{T_n\}$ be a sequence of Chatterjea type contraction mappings on X with some constant $\alpha \in (0, 1/2)$ which converges uniformly to T. Then T is also Chatterjea type contractive mapping with same constant α.*

Proof. Let W be a neighborhood of $\theta \in X$. Again let V be a neighborhood of $\theta \in X$ such that for some $x, y \in X$, $y - Tx \in V$. Then there exists a closed, convex, balanced and absorbing neighborhood U of $\theta \in X$ such that $y - Tx \in U \subset V$. Since the sequence $\{T_n\}$ converges uniformly to T so for all $x \in X$ and for each $k \in \mathbb{N}$, we have $Tx - T_n x \in \frac{1}{k} U$ whenever $n \geq m_k$, where $\{m_k\}_k$ is a strictly increasing sequence of $\mathbb{N}$. Then if $n \geq m_k$ we have, $y - T_n x = (y - Tx) + (Tx - T_n x) \in U + \frac{1}{k} U = (1 + \frac{1}{k}) U$ for all $k \geq 1$.
In particular, $y - T_{m_k} x \in (1 + \frac{1}{k}) U$ for all $k \geq 1$. Since T_{m_k} is Chatterjea type mapping, so $T_{m_k} x - T_{m_k} y - \alpha(x - T_{m_k} y) \in \alpha(1 + \frac{1}{k}) U$ for all $k \geq 1$.
Again since $T_n \to T$ uniformly so there exists $N \in \mathbb{N}$ such that for all $x \in X$, $T_{m_k} x - Tx \in \frac{1}{1-\alpha} V$ whenever $k \geq N$. Now for all $k \geq N$,

$$
\begin{aligned}
&Tx - Ty - \alpha(x - Ty)\\
&= (Tx - T_{m_k}x) + \{T_{m_k}x - T_{m_k}y - \alpha(x - T_{m_k}y)\} + (1-\alpha)(T_{m_k}y - Ty)\\
&\in \alpha\left(1 + \frac{1}{k}\right)U + W
\end{aligned}
$$

Letting $k \to \infty$ we have $Tx - Ty - \alpha(x - Ty) \in \alpha U + W$. Since W is arbitrary neighborhood of θ we have $Tx - Ty - \alpha(x - Ty) \in \alpha U \subset \alpha V$. Therefore T is a Chatterjea type contraction mapping with the constant α. □

Example 18. *Let $X = \mathbb{R}^{[0,1]}$. Then X is sequentially complete locally convex topological vector space which is not metrizable. Fix $F \in X$. Define $T_n^{(F)} : X \to X$ by $T_n^{(F)}(f) = \frac{f}{5} + \frac{F}{2^n}$ for all $f \in X$, for all $n = 1, 2, \cdots$. Then $\{T_n^{(F)}\}_n$ is a sequence of Chatterjea type contractive mappings with $\alpha = \frac{1}{6}$. Here the sequence $\{T_n^{(F)}\}_n$ converges uniformly to $T^F(f) = \frac{f}{5}$. Also $T^F(f)$ is a Chatterjea type map with the same constant $\alpha = \frac{1}{6}$.*

Theorem 19. *Let X be a sequentially complete locally convex topological vector space and $T : X \to X$ satisfies $Tx - Ty - \alpha(x - Ty) - \beta(x - y) \in \alpha V$, whenever $y - Tx \in V$ for all $x, y \in X$ and $0 \le \alpha, \beta < 1$ with $2\alpha + \beta < 1$. Then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$. Construct the iteration $x_n = Tx_{n-1}$ for all $n = 1, 2, \cdots$. Let V be an arbitrary neighbourhood of $\theta \in X$. Since $x_0 - x_1 \in X$ so there exists a $\lambda > 0$ such that $x_0 - x_1 \in \lambda V$, as without loss of generality V is assumed to be absorbing. Now $x_1 - Tx_0 = x_1 - x_1 = \theta \in \frac{1}{\alpha}W$ for every neighborhood W of $\theta \in X$. Then by hypothesis we have, $Tx_0 - Tx_1 - \alpha(x_0 - Tx_1) - \beta(x_0 - x_1) \in W$. Hence $x_1 - x_2 - \alpha(x_0 - x_2) - \beta(x_0 - x_1) \in W$, which in turn implies that $(1-\alpha)(x_1 - x_2) - (\alpha+\beta)(x_0 - x_1) \in W$ for every neighborhood W of $\theta \in X$.

Then $(1 - \alpha)(x_1 - x_2) - (\alpha + \beta)(x_0 - x_1) \in \bigcap_{W \in \mathcal{N}(\theta)} W = \{\theta\}$, implying that $x_1 - x_2 \in \left(\frac{\alpha+\beta}{1-\alpha}\right)(x_0 - x_1)$. Thus, $x_1 - x_2 \in \lambda\left(\frac{\alpha+\beta}{1-\alpha}\right)V$.

Similarly we can get $x_2 - x_3 \in \left(\frac{\alpha+\beta}{1-\alpha}\right)(x_1 - x_2) \in \lambda\left(\frac{\alpha+\beta}{1-\alpha}\right)^2 V$.

Proceeding in this way we get at the n-th stage $x_n - x_{n+1} \in \lambda\left(\frac{\alpha+\beta}{1-\alpha}\right)^n V$.

Since $2\alpha + \beta < 1$ we have $\frac{\alpha+\beta}{1-\alpha} < 1$ and so by Lemma 9, $\{x_n\}$ is a Cauchy sequence in X. As X is sequentially complete so $\{x_n\}$ is convergent in X and let $\lim_{n\to\infty} x_n = u \in X$.

Since $\lim_{n\to\infty} x_n = u$ so for every neighborhood V of $\theta \in X$ there exists $N \in \mathbb{N}$ such that $u - x_{n+1} \in \frac{\alpha+\beta}{\alpha}V$ for all $n \ge N$. That is for all $n \ge N$ $u - Tx_n \in \frac{\alpha+\beta}{\alpha}V$.

Thus for $n \ge N$, $Tx_n - Tu - \alpha(x_n - Tu) - \beta(x_n - u) \in (\alpha+\beta)V$, which implies $x_{n+1} - Tu - \alpha(x_n - Tu) - \beta(x_n - u) \in (\alpha+\beta)V$. Letting $n \to \infty$ we have, $u - Tu \in \frac{\alpha+\beta}{1-\alpha}V \subset V$. Since V is arbitrary we get $u = Tu$, showing that $u \in X$ is a

fixed point of T.
For uniqueness, suppose that $v \in X$ be another fixed point of T. That is $Tv = v$. Let W be a neighborhood of $\theta \in X$. Since W is absorbing, so there exists $\gamma > 0$ such that $u - v \in \gamma W$. *i.e.* $u - Tv \in \gamma W$. Then by hypothesis, $Tv - Tu - \alpha(v - Tu) - \beta(v - u) \in \alpha\gamma W$, which implies $v - u \in \frac{\alpha}{1-\alpha-\beta}\gamma W$. By using same argument we can have $v - u \in \left(\frac{\alpha}{1-\alpha-\beta}\right)^2 \gamma W$.
Proceeding in this way at the n-th step we get, $v - u \in \left(\frac{\alpha}{1-\alpha-\beta}\right)^n \gamma W$ for all $n = 1, 2, \cdots$. Since $2\alpha + \beta < 1$ so $\frac{\alpha}{1-\alpha-\beta} < 1$. Letting $n \to \infty$ we have $v = u$. Hence fixed point is unique. □

Example 20. *Consider $X = \mathbb{R}^{[0,1]}$, a sequentially complete locally convex topological vector space which is not metrizable. Fix $F \in X$. Define $T_F : X \to X$ by $T_F(f) = \frac{f}{2} + F$ for all $f \in X$. Then T_F satifies the condition of the above Theorem with constants $\alpha = \beta = \frac{1}{5}$. Also T_F has a unique fixed point $2F \in X$.*

Theorem 21. *Let X be a sequentially complete locally convex topological vector space. Let $T : X \to X$ be a mapping satisfying $T^p x - T^p y - \alpha(x - T^p y) \in \alpha V$ whenever $y - T^p x \in V$ where $0 \le \alpha < \frac{1}{2}$. Then T has a unique fixed point in X.*

Proof. Take $T^p = \phi$. Then $\phi(x) - \phi(y) - \alpha(x - \phi(y)) \in \alpha V$ whenever $y - \phi(x) \in V$. Then by Theorem 14, ϕ has a unique fixed point say $u \in X$. *i.e.* $T^p u = u$.
Now $Tu - u \in X$, *i.e.* $Tu - T^p u \in X$. Let W be a neighborhood of θ. Since W is absorbing so there exists $\gamma > 0$ such that $Tu - T^p u \in \gamma W$. Then by hypothesis $T^p u - T^p(Tu) - \alpha(u - T^p(Tu)) \in \alpha\gamma W$, which implies that $u - T(T^p u) - \alpha(u - T(T^p u)) \in \alpha\gamma W$ which implies, $u - Tu \in \frac{\alpha}{1-\alpha}\gamma W$.
It is a routine exercise to see that $u - Tu \in \left(\frac{\alpha}{1-\alpha}\right)^2 \gamma W$. Proceeding in this way at the n-th step we can get $u - Tu \in \left(\frac{\alpha}{1-\alpha}\right)^n \gamma W$, for all $n = 1, 2, \cdots$.
Letting $n \to \infty$ we get $u = Tu$. Consequently $u \in X$ is unique fixed point of T and uniqueness of u is trivial. □

Theorem 22. *Let X be a sequentially complete locally convex topological vector space and suppose that $T_1, T_2 : X \to X$ are two maps such that for a fixed positive integer p,*

$$T_1^p x - T_2^p y - \alpha(x - T_2^p y) \in \alpha V, \text{ whenever } y - T_1^p x \in V$$

and

$$T_2^p x - T_1^p y - \alpha(x - T_1^p y) \in \alpha V, \text{ whenever } y - T_2^p x \in V$$

for every neighborhood V of $\theta \in X$ and for $0 < \alpha < \frac{1}{2}$. Then T_1 and T_2 have a common fixed point in X.

Proof. Take $T_1^p = \phi_1$ and $T_2^p = \phi_2$. Then the conditions are reduced to

$$\phi_1 x - \phi_2 y - \alpha(x - \phi_2 y) \in \alpha V \text{ whenever } y - \phi_1 x \in V$$

and $$\phi_2 x - \phi_1 y - \alpha(x - \phi_1 y) \in \alpha V \text{ whenever } y - \phi_2 x \in V.$$

Then by Theorem 16, ϕ_1 and ϕ_2 have a common fixed point, say $u \in X$. That is $T_1^p u = u = T_2^p u$. Let V be a neighborhood of $\theta \in X$. Since V is absorbing so there exists $\gamma > 0$ such that $u - T_1 u \in \gamma V$, *i.e.*, $u - T_1(T_1^p u) \in \gamma V$ which in turn implies that $u - T_1^p(T_1 u) \in \gamma V$.
Then by hypothesis, $T_1^p(T_1 u) - T_2^p u - \alpha(T_1 u - T_2^p u) \in \alpha\gamma V$, *i.e.*, $T_1(T_1^p u) - T_2^p u - \alpha(T_1 u - T_2^p u) \in \alpha\gamma V$ implying that $u - T_1 u \in \frac{\alpha}{1-\alpha}\gamma V$.
Repeating this process we get, $u - T_1 u \in \left(\frac{\alpha}{1-\alpha}\right)^n \gamma V$. Letting $n \to \infty$, we have $u = T_1 u$, as $\frac{\alpha}{1-\alpha} < 1$. Similarly $u = T_2 u$. Hence $u \in X$ is a common fixed point of T_1 and T_2.

□

Acknowledgement. Sayantan Panja acknowledges financial support awarded by the Council of Scientific and Industrial Research, New Delhi, India through research fellowship for carrying out research work leading to the preparation of this manuscript.

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