FIXED POINT RESULTS FOR A CLASS OF ASYMPTOTICALLY REGULAR MAPS IN \( g \)-METRIC SPACE WITH ORDER \( n \)

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Abstract. In 2005, Lj. B. Ćirić studied fixed points of asymptotically regular mappings [Math. Commun. 10(2005), 111-114]. The results of that study were extended by several authors. The aim of this article is to extend the main results to the context of \( g \)-metric spaces with order \( n \), which was introduced by H. Choi, S. Kim, and S. Y. Yang in 2018.

1 Introduction

Let \( \mathbb{R}^+ \) be the set of nonnegative reals and let \( F_i : \mathbb{R}^+ \to \mathbb{R}^+ \) be a function such that \( F_i(0) = 0 \) and \( F_i \) is continuous at 0, \((i = 1, 2)\). In 2005, Lj. B. Ćirić [3] established the following result.

**Theorem 1.** ([3]) Let \((X, d)\) be a complete metric space and \( T \) be a self-mapping on \( X \) satisfying the following condition:

\[
d(Tx, Ty) \leq a_1 F_1[\min\{d(x, Tx), d(y, Ty)\}] + a_2 F_2[d(x, Tx) \cdot d(y, Ty)] \\
+ a_3 d(x, y) + a_4 [d(x, Tx) + d(y, Ty)] + a_5 [d(x, Ty) + d(y, Tx)] \tag{1.1}
\]

for all \( x, y \) in \( X \), where \( a_i = a_i(x, y) \) \((i = 1, 2, 3, 4, 5)\) are nonnegative functions for which there exist three constants \( K > 0 \) and \( \lambda_1, \lambda_2 \in (0, 1) \) such that the following inequalities,

\[
a_1(x, y), a_2(x, y) \leq K, \tag{1.2}
\]

\[
a_4(x, y) + a_5(x, y) \leq \lambda_1, \tag{1.3}
\]

\[
a_3(x, y) + 2a_5(x, y) \leq \lambda_2, \tag{1.4}
\]

are satisfied for all \( x, y \) in \( X \).

If \( T \) is asymptotically regular at some \( x_0 \) in \( X \), then the self-mapping \( T \) has a unique fixed point. Moreover, \( T \) is continuous at its unique fixed point.

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2 Preliminaries

In this section, we give some basic definitions and results on $g$-metric space with order $n$ [4].

For a set $X$, we denote $X^n := \prod_{i=1}^{n} X$.

**Definition 2.** Let $X$ be a nonempty set, and let the function $g : X^{n+1} \to [0, +\infty)$ satisfy the following properties:

1. $(g_1)$ $g(x_0, \ldots, x_n) = 0$ if and only if $x_0 = \ldots = x_n = 0$,
2. $(g_2)$ $g(x_0, \ldots, x_n) = g(x_{\sigma(0)}, \ldots, x_{\sigma(n)})$ for any permutation $\sigma$ on $\{0, 1, \ldots, n\}$,
3. $(g_3)$ $g(x_0, \ldots, x_n) \leq g(y_0, \ldots, y_n)$ for all $(x_0, \ldots, x_n), (y_0, \ldots, y_n) \in X^{n+1}$ with $\{x_i : i = 0, \ldots, n\} \subseteq \{y_i : i = 0, \ldots, n\}$,
4. $(g_4)$ (Triangle inequality) for all $x_0, \ldots, x_s, y_0, \ldots, y_t, w \in X$ with $s + t + 1 = n$
   
   $$g(x_0, \ldots, x_s, y_0, \ldots, y_t, w) \leq [g(x_0, \ldots, x_s, w, \ldots, w) + g(y_0, \ldots, y_t, w, \ldots, w)].$$

Then the function $g$ is called a generalized metric or more specifically, a $g$-metric with order $n$ on $X$, and the pair $(X, g)$ is called a $g$-metric space with order $n$.

The following theorem shows us that $g$-metrics generalize the notion of ordinary metric and $G$-metric introduced by Z. Mustafa and B. Sims [5].

**Theorem 3.** Let $X$ be a given nonempty set. The following are true.

(i) $d$ is a $g$-metric with order 1 on $X$ if and only if $d$ is a metric on $X$.

(ii) $d$ is a $g$-metric with order 2 on $X$ if and only if $d$ is a $G$-metric on $X$.

(iii) Define on $X \times X$ $d_g$ by

$$d_g(x, y) = g(x, y, \ldots, y) + g(y, x, \ldots, x),$$

then $d_g$ is a metric in $X$.

**Proposition 4.** Let $(X, g)$ be a $g$-metric space with order $m$. The following are true:

(i) $g(x, y, \ldots, y) \leq g(x, w, \ldots, w) + g(w, y, \ldots, y),$

(ii) $g(x, y, \ldots, y) \leq m \cdot g(y, x, \ldots, x),$

**Definition 5.** Let $(X, g)$ be a $g$-metric space. For $x_0 \in X$ and $r > 0$, the $g$-ball centered at $x_0$ with radius $r$ is

$$B_g(x_0, r) = \{y \in X : g(x_0, y, \ldots, y) < r\}.$$
Definition 6. Let \((X, g)\) be a \(g\)-metric space with order \(m\), and \((x_n)\) be a sequence of points of \(X\), a point \(x \in X\) is said to be the limit of the sequence \((x_n)\) if for all \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) such that

\[i_1, \ldots, i_m \geq N \Rightarrow g(x, x_{i_1}, \ldots, x_{i_m}) < \epsilon\]

and one can say that the sequence \((x_n)\) is \(g\)-convergent to \(x\).

Proposition 7. Let \((X, g)\) be a \(g\)-metric space with order \(m\), then the following are equivalent,

(i) \((x_n)\) is \(g\)-convergent to \(x\),

(ii) For a given \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(x_n \in B_g(x, \epsilon)\) for all \(n \geq N\).

(iii) \(g(x_n, x, \ldots, x) \to 0\), as \(n \to +\infty\), that is, for all \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(n \geq N \Rightarrow g(x_n, x, \ldots, x) < \epsilon\).

Definition 8. Let \((X, g)\) be a \(g\)-metric space with order \(m\), and \((x_n)\) be a sequence of points of \(X\). Then a sequence \((x_n)\) is called \(g\)-Cauchy if for all \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) such that

\[i_0, \ldots, i_m \geq N \Rightarrow g(x_{i_0}, \ldots, x_{i_m}) < \epsilon\]

Proposition 9. Let \((X, g)\) be a \(g\)-metric space. Then the following are equivalent.

(i) \((X, g)\) is \(g\)-Cauchy,

(ii) \(g(x_k, x_l, \ldots, x) \to 0\), as \(k, l \to +\infty\).

Definition 10. A \(g\)-metric space \((X, g)\) with order \(m\), is said to be complete if every \(g\)-Cauchy sequence in \((X, g)\) is \(g\)-convergent in \(X\).

Proposition 11. Let \((X, g)\) be a \(g\)-metric space with order \(m\). Then the following are equivalent.

(i) \((X, g)\) is complete,

(ii) \((X, d_g)\) is complete.

Definition 12. Let \((X, g)\) and \((X', g')\) be \(g\)-metric spaces with orders \(m\) and let \(T : (X, g) \to (X', g')\) be a self-mapping, then \(T\) is said to be \(g\)-continuous at a point \(a \in X\) if and only if, given \(\epsilon > 0\), there exists \(\delta > 0\) such that \(x_1, \ldots, x_n \in X\), and \(g(a, x_1, \ldots, x_n) < \delta\) implies \(g(Ta, Tx_1, \ldots, Tx_n) < \epsilon\). A self-mapping \(T\) is \(g\)-continuous at \(g\) if and only if it is \(g\)-continuous at all \(a \in X\).

Proposition 13. Let \((X, g)\) and \((X', g')\) be \(g\)-metric spaces with orders \(m\) and let \(T : (X, g) \to (X', g')\) be a self-mapping, then the following are equivalent.
(i) \( T \) is \( g \)-continuous,

(ii) For each point \( x \in X \) and for each sequence \( (x_n) \) in \( X \) \( g \)-converging to \( x \), \((T(x_n)) \) \( g \)-converges to \( T(x) \).

**Definition 14.** A self-mapping \( T \) on a \( g \)-metric space \((X, g)\) with order \( m \) is said to be \( g \)-asymptotically regular at a point \( x \) in \( X \), if

\[
\lim_{n \to +\infty} g(T^nx, T^{n+1}x, \ldots, T^{n+1}x) = 0,
\]

where \( T^nx \) denotes the \( n \)-th iterate of \( T \) at \( x \).

**Example 15.** Let \( \mathbb{R} \) be the set of all real numbers. Define \( g : \mathbb{R}^{m+1} \to \mathbb{R}^+ \) by

\[
g(x_0, \ldots, x_m) = \sum_{0 \leq i,j \leq m} |x_i - x_j|
\]

for all \( x_0, \ldots, x_m \in \mathbb{R} \). Then \((\mathbb{R}, g)\) is a \( g \)-metric space with order \( m \). Let \( T \) be a self-mapping on \( \mathbb{R} \) with \( Tx = ax \) \( |a| < 1 \). Then

\[
g(T^nx, T^{n+1}x, \ldots, T^{n+1}x) = m|a^n x - a^{n+1}x| = m|a|^n |x||a-1|,
\]

implying

\[
\lim_{n \to +\infty} g(T^nx, T^{n+1}x, \ldots, T^{n+1}x) = 0.
\]

Hence \( T \) is \( g \)-asymptotically regular at any point \( x \in \mathbb{R} \).

### 3 Main Result

The main result of this paper is the following theorem.

**Theorem 16.** Let \( F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) be a function such that \( F(t,0) = F(0,t) = 0 \) and \( F \) is continuous at \((t,0)\) and \((0,t)\) for all \( t \geq 0 \). Let \((X, g)\) a complete \( g \)-metric space with order \( m \), and \( T \) be a self-mapping on \( X \) satisfying the following condition:

\[
g(Tx, Ty, \ldots, Ty) \leq a_0 F \left( g(x, Tx, \ldots, Tx), g(y, Ty, \ldots, Ty) \right) + a_1 g(x, y, \ldots, y)
+ a_2 \left[ g(x, Tx, \ldots, Tx) + g(y, Ty, \ldots, Ty) \right]
+ a_3 \left[ g(x, Ty, \ldots, Ty) + g(y, Tx, \ldots, Tx) \right]
\]

for all \( x, y \) in \( X \), where \( a_i = a_i(x, y) \) are nonnegative functions such that \( a_i(x, y) = a_i(y, x) \) \((i = 0, 1, 2, 3)\) for which there exist three constants \( K > 0 \) and \( \lambda_1, \lambda_2 \in (0, 1) \) such that the following inequalities,

\[
a_0(x, y) \leq K,
\]

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\[ a_2(x, y) + a_3(x, y) \leq \lambda_1, \quad (3.3) \]

\[ a_1(x, y) + 2a_3(x, y) \leq \lambda_2, \quad (3.4) \]

are satisfied for all \( x, y \) in \( X \).

If \( T \) is \( g \)-asymptotically regular at some \( x_0 \) in \( X \). Then the self-mapping \( T \) has a unique fixed point. Moreover, \( T \) is continuous at its unique fixed point.

**Proof.** Let \( x_0 \) be a point of \( X \) at which \( T \) is \( g \)-asymptotically regular. Let \( (x_n) \) be a sequence defined by \( x_n = T^n x_0 \) for all \( n \in \mathbb{N} \). Then one can show that \( (x_n) \) is a Cauchy sequence, and that \( (x_n) \) converges to a fixed point. We prove the uniqueness of fixed point. Suppose that there exist two points \( u \) and \( v \) such that \( Tu = u \) and \( Tv = v \). From (3.1), with \( a_i = a_i(u, v) \),

\[
g(u, v, \ldots, v) = g(Tu, Tv, \ldots, Tv) \\
\leq a_0 F\left(g(u, Tu, \ldots, Tu), g(v, Tv, \ldots, Tv)\right) \\
+ a_1 g(u, v, \ldots, v) \\
+ a_2 \left[g(u, Tu, \ldots, Tu) + g(v, Tv, \ldots, Tv)\right] \\
+ a_3 \left[g(u, Tv, \ldots, Tv) + g(v, Tu, \ldots, Tu)\right] \\
= a_0 F\left(g(u, u, \ldots, u), g(v, v, \ldots, v)\right) \\
+ a_1 g(u, v, \ldots, v) \\
+ a_2 \left[g(u, u, \ldots, u) + \left[g(v, v, \ldots, v)\right]\right] \\
+ a_3 \left[g(u, v, \ldots, v) + g(v, u, \ldots, u)\right] \\
= a_1 g(u, v, \ldots, v) + a_3 \left[g(u, v, \ldots, v) + g(v, u, \ldots, u)\right].
\]

Similarly we have

\[
g(v, u, \ldots, u) \leq a_1 g(v, u, \ldots, u) + a_3 \left[g(v, u, \ldots, u) + g(u, v, \ldots, v)\right].
\]

Then

\[
\left[g(u, v, \ldots, v) + g(v, u, \ldots, u)\right] \\
\leq a_1 \left[g(u, v, \ldots, v) + g(v, u, \ldots, u)\right] \\
+ 2a_3 \left[g(u, v, \ldots, v) + g(v, u, \ldots, u)\right].
\]

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So,
\[(1 - \lambda_2)dg(u, v) = (1 - \lambda_2)[g(u, v, \ldots, v) + g(v, u, \ldots, u)] \leq 0,
\]
which implies \(u = v\).

For every nonnegative integer \(n\), we denote:
\[G_n = g(x_n, Tx_n, \ldots, Tx_n) \quad \text{and} \quad G'_n = g(Tx_n, x_n, \ldots, x_n).
\]

Using the triangle inequality, from (3.1) we have
\[g(x_n, x_m, \ldots, x_m) \leq g(x_n, x_{n+1}, \ldots, x_{n+1}) + g(x_{n+1}, x_{m+1}, \ldots, x_{m+1}) + g(x_{m+1}, x_m, \ldots, x_m) + G_n + g(Tx_n, Tx_m, \ldots, Tx_m) + G'_m,
\]
and
\[g(Tx_n, Tx_m, \ldots, Tx_m) \leq a_0 F\left(g(x_n, Tx_n, \ldots, Tx_n), g(x_m, Tx_m, \ldots, Tx_m)\right) + a_1 g(x_n, x_m, \ldots, x_m) + a_2 G_n + G_m + a_3 [g(x_n, Tx_m, \ldots, Tx_m) + g(x_m, Tx_n, \ldots, Tx_n)].\]

Note that by triangle inequality we have
\[g(x_n, Tx_m, \ldots, Tx_m) \leq g(x_n, x_m, \ldots, x_m) + g(x_m, Tx_m, \ldots, Tx_m)
= g(x_n, x_m, \ldots, x_m) + G_m,
\]
and
\[g(x_m, Tx_n, \ldots, Tx_n) \leq g(x_m, x_n, \ldots, x_n) + g(x_n, Tx_n, \ldots, Tx_n)
= g(x_m, x_n, \ldots, x_n) + G_n.
\]

Then (3.5) implies that
\[g(Tx_n, Tx_m, \ldots, Tx_m) \leq a_0 F\left(G_n, G_m\right) + a_1 g(x_n, x_m, \ldots, x_m) + (a_2 + a_3) [G_n + G_m] + a_3 [g(x_n, x_m, \ldots, x_m) + g(x_m, x_n, \ldots, x_n)].\]
So,
\[ g(x_n, x_m, \ldots, x_m) \leq G_n + g(Tx_n, Tx_m, \ldots, Tx_m) + G'_m \]
\[ \leq G_n + G'_m + a_0 F(G_m, G_n) + a_1 g(x_m, x_m, \ldots, x_m) \]
\[ + (a_2 + a_3) [G_n + G_m] \]
\[ + a_3 [g(x_n, x_m, \ldots, x_m) + g(x_m, x_m, \ldots, x_m)] . \]

Similarly we prove that
\[ g(x_m, x_n, \ldots, x_n) \leq G_m + g(Tx_m, Tx_n, \ldots, Tx_n) + G'_n \]
\[ \leq G_m + G'_n + a_0 F(G_m, G_n) + a_1 g(x_m, x_m, \ldots, x_m) \]
\[ + (a_2 + a_3) [G_m + G_n] \]
\[ + a_3 [g(x_m, x_n, \ldots, x_n) + g(x_n, x_m, \ldots, x_m)] . \]

Then
\[ \left[ g(x_m, x_n, \ldots, x_n) + g(x_n, x_m, \ldots, x_m) \right] \leq \left( G'_m + G'_n \right) \]
\[ + a_0 \left[ F(G_m, G_n) + F(G_n, G_m) \right] + (1 + 2a_2 + 2a_3) [G_m + G_n] \]
\[ + (a_1 + 2a_3) [g(x_m, x_n, \ldots, x_n) + g(x_n, x_m, \ldots, x_m)] . \]

Hence, from (3.2), (3.3) and (3.4), we obtain
\[ (1 - \lambda_2) d_g(x_n, x_m) \leq (1 - \lambda_2) \left[ g(x_n, x_m, \ldots, x_m) + g(x_m, x_m, \ldots, x_m) \right] \]
\[ \leq \left( G'_m + G'_n \right) + (1 + 2\lambda_1) [G_m + G_n] \]
\[ + K \left[ F(G_m, G_n) + F(G_n, G_m) \right] . \]

Since
\[ \lim_{n \to +\infty} g(x_n, Tx_n, \ldots, Tx_n) = 0 \]
and $F$ is continuous at $(0,0)$, then by taking the limit as $m$ tends to infinity we obtain
\[ \lim_{n,m \to +\infty} (1 - \lambda_2) d_g(x_n, x_m) = 0, \]

implying
\[ \lim_{n,m \to +\infty} d_g(x_n, x_m) = 0. \]
So,

$$\lim_{n,m \to +\infty} g(x_n, x_m, \ldots, x_m) = 0,$$

which implies that \((x_n)\) is a \(g\)-Cauchy sequence. Since \((X, g)\) is complete, then \((x_n)\) is \(g\)-convergent to limit (say) \(u\) in \(X\).

Now we show that \(u\) is equal to the unique fixed point \(z\) of \(T\). We start by proving that \(T u = u\). To get a contradiction, let us suppose that \(g(u, Tu, \ldots, Tu) > 0\). Then, from (3.1) and triangle inequality we have

$$g(u, Tu, \ldots, Tu) \leq g(u, x_n, \ldots, x_n) + g(x_n, Tx_n, \ldots, Tx_n)$$

and

$$g(Tx_n, Tu, \ldots, Tu) \leq a_0 F(G_n, g(u, Tu, \ldots, Tu))$$

$$+ a_1 g(x_n, u, \ldots, u)$$

$$+ a_2 [G_n + g(u, Tu, \ldots, Tu)]$$

$$+ a_3 [g(x_n, Tu, \ldots, Tu) + g(u, Tx_n, \ldots, Tx_n)].$$

And we have

$$g(x_n, Tu, \ldots, Tu) \leq g(x_n, u, \ldots, u) + g(u, Tu, \ldots, Tu)$$

and

$$g(u, Tx_n, \ldots, Tx_n) \leq g(u, x_n, \ldots, x_n) + g(x_n, Tx_n, \ldots, Tx_n)$$

$$= g(u, x_n, \ldots, x_n) + G_n.$$  

Then

$$g(Tx_n, Tu, \ldots, Tu) \leq a_0 F(G_n, g(u, Tu, \ldots, Tu))$$

$$+ a_1 g(x_n, u, \ldots, u) + a_2 [G_n + g(u, Tu, \ldots, Tu)]$$

$$+ a_3 [g(x_n, u, \ldots, u) + g(u, x_n, \ldots, x_n)$$

$$+ G_n + g(u, Tu, \ldots, Tu))$$

$$\leq a_0 F(G_n, g(u, Tu, \ldots, Tu)) + (a_2 + a_3) G_n$$

$$+ (a_2 + a_3) g(u, Tu, \ldots, Tu)$$

$$+ (a_1 + a_3) g(x_n, u, \ldots, u) + a_3 g(u, x_n, \ldots, x_n).$$

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Therefore, from (3.2), (3.3) and (3.4), we have
\[
g(Tx_n, Tu, \ldots, Tu) \leq KF(G_n, g(u, Tu, \ldots, Tu)) + \lambda_1 G_n
\]
\[
+ \lambda_1 g(u, Tu, \ldots, Tu)
\]
\[
+ \lambda_2 g(x_n, u, \ldots, u) + \lambda_2 g(u, x_n, \ldots, x_n),
\]
then
\[
g(u, Tu, \ldots, Tu) \leq g(u, x_n, \ldots, x_n) + G_n + \lambda_1 G_n
\]
\[
+ KF(G_n, g(u, Tu, \ldots, Tu))
\]
\[
+ \lambda_1 g(u, Tu, \ldots, Tu) + \lambda_2 g(x_n, u, \ldots, u) + \lambda_2 g(u, x_n, \ldots, x_n)
\]
\[
= KF(G_n, g(u, Tu, \ldots, Tu)) + (1 + \lambda_1)G_n + \lambda_1 g(u, Tu, \ldots, Tu)
\]
\[
+ \lambda_2 g(x_n, u, \ldots, u) + (1 + \lambda_2)g(u, x_n, \ldots, x_n).
\]
Taking the limit and using the continuity of $F$ at $(0, g(u, Tu, \ldots, Tu))$, we get
\[
g(u, Tu, \ldots, Tu) \leq \lambda_1 g(u, Tu, \ldots, Tu) < g(u, Tu, \ldots, Tu),
\]
which is a contradiction. So, $g(u, Tu, \ldots, Tu) = 0$, that is $Tu = u$. By uniqueness of $z$, we must have $z = u$. We conclude that the self-mapping $T$ has a unique fixed point in $X$. To prove that $T$ is continuous at $z$, let $(u_n)$ be a sequence such that $u_n \to z = Tz$. Then from (3.1)
\[
g(z, Tu_n, \ldots, Tu_n) = g(Tz, Tu_n, \ldots, Tu_n)
\]
\[
\leq a_0 F(0, g(u_n, Tu_n, \ldots, Tu_n))
\]
\[
+ a_1 g(z, u_n, \ldots, u_n)
\]
\[
+ a_2 g(u_n, Tu_n, \ldots, Tu_n)
\]
\[
+ a_3 \left[ g(u_n, z, \ldots, z) + g(z, Tu_n, \ldots, Tu_n) \right],
\]
by using the fact that $F(t, 0) = 0$, and the triangle inequality we have
\[
g(u_n, Tu_n, \ldots, Tu_n) \leq g(u_n, z, \ldots, z) + g(z, Tu_n, \ldots, Tu_n),
\]
then
\[
g(z, Tu_n, \ldots, Tu_n) = g(Tz, Tu_n, \ldots, Tu_n)
\]
\[
\leq a_1 g(z, u_n, \ldots, u_n)
\]
\[
+ (a_2 + a_3) g(u_n, z, \ldots, z) + (a_2 + a_3) g(z, Tu_n, \ldots, Tu_n).
\]
Hence, using (3.2), (3.3) and (3.4)
\[(1 - \lambda_1)g(z, T u_n, \ldots, T u_n) \leq \lambda_1 g(u_n, z, \ldots, z) + \lambda_2 g(z, u_n, \ldots, u_n).\]

By letting \(n\) go to infinity we obtain
\[
\limsup_n g(z, T u_n, \ldots, T u_n) = 0,
\]
which implies that \(\lim_{n \to +\infty} T u_n = z\). This completes the proof. \(\square\)

4 Consequences and Applications

We have the following corollaries.

**Corollary 17.** Let \(F_i : \mathbb{R}^+ \to \mathbb{R}^+\) be a function such that \(F_i(0) = 0\) and \(F_i\) is continuous at 0 for \(i = 1, 2\).

Let \((X, g)\) a complete \(g\)-metric space with order \(m\), and \(T\) be a self-mapping on \(X\) satisfying the following condition:
\[
g(Tx, Ty, \ldots, Ty) \leq b_1 F_1 \left( \min \left[ g(x, Tx, \ldots, Tx), g(y, Ty, \ldots, Ty) \right] \right)
+ b_2 F_2 \left[ g(x, Tx, \ldots, Tx) g(y, Ty, \ldots, Ty) \right]
+ b_3 g(x, y, \ldots, y)
+ b_4 \left[ g(x, Tx, \ldots, Tx) + g(y, Ty, \ldots, Ty) \right]
+ b_5 \left[ g(x, Ty, \ldots, Ty) + g(y, Tx, \ldots, Tx) \right]
\]
for all \(x, y\) in \(X\), where \(b_i = b_i(x, y)\) \((i = 1, 2, 3, 4, 5)\) are nonnegative functions such that \(b_i(x, y) = b_i(y, x)\) for all \(x, y \in X\), which there exist three constants \(K > 0\) and \(\lambda_1, \lambda_2 \in (0, 1)\) such that the following inequalities,
\[
\begin{align*}
    b_1(x, y), b_2(x, y) & \leq K, \\
    b_4(x, y) + b_5(x, y) & \leq \lambda_1, \\
    b_3(x, y) + 2b_5(x, y) & \leq \lambda_2,
\end{align*}
\]
are satisfied for all \(x, y\) in \(X\). If \(T\) is \(g\)-asymptotically regular at some \(x_0\) in \(X\). Then the self-mapping \(T\) has a unique fixed point. Moreover, \(T\) is continuous at its unique fixed point.

**Proof.** The proof follows from Theorem 16 by considering the functions:
\[
F(s, t) := F_1(\min(s, t)) + F_2(st),
a_0(x, y) := \max(b_1(x, y), b_2(x, y)), \text{ and } a_1 := b_3, a_2 := b_4, a_3 := b_5.
\]
\(\square\)
Corollary 18. Let $(X, g)$ be a complete $g$-metric space with order $m$, and $T$ be a self-mapping on $X$ satisfying the condition, for all $x, y$ in $X$,

$$g(Tx, Ty, \ldots, Ty) \leq \alpha [g(x, Tx, \ldots, Tx) + g(y, Ty, \ldots, Ty)]$$

where $\alpha \in (0, 1)$. If $T$ is $g$-asymptotically regular at some $x_0$, then the self-mapping $T$ has a unique fixed point. Moreover, $T$ is continuous at its unique fixed point.

Proof. The proof follows from Theorem 16 by considering the function $F(s, t) := 0$ and $a_2(x, y) := \alpha$, and $a_1(x, y) := 0$ and $a_2(x, y) := a_3(x, y) := 0$.

Corollary 19. Let $(X, g)$ be a complete $g$-metric space with order $m$, and $T$ a self-mapping on $X$ satisfying the condition, for all $x, y$ in $X$,

$$g(Tx, Ty, \ldots, Ty) \leq pg(x, y, \ldots, y) + q [g(x, Tx, \ldots, Tx) + g(y, Ty, \ldots, Ty)] + r [g(x, Ty, \ldots, Ty) + g(y, Tx, \ldots, Tx)],$$

where $p, q,$ and $r$ are fixed nonnegative real numbers such that $q + r < 1$ and $p + 2r < 1$. If $T$ is $g$-asymptotically regular at some $x_0$. Then the self-mapping $T$ has a unique fixed point. Moreover, $T$ is continuous at its unique fixed point.

Proof. The proof follows from Theorem 16 by considering the function $F(s, t) := 0$ and $a_1(x, y) := p$, and $a_2(x, y) := q$ and $a_3(x, y) := r$ and $\lambda_1 = q + r$ and $\lambda_2 = p + 2r$.

Corollary 20. Let $\alpha \geq 0$, and $\beta \in (0, 1)$. Let $(X, G)$ be a $G$-complete $G$-metric space and $T$ a self-mapping on $X$ satisfying the condition,

$$G(Tx, Ty, \ldots, Ty) \leq \beta G(x, y, \ldots, y).$$

$$+ \alpha \min\{G(x, Tx, \ldots, Tx), G(y, Ty, \ldots, Ty)\} + G(x, Tx, \ldots, Tx)G(y, Ty, \ldots, Ty)$$

$$1 + G(x, y, \ldots, y)$$

for all $x, y$ in $X$. If $T$ is $g$-asymptotically regular at some $x_0$ in $X$, then the self-mapping $T$ has a unique fixed point. Moreover, $T$ is continuous at its unique fixed point.

Proof. The proof follows from Theorem 16 by considering the function $F(s, t) := \alpha [\min\{s, t\} + st]$ and $a_0(x, y) := \frac{1}{1 + G(x, y, \ldots, y)}$, $a_1(x, y) := \beta$, and $a_2(x, y) := 0$ and $a_3(x, y) := 0$.

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