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THE HILBERT TRANSFORM

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Abstract. The Hilbert transform is essentially the only singular operator in one dimension. This undoubtedly make it one of the most important linear operator in harmonic analysis. This is an expository paper about the Hilbert transform aimed to anyone that has even scratched the surface of the theory of integration, and functional analysis as well as a basic rudiments of Fourier transform. We provide a systematic (Although by no means complete) account of the basic results on the Hilbert transform. We want to point out that we present a friendly proof of the remarkable result due to Stein and Weiss (1959) (see [24]) and we use it combined with the Cavalieri principle to obtain an exact formula for the L_p-norm of $H(\chi_E)$.

1 Introduction

The Hilbert transform of a sufficiently well-behaved function $f(x)$ is defined to be

$$\begin{aligned} Hf(x) &= P.V. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy \\ &= \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{|x-y|>\epsilon} \frac{f(y)}{x-y} dy \end{aligned} \tag{1}$$

It is not immediately not clear that $Hf(x)$ is well-defined even for nice functions [4]. Though (1) "almost" looks like an ordinary convolution, there are however certain technical subtleties associated with the definition. The primitive idea behind the definition of the transform is quite simple, namely to transform $f(x)$ by convolving with the kernel $\frac{1}{\pi x}$. It is in doing so rigorously that one encounters technical difficulties the kernel fails to be absolutely integrable owing to its slow decay and more importantly, due to the singularity at the origin. As we saw in the example on subsection 1.The limiting argument in (1) is used to avoid the singularity by truncating the kernel around the origin in a systematic fashion. As will be shown shortly, this indeed works for sufficiently regular functions. The other pathology,

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namely the slow decay of the kernel, can be circumvented relatively easily, simple by restricting the domain of (1)

to functions having a sufficiently fast decay.

From a historical point of view, the Hilbert transform originated in the work of David Hilbert on integral equation and boundary value in 1905 (see [18]).

The Hilbert transform was a motivating example for Antoni Zygmund and Alberto Calderón, during their study of singular integrals (see [7]) since then theory has expanded widely and found many applications in both mathematics and physics.

The topic discussed in this presentation does not follow any specifical source. However, we strongly recommend the reader to see the monumental work done by King F.W in [17] and the reference therein.

Since the information about the Hilbert transform is often scattered in books about signal processing. Their authors frequently use mathematical formulas without explaining them thoroughly to the reader, in what we will try to fill those gaps, in that sense we start by reviewing the notion of the Cauchy principal value needed in the definition of the Hilbert transform. After this we proceed to exam some basic properties of the Hilbert transform, most of which will be proven in detail. We also calculate the Hilbert transform of some functions to get acquainted with its use.

We treat the Hilbert transform as an operator to perform functional analysis technique, among other things, we prove that this operator is a one to one bounded and unitary operator in $L_2(\mathbb{R})$, more over, we prove that the operator H is onto and so has inverse namely $H^{-1} = -H$ and precisely its adjoint H^* is given by $H^* = H^{-1} = -H$. We end this paper by calculating the exact L_p-norm of $H(\chi_E)$.

2 The Fourier transform in $L_1(\mathbb{R})$

We consider real or complex-value functions f defined on $\mathbb{R}$. The Fourier transform of $f(x)$ is the function $\mathcal{F}f(\xi)$ or $\widehat{f}(\xi)$ whichever is more convenient in context is defined by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\, dx. \tag{2}$$

The inverse Fourier transform of a function $g(\xi)$ is

$$\mathcal{F}^{-1}g(x) = g^{v}(x) = \int_{\mathbb{R}} g(\xi)e^{2\pi i x\xi}\, d\xi.$$

The Fourier transform, or the inverse transform of a real-value function is (in general) complex valued.

Throughout this text we will use the following properties of the Fourier transform

1. $\widehat{f'}(\xi) = (2\pi i\xi)\widehat{f}(\xi)$

2. If $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ then $||\widehat{f}||_2 = ||f||_2$ (Plancharel).

   Plancharel allow us to extend (2) to $L_2(\mathbb{R})$ as a unique linear operator.

3. $\int_{\mathbb{R}} f(x)\overline{g(x)}\ dx = \int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\ d\xi$ (Parseval).

3 The Cauchy principal value

Let us consider a real-valued function f and its integral over an interval $[a, b]$

$$\int_a^b f(x)\ dx.$$

Suppose that for some $x_0 \in [a, b]$, f is unbounded i.e

$$\lim_{x \to x_0} \sup |f(x)| = \infty.$$

Then one usually considers the integral as

$$\int_a^b f(x)\ dx = \lim_{\epsilon_1 \to 0} \int_a^{x_0 - \epsilon_1} f(x)\ dx + \lim_{\epsilon_2 \to 0} \int_{x_0+\epsilon_2}^b f(x)\ dx$$

where the limits are taken independently of each other. These limits may however not exist. Another possibility is to look at a symmetric limit, called the Cauchy principal value of the integral

$$\lim_{\epsilon \to 0} \left[\int_a^{x_0-\epsilon} f(x)\ dx + \int_{x_0+\epsilon}^b f(x)\ dx \right] = P.V. \int_a^b f(x)\ dx$$

which may exists even if the individual limits do not. This is due to the fact that using a single parameter in the limit allows cancellations between the integrals. The Cauchy principal value of an improper integral

$$\int_{-\infty}^{\infty} f(x)\ dx$$

is defined in a similar fashion

$$P.V. \int_{-\infty}^{\infty} f(x)\ dx = \lim_{a \to \infty} \int_{-a}^{a} f(x)\ dx.$$

It should be noted that even both the Cauchy principal value and the non-symmetric limit of an integral exist they do not need to yield the same result (see [1]).

The Cauchy principal value is a useful tool enables one to extract finite and meaningful quantities otherwise ill-defined expression.

Now, let us consider for example the integral

$$\int_0^6 \frac{dx}{5-x}$$

where the integral has a singularity at $x = 5$. The standard non-symmetric limiting procedure result in

$$\begin{aligned}\int_0^6 \frac{dx}{5-x} &= \lim_{\epsilon_1\to 0}\int_0^{5-\epsilon_1}\frac{dx}{5-x} + \lim_{\epsilon_2\to}\int_{5+\epsilon_2}^6 \frac{dx}{5-x}\\ &= -\lim_{\epsilon_1\to 0}\left[\ln\left(\frac{\epsilon_1}{5}\right)\right] - \lim_{\epsilon_2\to 0}\left[\ln\left(\frac{1}{\epsilon_2}\right)\right],\end{aligned}$$

an expression formally of the form $\infty - \infty$ and hence not defined. The Cauchy principal value of the integral gives on the other hand the neat result

$$\begin{aligned}P.V.\int_0^6 \frac{dx}{5-x} &= \lim_{\epsilon\to 0}\left(\int_0^{5-\epsilon}\frac{dx}{5-x} + \int_{5+\epsilon}^6 \frac{dx}{5-x}\right)\\ &= -\lim_{\epsilon\to 0}\left[\ln\left(\frac{\epsilon}{5}\right) + \ln\left(\frac{1}{\epsilon}\right)\right]\\ &= -\lim_{\epsilon\to 0}\ln\left(\frac{1}{5}\right)\\ &= \ln 5.\end{aligned}$$

The symmetry of the limit is indeed important for the consistency of the results, if the upper limit of the first integral in the above calculation would have been $5 - 2\epsilon$ while the other integral still had $5 + \epsilon$ as its lower limits, the result would have been $\ln(2/5)$, which has really non-meaning at all.

For further reading on the Cauchy integral, see ([22], [15],[19],[3],[10]).

4 The class of Schwartz functions

We now introduce the class of Schwartz function on $\mathbb{R}$. Roughly speaking, a function is Schwartz if it is smooth and all of its derivative decay faster than the reciprocal of any polynomial at infinity. More precisely, we give the following definition.

Definition 4.1. A C^∞ complex-value function f on $\mathbb{R}$ is called a Schwartz function if for every pair $\alpha, \beta \in \mathbb{N}$ there exists a positive constant $C_{\alpha,\beta}$ such that

$$\rho_{(\alpha,\beta)}(f) = \sup_{x\in\mathbb{R}} |x^\alpha \partial^\beta f(x)| = C_{\alpha,\beta} < \infty.$$

The quantities $\rho_{\alpha\beta}(f)$ are called the Schwartz seminorms of f. The set of all Schwartz function on $\mathbb{R}$ is denoted by $S(\mathbb{R})$.

In here, why do we use Schwartz class instead of L_2 class. Here goes the answer. Let $f \in S(\mathbb{R})$, then

$$\begin{aligned} ||f||_2^2 &= \int_{\mathbb{R}} |f(x)|^2 \, dx \\ &= \int_{|x|\leq 1} |f(x)|^2 \, dx + \int_{|x|>1} |f(x)|^2 \, dx \\ &= 2\left(\sup_{x\in\mathbb{R}} |f(x)|\right)^2 + \sup_{x\in\mathbb{R}} \left||x|^{2N} f(x)\right| \int_{|x|>1} \frac{dx}{|x|^{2N}} \\ &< \infty, \end{aligned}$$

since

$$\int_{|x|>1} \frac{dx}{|x|^{2N}} = \int_1^\infty \frac{dr}{r^{2N}} = \left.\frac{r^{1-2N}}{1-2N}\right|_1^\infty < \infty$$

if

$$1 - 2N < 0 \Rightarrow 1 < 2N \Rightarrow N > \frac{1}{2}.$$

In this way we have proof that $f \in L_2(\mathbb{R})$ and so

$$S(\mathbb{R}) \subset L_2(\mathbb{R}).$$

For further results reading on the Schwartz class see [5] and [12].

5 The Hilbert Transform

The Hilbert transform of a sufficiently well-behaved function $f(x)$ is defined to be

$$\begin{aligned} Hf(x) &= P.V. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy \\ &= \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{|x-y|>\epsilon} \frac{f(y)}{x-y} dy \\ &= -\lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{|t|>\epsilon} \frac{f(x-t)}{t} dt. \end{aligned} \tag{3}$$

Now, we will check that (3) is well defined. In order to do that, let us take f belonging to the Schwartz class $S = S(\mathbb{R})$. Then

$$Hf(x) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{|x-t|>\epsilon} \frac{f(t)}{x-t} dt$$

$$
\begin{aligned}
&= \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int\limits_{\epsilon<|x-t|<1} \frac{f(t)}{x-t} dt + \frac{1}{\pi} \int\limits_{|x-t|>1} \frac{f(t)}{x-t} dt \\
&= \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int\limits_{\epsilon<|x-t|<1} \left(\frac{f(t)-f(x)}{x-t} + \frac{f(x)}{x-t} \right) dt + \frac{1}{\pi} \int\limits_{|x-t|>1} \frac{f(t)}{x-t} dt \\
&= \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int\limits_{\epsilon<|x-t|<1} \frac{f(t)-f(x)}{x-t} dt + \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int\limits_{\epsilon<|x-t|<1} \frac{f(x)}{x-t} dt + \frac{1}{\pi} \int\limits_{|x-t|>1} \frac{f(t)}{x-t} dt \\
&= -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \int\limits_{\epsilon<|x-t|<1} \frac{f(t)-f(x)}{t-x} dt + \frac{1}{\pi} \int\limits_{|x-t|>1} \frac{f(t)}{x-t} dt
\end{aligned}
$$

where we used the fact that

$$
\int\limits_{\epsilon<|x-t|<1} \frac{dt}{x-t} = \int\limits_{\epsilon<|y|<1} \frac{dy}{y} = 0
$$

because $\frac{1}{y}$ is an odd function.

Then

$$
\begin{aligned}
|Hf(x)| &\leq \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int\limits_{\epsilon<|x-t|<1} \left| \frac{f(t)-g(x)}{t-x} \right| dt + \frac{1}{\pi} \int\limits_{|x-t|>1} \frac{|f(t)|}{|x-t|} dt \\
&\leq \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int\limits_{\epsilon<|x-t|<1} \left| \frac{f(t)-f(x)}{t-x} \right| dt + \frac{1}{\pi} \int\limits_{|x-t|>1} \frac{|x-t||f(t)|}{|x-t|^2} dt.
\end{aligned}
$$

Now, by the mean valued theorem there exists θ_t, between t and x such that

$$
\left| \frac{f(t)-f(x)}{t-x} \right| = |f'(\theta_t)| \leq ||f'||_\infty. \tag{4}
$$

Thus

$$
\begin{aligned}
|Hf(x)| &\leq \frac{1}{\pi} \int\limits_{|x-t|<1} ||f'||_\infty dt + \frac{1}{\pi} \sup_{x \in \mathbb{R}} |x||f(x)| \\
&\leq \frac{2}{\pi} ||f'||_\infty + \frac{1}{\pi} \rho_{(1,0)}(f) \\
&\leq \frac{2}{\pi} \left(||f'||_\infty + \frac{1}{\pi} \rho_{(1,0)}(f) \right).
\end{aligned}
$$

Thus $Hf(x)$ is indeed well-defined for all x.

Also, observe that

$$\begin{aligned}
Hf(x) &= P.V.\frac{1}{\pi}\int_{\infty}^{\infty}\frac{f(y)}{x-y}dy \\
&= \lim_{\epsilon\to 0^+}\frac{1}{\pi}\int_{|x-y|>\epsilon}\frac{f(y)}{x-y}dy \\
&= -\lim_{\epsilon\to 0^+}\frac{1}{\pi}\int_{|t|>\epsilon}\frac{f(x-t)}{t}dt \\
&= -\lim_{\epsilon\to 0^+}\frac{1}{\pi}\int_{\{t>\epsilon\}\cup\{t<-\epsilon\}}\frac{f(x-t)}{t}dt \\
&= -\lim_{\epsilon\to 0^+}\frac{1}{\pi}\left[\int_{\{t>\epsilon\}}\frac{f(x-t)}{t}dt+\int_{\{t<-\epsilon\}}\frac{f(x-t)}{t}dt\right] \\
&= -\frac{1}{\pi}\lim_{\epsilon\to 0^+}\left[\int_{\epsilon}^{\infty}\frac{f(x+t)}{t}dt-\int_{\epsilon}^{\infty}\frac{f(x-t)}{t}dt\right] \\
&= -\frac{1}{\pi}\lim_{\epsilon\to 0^+}\int_{\epsilon}^{\infty}\frac{f(x+t)-f(x-t)}{t}dt.
\end{aligned}$$

Thus

$$Hf(x) = -\frac{1}{\pi}\lim_{\epsilon\to 0^+}\int_{\epsilon}^{\infty}\frac{f(x+t)-f(x-t)}{t}dt. \tag{5}$$

Consider the characteristic function $\chi_{[a,b]}$ of an interval $[a,b]$. Let us find out the Hilbert transform of $\chi_{[a,b]}$. In fact

$$\begin{aligned}
H\chi_{[a,b]}(x) &= \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|y|>\epsilon}\frac{\chi_{[a,b]}(x-y)}{y}dy \\
&= \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{\substack{|y|>\epsilon \\ b-x\leq y\leq x-a}}\frac{dy}{y}
\end{aligned}$$

thus, we only need to consider there cases

I) $x-b>0$

II) $x-a<0$

III) $x-b<0<x-a$

For the first two cases

$$\begin{aligned} H\chi_{[a,b]}(x) &= \frac{1}{\pi}\int_{x-b}^{x-a}\frac{dy}{y} \\ &= \frac{1}{\pi}\ln\left|\frac{x-a}{x-b}\right|. \end{aligned}$$

For the third case, without lost of generality, we can assume $\epsilon < \min(|x-a|,|x-b|)$,

$$\begin{aligned} H\chi_{[a,b]}(x) &= \frac{1}{\pi}\lim_{\epsilon\to 0}\left(\int_{x-b}^{-\epsilon}\frac{dy}{y}+\int_{\epsilon}^{x-a}\frac{dy}{y}\right) \\ &= \frac{1}{\pi}\lim_{\epsilon\to 0}\left[\ln|y|\Big|_{x-b}^{-\epsilon}+\ln|y|\Big|_{\epsilon}^{x-a}\right] \\ &= \frac{1}{\pi}\lim_{\epsilon\to 0}\left[\ln|-\epsilon|-\ln|x-b|+\ln|x-a|-\ln|\epsilon|\right] \\ &= \frac{1}{\pi}\ln\left[\frac{x-a}{x-b}\right]. \end{aligned}$$

Note that $H\chi_{[a,b]}(x)$ blows up logarithmically for x near the points a and b and decays like $\frac{1}{x}$ as $x\to\pm\infty$, for $a=1$ and $b=3$ we have

$$H\chi_{[1,3]}(x) = \frac{1}{\pi}\log\left|\frac{x-1}{x-3}\right|.$$

Next, observe the following graph.

Now, observe that

$$\begin{aligned} H\chi_{[a,b]}(x) &= \frac{1}{\pi}\log\left|\frac{x-a}{x-b}\right| \\ &= \frac{1}{\pi}\log\left|1+\frac{b-a}{x-b}\right| \\ &\geq \frac{1}{\pi}\log\left|\frac{b-a}{x-b}\right| \\ &\geq \frac{1}{\pi}\log\frac{|b-a|}{|x|-|b|}\to\infty \end{aligned}$$

as $|x|\to\infty$, this tell us that $Hf(x)\notin L_1(\mathbb{R})$.

For further results on the Hilbert transform see [25] and [14].

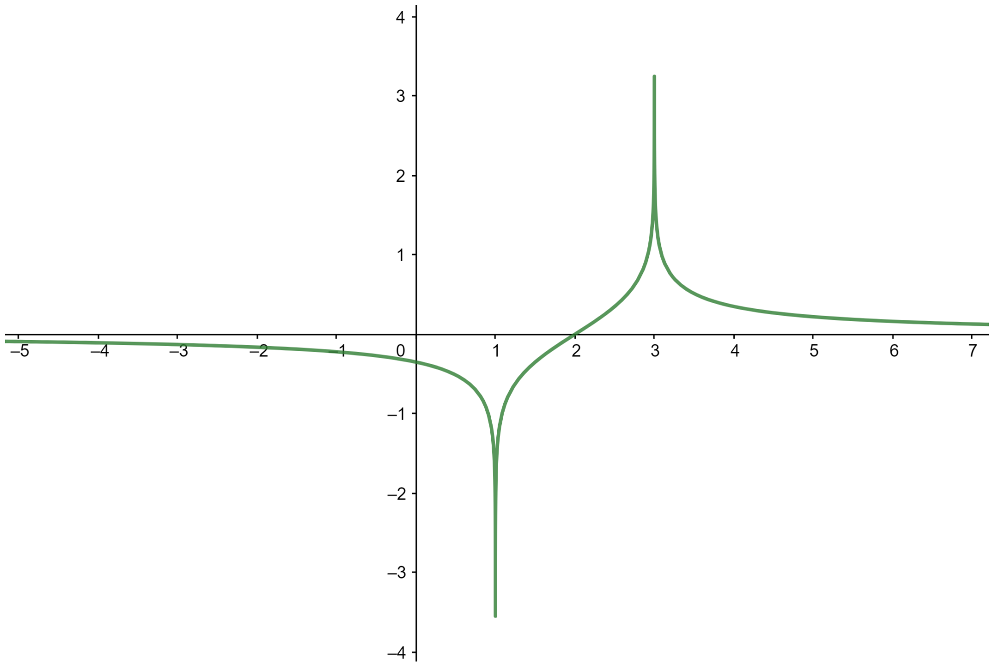

Figure 1: $H\chi_{[1,3]}(x) = \frac{1}{\pi}\log\left|\frac{x-1}{x-3}\right|$

6 Some Hilbert transform

Let us calculate the Hilbert transform of some basic functions.

6.1 A constant function

Let $f : \mathbb{R} \to \mathbb{R}$ be a constant function, $f(x) = c \in \mathbb{R}$. The Hilbert transform of f is then

$$
\begin{aligned}
Hf(x) &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy \\
&= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{c}{x-y} dy \\
&= \frac{c}{\pi} \lim_{\epsilon \to 0} \left(\int_{t-\frac{1}{\epsilon}}^{t-\epsilon} \frac{dy}{x-y} + \int_{t+\epsilon}^{t+\frac{1}{\epsilon}} \frac{dy}{x-y} \right) \\
&= -\frac{c}{\pi} \lim_{\epsilon \to 0} \left[\ln|\epsilon| - \ln\left|\frac{1}{\epsilon}\right| + \ln\left|\frac{1}{\epsilon}\right| - \ln|\epsilon| \right] \\
&= 0.
\end{aligned}
$$

6.2 The Hilbert transform of $\sin x$ and $\cos x$

Note that $\sin(\cdot)$ and $\cos(\cdot)$ belong to C^∞. Hence

$$\begin{aligned}
H(\sin x) &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\sin(y)}{x-y} dy \\
&= -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\sin(x+t)}{t} dt \\
&= -\frac{\cos x}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\sin t}{t} dt - \frac{\sin x}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\cos t}{t} dt.
\end{aligned}$$

From the odd property of $\frac{\cos t}{t}$ it follows

$$P.V. \int_{-\infty}^{\infty} \frac{\cos t}{t} dt = 0.$$

Hence

$$\begin{aligned}
H(\sin x) &= -\frac{\cos x}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\sin t}{t} dt \\
&= -\frac{\cos x}{\pi} \int_{-\infty}^{\infty} \frac{\sin t}{t} dt \\
&= -\frac{\cos x}{\pi} \cdot \pi \\
&= -\cos x
\end{aligned}$$

i.e

$$H(\sin x) = -\cos x.$$

Now, let us calculate

$$\begin{aligned}
H(\cos x) &= H(\sin(x + \pi/2)) \\
&= -\cos(x + \pi/2) \\
&= \sin x
\end{aligned}$$

i.e

$$H(\cos x) = \sin x.$$

6.3 Relationship between the Hilbert transform and the Laplace transform

The Laplace transform of the function f is defined by

$$\mathcal{L}f(x) = \int_0^\infty f(s)e^{-xs}\,ds.$$

Next, we shall exam the connection between the Hilbert and Laplace transform. Let f be a function such that $f(x) = 0$ for $x < 0$. Then using Fubini's theorem we have

$$\begin{aligned} Hf(-x) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{-x-y}\,dy \\ &= -\frac{1}{\pi}\int_0^\infty \frac{f(y)}{x+y}\,dy \end{aligned}$$

thus

$$\begin{aligned} -\pi Hf(-x) &= \int_0^\infty \frac{f(y)}{x+y}\,dy \\ &= \int_0^\infty f(y)\left(\int_0^\infty e^{-(x+y)t}\,dt\right)\,dy \\ &= \int_0^\infty f(y)\left(\int_0^\infty e^{-xt}e^{-yt}\,dt\right)\,dy \\ &= \int_0^\infty e^{-xt}\left(\int_0^\infty f(y)e^{-yt}\,dy\right)\,dt \\ &= \int_0^\infty e^{-xt}\mathcal{L}f(t)\,dt \\ &= \mathcal{L}(\mathcal{L}f)(x). \end{aligned}$$

Thus

$$\pi Hf(-x) = -\mathcal{L}(\mathcal{L}f)(x).$$

Example. Let

$$f(x) = \begin{cases} e^{-ax} & 0 \le x < \infty \\ 0 & -\infty < x < 0 \end{cases}$$

where $a > 0$. The Laplace transform of f is given by

$$\begin{aligned} \mathcal{L}f(t) &= \int_0^\infty f(s)e^{-st}\,ds \\ &= \frac{1}{a+t}. \end{aligned}$$

Now,

$$\begin{aligned}
-\pi Hf(-x) &= \mathcal{L}(\mathcal{L}f)(x) \\
&= \mathcal{L}\left(\left\{\frac{1}{a+x}\right\}\right) \\
&= e^{ax}\int_{ax}^{\infty}\frac{e^{-t}}{t}\,dt.
\end{aligned}$$

6.4 The Hilbert transform of $\frac{1}{x+i\alpha}$ and the Poisson kernel

As a final elementary example, consider the evaluation of $H(\frac{1}{x+i\alpha})$, where α is real. A partial fraction expansion yields

$$\begin{aligned}
H(\frac{1}{x+i\alpha}) &= \frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{dy}{(y+i\alpha)(x-y)} \\
&= \frac{1}{\pi(x+i\alpha)}P.V.\int_{-\infty}^{\infty}\left(\frac{1}{y+i\alpha}+\frac{1}{x-y}\,dy\right) \\
&= \frac{1}{\pi(x+i\alpha)}P.V.\int_{-\infty}^{\infty}\left(\frac{y-i\alpha}{y^2+\alpha^2}+\frac{1}{x-y}\,dy\right) \\
&= \frac{1}{\pi(x+i\alpha)}P.V.\int_{-\infty}^{\infty}\left(\frac{y}{y^2+\alpha^2}-\frac{i\alpha}{y^2+\alpha^2}+\frac{1}{x-y}\,dy\right) \\
&= \frac{i\alpha}{\pi(x+i\alpha)}P.V.\int_{-\infty}^{\infty}\frac{dy}{y^2+\alpha^2}
\end{aligned}$$

where the final result follows from subsection 3.1 and the odd character of the first term in the integral. Now using a change of integration variable and paying attention to the sign of α gives

$$\begin{aligned}
H(\frac{1}{x+i\alpha}) &= -\frac{i\alpha}{\pi(x+i\alpha)}\begin{cases}\frac{1}{\alpha}\int_{-\infty}^{\infty}\frac{dt}{t^2+1} & \alpha>0\\ -\frac{1}{\alpha}\int_{-\infty}^{\infty}\frac{dt}{t^2+1} & \alpha<0\end{cases} \\
&= -\frac{i\alpha}{\pi(x+i\alpha)}\begin{cases}\frac{\pi}{\alpha} & \alpha>0\\ -\frac{\pi}{\alpha} & \alpha<0\end{cases} \\
&= \begin{cases}\frac{-i}{x+i\alpha} & \alpha>0\\ \frac{i}{x+i\alpha} & \alpha<0.\end{cases}
\end{aligned}$$

What happens for the case $\alpha = 0$?

Definition 6.2. The truncated Hilbert transform of $f \in S(\mathbb{R})$ (at height ϵ) is defined by

$$H_\epsilon(f)(x) = \frac{1}{\pi} \int\limits_{|y|\geq\epsilon} \frac{f(x-y)}{y}\, dy$$
$$= \frac{1}{\pi} \int\limits_{|x-y|\geq\epsilon} \frac{f(y)}{x-y}\, dy.$$

Note that
$$Hf(x) = \lim_{\epsilon\to0^+} H_\epsilon(f)(x).$$

Note that for given $x \in \mathbb{R}$, $Hf(x)$ is defined for all integrable function f on $\mathbb{R}$ that satisfy a Hölder condition near the point x, that is

$$|f(x) - f(y)| \leq C_x|x-y|^{\epsilon_x}$$

for some $C_x > 0$ and $\epsilon_x > 0$ whenever $|x-y| < \delta_x$.

Indeed, suppose that this is the case, then

$$H_\epsilon f(x) = \frac{1}{\pi} \int\limits_{\epsilon<|x-y|<\delta_x} \frac{f(y)}{x-y}\, dy + \frac{1}{\pi} \int\limits_{|x-y|\geq\delta_x} \frac{f(y)}{x-y}\, dy$$
$$= \frac{1}{\pi} \int\limits_{\epsilon<|x-y|<\delta_x} \frac{f(y)-f(x)}{x-y}\, dy + \frac{1}{\pi} \int\limits_{|x-y|\geq\delta_x} \frac{f(y)}{x-y}\, dy.$$

Both integrals converges absolutely and hence the limit of $H_\epsilon f(x)$ exists as $\epsilon \to 0$. Therefore, the Hilbert transform of piecewise smooth integrable function is well defined at all points of Hölder-Lipschitz continuity of the function. On the other hand observe that $H_\epsilon f(x)$ is well defined for all $f \in L_p$, $1 \leq p < \infty$ as it follows from the Hölder inequality since $\frac{1}{x}$ is in L_q on the set $|x| \geq \epsilon$.

Let us consider the Poisson kernel $P(x) = \frac{1}{1+x^2}$, recalling that $P.V. \int_{-\infty}^{\infty} \frac{dy}{x-y} = 0$. we can write

$$HP(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{P(y)}{x-y} dy$$
$$= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{dy}{(1+y^2)(x-y)} dy$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x-y} \left[\frac{1}{1+y^2} - \frac{1}{1+x^2}\right]\, dy$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x-y} \left[\frac{x^2-y^2}{(1+y^2)(1+x^2)}\right]\, dy$$

$$= \frac{1}{\pi(1+x^2)} \int_{-\infty}^{\infty} \frac{(x-y)(x+y)}{(x-y)(1+y^2)}\, dy$$
$$= \frac{x}{\pi(1+x^2)} \int_{-\infty}^{\infty} \frac{dy}{(1+y^2)} + \frac{1}{\pi(1+x^2)} \int_{-\infty}^{\infty} \frac{y}{(1+y^2)}\, dy$$
$$= \frac{\pi x}{\pi(1+x^2)}$$
$$= \frac{x}{1+x^2}.$$

$HP(x) = \frac{x}{1+x^2}$ is some time called the "conjugate Poisson kernel".

Note, in the figure (2) we can observe the behavior of the function P and its Hilbert transform HP.

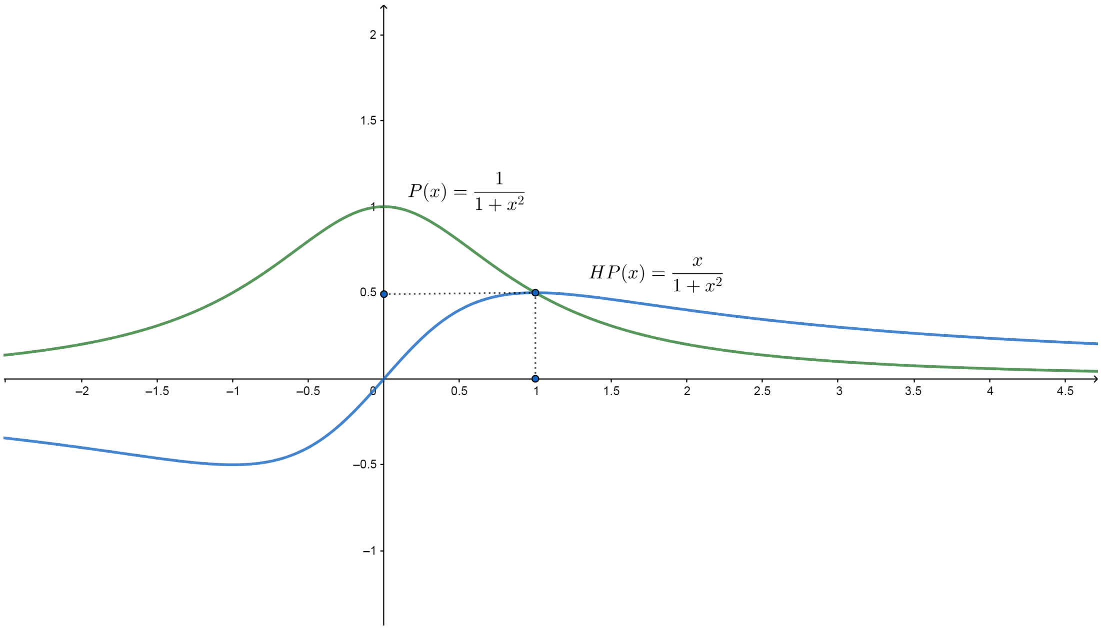

Figure 2: $P(x)$ and $HP(x)$

Example. Consider

$$f(x) = \frac{a}{a^2+x^2} \qquad \text{for} \qquad a > 0.$$

The Fourier transform of f is given by

$$\mathcal{F}f(x) = a \int_{-\infty}^{\infty} \frac{e^{-ixt}}{a^2+t^2}\, dt$$
$$= 2a \int_{0}^{\infty} \frac{\cos(xt)}{a^2+t^2}\, dt$$

$$= \pi e^{-a|x|}.$$

In the last equality we used the residue theorem.

Next,

$$\begin{aligned} Hf(x) &= -i\mathcal{F}^{-1}\left(\left\{(\operatorname{sgn} y)\pi e^{-a|y|}\right\}\right)(x) \\ &= -\frac{i}{2\pi}\int_{-\infty}^{\infty}(\operatorname{sgn} y)\pi e^{-a|y|}e^{ixy}\,dy \\ &= \frac{1}{2}\int_{-\infty}^{\infty}(\operatorname{sgn} y)\sin(xy)e^{-a|y|}\,dy \\ &= \int_{0}^{\infty}\sin(xy)e^{-ay}\,dy \\ &= \frac{x}{a^2+x^2}. \end{aligned}$$

Thus

$$H\left(\frac{a}{a^2+x^2}\right) = \frac{x}{a^2+x^2}.$$

6.5 Hilbert transform of periodic functions

Let us begin with the following observation. If one would formally differentiate the well-known Fourier series

$$\log\left|\sin\left(\frac{x}{2}\right)\right| = -\log 2 - \sum_{n=1}^{\infty}\frac{\cos(nx)}{n}, \quad x \neq 2\pi k, \quad k \in \mathbb{Z}$$

one would get

$$\frac{1}{2}\cot\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty}\sin(nx).$$

Now, as it happens, one cannot just differentiate Fourier series as one wish. Note that, for general $x \in (0, 2\pi)$, the terms $\sin(nx) \not\to 0$, so the series in the right- hand side does not converge. One way out of this would be to interpret everything in terms of distributions, as will we do it.

Now, one can also define the Hilbert transform for periodic functions. Suppose $u(t)$ is a functions with period $2T$. Such that $u(t)$ is absolutely continuous in $[-T, T]$ and $u'(t) \in L_2([-T, T])$. Then it can be expressed as a Fourier series

$$u(t) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{i\pi nt}{T}}$$

where each coefficient C_n for a fixed n is given by

$$C_n = \frac{1}{2T}\int_{-T}^{T} u(s)e^{\frac{-i\pi ns}{T}}\,ds.$$

With the fact that $H(C) = 0$ for any constant C and $H(e^{ist}) = -i\,\mathrm{sgn}(s)e^{ist}$ for any real number s we get

$$\begin{aligned}
H(u(t)) &= H(C_0) + H\left(\sum_{n=1}^{\infty} C_n e^{\frac{i\pi nt}{T}}\right) + H\left(\sum_{n=-\infty}^{1} C_n e^{\frac{i\pi nt}{T}}\right) \\
&= \sum_{n=1}^{\infty} C_n H\left(e^{\frac{i\pi nt}{T}}\right) + \sum_{n=1}^{\infty} C_n H\left(e^{\frac{-i\pi n}{T}}\right) \\
&= -i\sum_{n=1}^{\infty} C_n \left(e^{\frac{i\pi nt}{T}} - e^{\frac{-i\pi nt}{T}}\right) \\
&= -\frac{i}{2T}\sum_{n=1}^{\infty} \left(\int_{-T}^{T} u(s)e^{\frac{-i\pi nt}{T}}\right)\left(e^{\frac{i\pi nt}{T}} - e^{\frac{-i\pi nt}{T}}\right) \\
&= \frac{-i}{2T}\int_{-T}^{T} u(s)\left(\sum_{n=1}^{\infty} e^{\frac{i\pi n(t-s)}{T}} - e^{\frac{-i\pi n(t-s)}{T}}\right) ds \\
&= \frac{1}{2T}\int_{-T}^{T} u(s)\sum_{n=1}^{\infty} 2\sin\left(\frac{\pi n(t-s)}{T}\right) ds \\
&= \frac{1}{2T} P.V \int_{-T}^{T} u(s)\cot\left(\pi\left(\frac{t-s}{T}\right)\right) ds.
\end{aligned}$$

Now, the following definition of a periodic Hilbert transform makes sense

Definition 6.3 (Hilbert transform of a periodic function)**.** Let $u(t)$ be a periodic function with periodicity $2T$. Then $H_T(u(t))$ is the periodic Hilbert transform of $u(t)$ given by

$$H_T(u(t)) = \frac{1}{2T} P.V \int_{-T}^{T} u(s)\cot\left(\pi\left(\frac{t-s}{T}\right)\right) ds.$$

Note. If $T \to \infty$, then $u(t)$ has infinite periodicity and is then, obviously, an non-periodicity function. What happen to H_T when $T \to \infty$? The following result answer that question.

Theorem 6.1. *Let u be a periodic function with periodicity $2T$. Then*

$$\lim_{T\to\infty} H_T(u(t)) = \frac{1}{\pi} P.V \int_{\infty}^{\infty} \frac{u(s)}{t-s}\, ds.$$

Proof. We start by evaluating the limit of the integrand when $T \to \infty$.

$$\lim_{T\to\infty} \frac{\cot\left(\pi\left(\frac{t-s}{T}\right)\right)}{2T} = \lim_{T\to\infty} \frac{\cos\left(\pi\left(\frac{t-s}{T}\right)\right)}{\sin\left(\pi\left(\frac{t-s}{T}\right)\right)} \cdot \frac{1}{2T}$$

$$= \lim_{y\to o^+} \frac{\cos y}{\sin y} \cdot \frac{y}{\pi(t-s)}$$
$$= \frac{1}{\pi(t-s)} \cdot 1$$
$$= \frac{1}{\pi(t-s)}$$

Above we made a change of variable $y = \pi\frac{t-s}{2T}$ and as $T \to \infty$ we get $y \to 0^+$. And so

$$\lim_{T\to\infty} H_T(u(t)) = \lim_{T\to\infty} \frac{1}{2T} P.V \int_{-T}^{T} u(s) \cot\left(\pi\left(\frac{t-s}{T}\right)\right) ds$$
$$= \lim_{T\to\infty} P.V \int_{-T}^{T} u(s) \frac{\cot\left(\pi\left(\frac{t-s}{T}\right)\right)}{2T} ds$$
$$= \frac{1}{\pi} P.V \int_{-\infty}^{\infty} \frac{u(s)}{t-s} ds.$$

We recognize this last expression as the Hilbert transform on the real line. Clearly $H_T \to H$ as $T \to \infty$, as should be expected [17]. □

Find the Hilbert transform of the periodic function $\sin t$.
Solution.

$$H_\pi(\sin t) = \frac{1}{2\pi} P.V \int_{-\pi}^{\pi} \sin t \cot\left(\frac{x-t}{2}\right) dt$$
$$= \frac{1}{2\pi} P.V \int_{-\pi}^{\pi} \sin(x-t) \cot\left(\frac{t}{2}\right) dt$$
$$= \frac{1}{2\pi} P.V \int_{-\pi}^{\pi} (\sin x \cos t - \cos x \sin t) \cot\left(\frac{t}{2}\right) dt$$
$$= -\frac{1}{2\pi} \cos x \int_{-\pi}^{\pi} \sin t \cot\left(\frac{t}{2}\right) dt$$
$$= -\frac{\cos x}{\pi} \int_{0}^{\pi} \sin t \cot\left(\frac{t}{2}\right) dt$$
$$= -\frac{\cos x}{\pi} \int_{0}^{\pi} 2\cos^2\left(\frac{t}{2}\right) dt$$
$$= -\frac{\cos x}{\pi} \int_{0}^{\pi} (\cos t + 1)\, dt$$
$$= -cosx.$$

Hence

$$H_\pi(\sin t) = -cosx.$$

For additional reading on the Hilbert transform of periodic functions see [6] and [20].

For a discussion of the Hilbert transform of periodic functions with arbitrary period see [20] and [21].

7 Properties

In this subsection we will take close look at some of the basic properties of the Hilbert transform on the real line.

We start with a simple result concerning the character of the Hilbert transform as an operator.

Theorem 7.2. *The Hilbert transform is linear.*

Proof. Let f and g belonging to $S(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$. Straight from the definition we obtain

$$\begin{aligned} H(\alpha f + \beta g)(x) &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\alpha f(y) + \beta g(y)}{x - y}\, dy \\ &= \frac{\alpha}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x-y}\, dy + \frac{\beta}{\pi} P.V. \int_{-\infty}^{\infty} \frac{g(y)}{x-y}\, dy \\ &= \alpha H(f)(x) + \beta Hg(x). \end{aligned}$$

□

Another slightly less obvious property is that Hilbert transform commute with translation and positive dilations.

Theorem 7.3. *Let τ_a be the translation operator defined by $\tau_a f(x) = f(x-a)$ and let S_a for $a > 0$ be the dilation operator $S_a f(x) = f(ax)$. Then $H\tau_a f = \tau_a Hf$ and $HS_a f = S_a Hf$.*

Proof. By a simple change of variables

$$\begin{aligned} H\tau_a f(x) &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y-a)}{x-y}\, dy \\ &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(u)}{x-a-u}\, du \\ &= Hf(x-a) \end{aligned}$$

$$= \tau_a Hf(x).$$

And

$$\begin{aligned} HS_a f(x) &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(ay)}{x-y}\, dy \\ &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(u)}{ax-u}\, du \\ &= S_a Hf(x). \end{aligned}$$

□

Theorem 7.4. *Let R be the reflexion operator $Rf(x) = f(-x)$. Then HRf(x)=-RHf(x).*

Proof. By a change of variable we have

$$\begin{aligned} HRf(x) &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(-y)}{x-y}\, dy \\ &= -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(u)}{-x-u}\, du \\ &= -Hf(-x) \\ &= -RHf(x). \end{aligned}$$

□

There is no simple formula for the Hilbert transform of a product of two functions. However, we will discuss the special cases of the Hilbert transform of $xf(x)$ an $\frac{f(x)}{x}$.

Theorem 7.5. *Let f be integrable. Then*

$$H(xf(x)) = xHf(x) - \frac{1}{\pi} \int_{-\infty}^{\infty} f(y)\, dy. \tag{6}$$

If $\frac{f(x)}{x}$ is integrable. Then

$$H\left(\frac{f(x)}{x}\right) = \frac{Hf(x) - Hf(0)}{x}.$$

Proof.

$$\begin{aligned} H(xf(x)) &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{yf(y)}{x-y}\, dy \\ &= \frac{x}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x-y}\, dy - \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{(x-y)f(y)}{x-y}\, dy \\ &= xHf(x) - \frac{1}{\pi} \int_{-\infty}^{\infty} f(y)\, dy. \end{aligned}$$

And by using (6) on the function $\frac{f(x)}{x}$ we have

$$\begin{aligned} Hf(x) &= xH\left(\frac{f(x)}{x}\right) - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{y}\, dy \\ &= xH\left(\frac{f(x)}{x}\right) - Hf(0), \end{aligned}$$

then

$$H\left(\frac{f(x)}{x}\right) = \frac{Hf(x) - Hf(0)}{x}.$$

□

Theorem 7.6. *If f is an odd function, i.e $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Then its Hilbert transform is an even function, that is $Hf(x) = Hf(-x)$. And vice versa.*

Proof. Let f be an odd function, i.e $f(-x) = -f(x)$. We like to prove that

$$H_\epsilon f(x) = H_\epsilon f(-x).$$

In fact,

$$\begin{aligned} H_\epsilon f(x) &= \frac{1}{\pi} \int_{|y|>\epsilon} \frac{f(x-y)}{y}\, dy \\ &= \frac{1}{\pi}\left(\int_{-\infty}^{-\epsilon} \frac{f(x-y)}{y}\, dy + \int_{\epsilon}^{\infty} \frac{f(x-y)}{y}\, dy \right) \\ &= \frac{1}{\pi}\left(\int_{\infty}^{\epsilon} \frac{f(x+y)}{y}\, dy + \int_{-\epsilon}^{-\infty} \frac{f(x+y)}{y}\, dy \right) \end{aligned}$$

$$= -\frac{1}{\pi}\left(\int_{\epsilon}^{\infty} \frac{f(x+y)}{y}\, dy + \int_{-\infty}^{-\epsilon} \frac{f(x+y)}{y}\, dy\right)$$
$$= \frac{1}{\pi}\int_{|y|>\epsilon} \frac{-f(x+y)}{y}\, dy$$
$$= \frac{1}{\pi}\int_{|y|>\epsilon} \frac{f(-x-y)}{y}\, dy$$
$$= H_\epsilon f(-x).$$

Thus

$$Hf(x) = \lim_{\epsilon\to 0} H_\epsilon f(x) = \lim_{\epsilon\to 0} H_\epsilon f(-x) = Hf(-x).$$

□

Now, we study the convolution property of the Hilbert transform.

Theorem 7.7. *$H(f * g)(x) = (Hf * g)(x) = (f * Hg)(x)$.*

Proof. By Fubini's theorem on the one hand, we have

$$\begin{aligned} H(f*g)(x) &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{(f*g)(y)}{x-y}\, dy \\ &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{1}{x-y}\left(\int_{-\infty}^{\infty} f(s)g(y-s)\, ds\right) dy \\ &= \int_{-\infty}^{\infty} f(s)\left(\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{g(y-s)}{x-y}\, dy\right) ds \\ &= \int_{-\infty}^{\infty} f(s)\left(\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{g(y)}{x-s-y}\, dy\right) ds \\ &= \int_{-\infty}^{\infty} f(s) Hg(x-s)\, ds \\ &= (f * Hg)(x). \end{aligned} \tag{7}$$

On the other hand

$$H(f*g)(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{(f*g)(y)}{x-y}\, dy$$

$$
\begin{aligned}
&= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{1}{x-y} \left(\int_{-\infty}^{\infty} f(y-s)g(s)\, ds \right) dy \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y-s)}{x-y}\, dy \right) g(s)\, ds \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x-s-y}\, dy \right) g(s)\, ds \\
&= \int_{-\infty}^{\infty} Hf(x-s)g(s)\, ds \\
&= (Hf * g)(x). \qquad (8)
\end{aligned}
$$

Finally, combining (7) and (8) we have

$$H(f * g)(x) = (Hf * g)(x) = (f * Hg)(x).$$

□

Theorem 7.8. *Let $f \in L_2(\mathbb{R})$. Then $f(x)$ and $Hf(x)$ are orthogonal functions, that is*

$$\langle f, Hf \rangle = 0.$$

Proof. Using the Parseval identity we get

$$
\begin{aligned}
\langle f, Hf \rangle &= \int_{-\infty}^{\infty} f(x)\overline{Hf(x)}\, dx \\
&= \int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{Hf}(\xi)}\, d\xi \\
&= \int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{[-i\text{sign}\xi \widehat{f}(\xi)]}\, d\xi \\
&= i \int_{-\infty}^{\infty} \text{sign}\xi \widehat{f}(\xi)\overline{\widehat{f}(\xi)}\, d\xi \\
&= i \int_{-\infty}^{\infty} \text{sign}\xi |\widehat{f}(\xi)|^2\, d\xi.
\end{aligned}
$$

Since $|\widehat{f}(\xi)|^2 = |\widehat{f}(-\xi)|^2$, therefore sign$|\widehat{f}(\xi)|^2$ is an odd function since signξ is odd and $|\widehat{f}(\xi)|^2$ even with the symmetric interval of integration, the integral is zero and we have the desired result,

$$\langle f, Hf \rangle = 0$$

which means that $f(x)$ and $Hf(x)$ are orthogonal functions, and so the proof is complete. □

7.1 The Hilbert transform product

The Hilbert transform of a product of two functions is now considered. The following result is due to Bedrosian (see [2]) and is referred to as the Hilbert transform product, or Bedrosian's theorem . Let f and g two function belonging to $L_2(\mathbb{R})$. Suppose that the Fourier transform of $f(x)$, denote by $\mathcal{F}(\xi)$, vanishes for $|\xi| > a$ with $a > 0$, and the Fourier transform of $g(x)$, denote by $\mathcal{G}(\xi)$, vanishes for $|\xi| < a$, then

$$H\{f(x)g(x)\} = f(x)Hg(x).$$

The Hilbert transform of the product is given in terms of the Fourier transform $\mathcal{F}$ and $\mathcal{G}$. The Fubini theorem allow us to perform

$$\begin{aligned} H\{f(x)g(x)\} &= H\left[\int_{-\infty}^{\infty} \mathcal{F}(\xi)e^{i\xi x}\, d\xi \int_{-\infty}^{\infty} \mathcal{G}(t)e^{itx}\, dt\right] \\ &= \int_{-\infty}^{\infty} \mathcal{F}(\xi)\, d\xi \int_{-\infty}^{\infty} \mathcal{G}(t)H\{e^{i(t+\xi)x}\, dt\} \\ &= \int_{-\infty}^{\infty} \mathcal{F}(\xi)\, d\xi \int_{-\infty}^{\infty} \mathcal{G}(t)\{-i\,\mathrm{sgn}(\xi+t)e^{i(t+\xi)x}\, dt\}. \end{aligned}$$

The last integral simplifies on noting that the support of $\mathcal{F}(\xi)$ is $[-a, a]$ and the support of $\mathcal{G}(\xi)$ is $(-\infty, -a] \cup [a, \infty)$, hence

$$\begin{aligned} H\{f(x)g(x)\} &= -i\int_{-a}^{a} \mathcal{F}(x)e^{i\xi x}\, d\xi \left[\int_{-\infty}^{-a} \mathcal{G}(t)e^{ixt}\,\mathrm{sgn}(t+\xi)\, dt + \int_{a}^{\infty} \mathcal{G}(t)e^{ixt}\,\mathrm{sgn}(t+\xi)\, dt\right] \\ &= -i\int_{-a}^{a} \mathcal{F}\, d\xi \left[\int_{-\infty}^{-a+\xi} \mathcal{G}(\nu-\xi)e^{ix\nu}\,\mathrm{sgn}(\nu)\, d\nu + \int_{a+\xi}^{\infty} \mathcal{G}(\nu-\xi)e^{ix\nu}\,\mathrm{sgn}(\nu)\, d\nu\right] \\ &= -i\int_{-a}^{a} F(\xi)e^{ix\xi}\, d\xi \left[-\int_{-\infty}^{-a} \mathcal{G}(y)e^{ixy}\, dy + \int_{a}^{\infty} \mathcal{G}(y)e^{ixy}\, dy\right] \\ &= f(x)\left[\int_{-\infty}^{-a} \mathcal{G}(y)(-i\,\mathrm{sgn}\, y)e^{ixy}\, dy + \int_{a}^{\infty} \mathcal{G}(y)(-i\,\mathrm{sgn}\, y)e^{ixy}\, dy\right] \\ &= f(x)\left[\int_{-\infty}^{-a} \mathcal{G}(y)H\{ixy\}\, dy + \int_{a}^{\infty} \mathcal{G}(y)H\{ixy\}\, dy\right] \\ &= f(x)\left[H\left[\int_{-\infty}^{-a} \mathcal{G}(y)e^{ixy}\, dy + \int_{a}^{\infty} \mathcal{G}(y)e^{ixy}\, dy\right]\right] \end{aligned}$$

$$= f(x)H\left(\int_{-\infty}^{\infty} \mathcal{G}(y)e^{ixy}\right)$$
$$= f(x)Hg(x).$$

And so

$$H\{f(x)g(x)\} = f(x)Hg(x). \tag{9}$$

Remark. The restriction on the class of functions for which the equation (9) holds is rather severe, but there is an important practical applications.

The following examples will illustrate the approach. Let functions

$$\sin c(x) = \frac{\sin(\pi x)}{\pi x}.$$

The Fourier transform of $\sin c(ax)$, where a is a real number, is given by

$$\mathcal{F}\{\sin c(ax)\} = \frac{1}{2a}\{\text{sgn}(\pi a + x) + \text{sgn}(\pi a - x)\}.$$

The $\sin c$ function therefore satisfies the condition that its Fourier transform has a support of a finite interval around the origin, specifically $(-\pi a, \pi a)$ from which if follows that

$$\begin{aligned} H(\sin(ax)\sin c(bx)) &= H(\sin(ax))\sin c(bx) \\ &= -\cos(ax)\sin c(bx) \end{aligned}$$

for $0 < b\pi < a$.

In a similar fashion

$$\begin{aligned} H(\cos(ax)\sin c(bx)) &= H(\cos(ax))\sin c(bx) \\ &= \sin(ax)\sin c(bx) \end{aligned}$$

for $0 < b\pi < a$.

Theorem 7.9. *(Hilbert inversion theorem) Given $f \in L_2(\mathbb{R})$, then*

$$f(x) = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{Hf(y)}{x-y}\, dy.$$

Proof. Using the Fourier Inversion theorem, we can rewrite f as follows;

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{-ix\xi}\, d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(\mu) e^{i\xi\mu} \, d\mu \right) e^{-ix\xi} \, d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \left(\int\limits_{-\infty}^{\infty} \{ f(\mu) \cos(\xi\mu) + i f(\mu) \sin(\xi\mu) \} \right) \, d\mu \times (\cos(x\xi) - i \sin(x\xi)) \, d\xi.$$

If we consider the case of real valued functions we have

$$f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \left(\int\limits_{-\infty}^{\infty} f(\mu) \cos(\xi\mu) d\mu \right) \cos(x\xi) d\xi + \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \left(\int\limits_{-\infty}^{\infty} f(\mu) \sin(\xi\mu) d\mu \right) \sin(x\xi) d\xi.$$

Thus, we can finally write

$$f(x) = \int_0^\infty \{ a(\xi) \cos(x\xi) + b(\xi) \sin(x\xi) \} \, d\xi$$

where

$$a(\xi) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} f(\mu) \cos(\xi\mu) d\mu$$

and

$$b(\xi) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} f(\mu) \sin(\xi\mu) d\mu.$$

Now, we define

$$u(x, y) = \int_0^\infty \{ a(\xi) \cos(x\xi) + b(\xi) \sin(x\xi) \} e^{-y\xi} \, d\xi$$

observe that for $y \geq 0$, $u(x, y)$ is well defined and its the real part of

$$\phi(z) = \int_0^\infty \{ a(\xi) - i b(\xi) \} e^{iz\xi} \, d\xi$$

where $z = x + iy$. The imaginary part of $\phi(z)$ is then

$$v(x, y) = -\int_0^\infty \{ b(\xi) \cos(x\xi) - a(\xi) \sin(x\xi) \} \, d\xi$$

writing $g(x) = -v(x, 0)$, we have

$$g(x) = \int_0^\infty \{ b(\xi) \cos(x\xi) - a(\xi) \sin(x\xi) \} \, d\xi$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \{f(\mu)(\sin(\mu\xi)\cos(\mu\xi) - \cos(\mu\xi)\sin(\mu\xi))\}\, d\mu d\xi$$
$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \{f(\mu)\sin[(\mu - x)\xi]\}\, d\mu d\xi.$$

This can be written as

$$g(x) = \lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^\lambda \int_{-\infty}^\infty \{f(\mu)\sin[(\mu - x)\xi]\}\, d\mu d\xi.$$

Applying Fubini and solving for ξ gives

$$g(x) = \lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-\infty}^\infty \left[f(\mu)\left(\frac{-\cos(\mu - x)\xi}{\mu - x}\right)\right]_0^\lambda d\mu$$
$$= \lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-\infty}^\infty f(\mu)\left[\frac{1-\cos(\mu - x)\lambda}{\mu - x}\right] d\mu$$
$$= \lim_{\lambda\to\infty}\frac{1}{\pi}\left(\int_0^\infty \frac{1-\cos(\mu - x)\lambda}{\mu - x} f(\mu)\, d\mu + \int_0^\infty \frac{1-\cos(-\mu_0 - x)\lambda}{-\mu_0 - x} f(\mu)\, d\mu_0\right).$$

Now using the change of coordinates $\mu - x = t$ and $\mu_0 + x = t_0$ it becomes

$$g(x) = \frac{1}{\pi}\left(\int_0^\infty \frac{1-\cos(\lambda t)}{t} f(x+t)dt - \int_0^\infty \frac{1-\cos(\lambda t_0)}{t_0} f(x - t_0)dt_0\right)$$
$$= \lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^\infty \frac{1-\cos(\lambda t)}{t}\left(f(x+t) - f(x-t)\right)dt$$
$$= \frac{1}{\pi}\int_0^\infty \frac{(f(x+t) - f(x-t)}{t}\, dt - \lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^\infty \frac{\cos(\lambda t)}{t}\left(f(x+t) - f(x-t)\right)dt.$$

Now, let us take care about

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^\infty \frac{\cos(\lambda t)}{t}\left(f(x+t) - f(x-t)\right)dt.$$

In order to do that set $F(t,x) = f(x+t) - f(x-t)$ applying the Riemann-Lebesgue lemma we have

$$\lim_{\lambda\to\infty}\int_0^\infty \frac{\cos(\lambda t)}{t} F(t,x)\, dt = \lim_{\lambda\to\infty}\int_0^\infty \frac{\cos(u)}{u} F\left(\frac{u}{\lambda}, x\right) = 0\, dt$$

since $F(0,x) = 0$. Thus

$$g(x) = \frac{1}{\pi}\int_0^\infty \frac{f(x+t) - f(x-t)}{t}\, dt.$$

In a similar manner

$$f(x) = -\frac{1}{\pi}\int_0^\infty \frac{g(x+t)-g(x-t)}{t}\, dt.$$

Now, note that

$$\begin{aligned}
f(x) &= -\frac{1}{\pi}\int_0^\infty \frac{g(x+t)-g(x-t)}{t}\, dt \\
&= -\frac{1}{\pi}\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{g(x+t)-g(x-t)}{t}\, dt \\
&= -\frac{1}{\pi}\lim_{\epsilon\to 0}\left[\int_\epsilon^\infty \frac{g(x+t)}{t}\, dt - \int_\epsilon^\infty \frac{g(x-t)}{t}\, dt\right] \\
&= -\frac{1}{\pi}\lim_{\epsilon\to 0}\left[\int_{x+\epsilon}^\infty \frac{g(y)}{y-x}\, dy + \int_{-\infty}^{x-\epsilon} \frac{g(y)}{y-x}\, dy\right] \\
&= P.V.\int_{-\infty}^\infty \frac{g(y)}{y-x}\, dy.
\end{aligned}$$

In a similar way, we have

$$g(x) = P.V.\int_{-\infty}^\infty \frac{f(z)}{z-y}\, dz.$$

Finally

$$\begin{aligned}
f(x) &= -\frac{1}{\pi}P.V.\int_{-\infty}^\infty \frac{g(y)}{y-x}\, dy \\
&= -\frac{1}{\pi}P.V.\int_{-\infty}^\infty \frac{1}{y-x}\left(P.V.\int_{-\infty}^\infty \frac{f(z)}{z-y}\, dz\right) dy \\
&= -\frac{1}{\pi}P.V.\int_{-\infty}^\infty \frac{Hf(y)}{y-x}\, dy.
\end{aligned}$$

□

The following result will have an important role in the development of the next result.

Theorem 7.10.

$$\mathcal{F}\left(\frac{1}{x}\right) = -i\pi\,\mathrm{sgn}(\xi)$$

where sgn *is the signum function*

$$\mathrm{sgn}(x) = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0. \end{cases}$$

Proof.

$$\begin{aligned}\mathcal{F}\left(\frac{1}{x}\right) &= P.V.\int_{-\infty}^{\infty}\frac{1}{x}e^{-2\pi i x\xi}\,dx\\ &= P.V.\int_{-\infty}^{\infty}\frac{\cos(2\pi x\xi)-i\sin(2\pi x\xi)}{x}dx\\ &= P.V.\int_{-\infty}^{\infty}\frac{\cos(2\pi x\xi)}{x}\,dx - iP.V.\int_{-\infty}^{\infty}\frac{\sin(2\pi x\xi)}{x}\,dx.\end{aligned}$$

The first integral vanishes as a principal value, because the integral is a an odd function. The latter one has no divergence since the integrand approaches the value 1 near the origin and so the principal value plays no role. Instead, it has to be considered separately for different values of ξ. For $\xi = 0$ the integral vanish trivially. Assuming that $\xi < 0$ and making a change of variable $t = 2\pi x\xi$, the integral because becomes

$$\int_{-\infty}^{\infty}\frac{\sin(2\pi x\xi)}{x}\,dx = -\int_{\infty}^{\infty}\frac{\sin t}{t}\,dt = -\pi.$$

For $\xi > 0$ the same calculation yields

$$\int_{-\infty}^{\infty}\frac{\sin(2\pi x\xi)}{x} = \pi.$$

Combining all these results we can conclude that

$$\mathcal{F}\left(\frac{1}{x}\right) = -i\pi\,\mathrm{sgn}(\xi).$$

□

Some time the Hilbert transform is define by the following result.

Theorem 7.11. *Let $f \in L_2(\mathbb{R})$ and consider*

$$Hf(x) = (f * g)(x) \tag{10}$$

where $g(x) = \frac{1}{\pi x}$. Then

$$\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\widehat{f}(\xi).$$

Proof. Taking the Fourier transform of (10) we have

$$\begin{aligned}\widehat{Hf}(\xi) &= \widehat{f * g}(\xi)\\ &= \widehat{f}(\xi)\widehat{g}(\xi)\\ &= \frac{1}{\pi}(-i\ \pi\ \ \mathrm{sgn}(\xi))\widehat{f}(\xi)\\ &= -i\ \ \mathrm{sgn}(\xi)\widehat{f}(\xi).\end{aligned}$$

□

7.2 Differentiation property of the Hilbert transform in L_2

Theorem 7.12. *Let $f \in L_2(\mathbb{R})$ such that f is differentiable with $f' \in L_2(\mathbb{R})$ and Hf be differentiable with $(Hf)' \in L_2(\mathbb{R})$. Then*

$$H\left(\frac{df(x)}{dx}\right) = \frac{d}{dx}H(f(x)).$$

Proof. By the differentiation property of the Fourier transform and Theorem 7.11 we get by some simple algebra

$$\begin{aligned}\widehat{Hf'(\xi)} &= (-i\text{sign}\xi)\widehat{f'(\xi)} \\ &= (-i\text{sign}\xi)(2\pi i\xi)\widehat{f}(\xi) \\ &= 2\pi i\xi(-i\text{sign}\xi)\widehat{f}(\xi) \\ &= 2\pi i\xi\widehat{H}(f)(\xi) \\ &= \widehat{H'(f)(\xi)}.\end{aligned}$$

Taking the inverse Fourier transform we obtain

$$H\left(\frac{df(x)}{dx}\right) = \frac{d}{dx}H(f(x))$$

and the proof is complete. □

Example. Let $f(x) = \frac{x}{1+x^2}$, then

$$\begin{aligned}H\left(\frac{x}{1+x^2}\right) &= H\left(-\frac{1}{2}\frac{d}{dx}\left(\frac{1}{1+x^2}\right)\right) \\ &= -\frac{1}{2}\frac{d}{dx}\left(H\left(\frac{1}{1+x^2}\right)\right) \\ &= -\frac{1}{2}\frac{d}{dx}\left(\frac{x}{1+x^2}\right) \\ &= \frac{x^2-1}{2(1+x^2)^2}.\end{aligned}$$

8 The Hilbert transform as an operator in L_2

It is just time to do some functional analysis.

Definition 8.4. If $\mathcal{H}$ is a Hilbert space and $T \in B(\mathcal{H})$, then T is unitary if $TT^* = T^*T = I$. Here T^* stand for the adjoint of T.

- In what follows, we consider the Hilbert transform H as it is, an operator.

- Note

$$\begin{aligned}||Hf||_{L_2} &= ||\widehat{Hf}||_{L_2}\\ &= ||-i\ \text{sign}(\cdot)\widehat{f}(\cdot)||_{L_2}\\ &= ||\widehat{f}||_{L_2} = ||f||_{L_2}\end{aligned}$$

Then $H \in B(\mathcal{H})$.

Also observe that if $Hf_1 = Hf_2$, then

$$0 = ||Hf_1 - Hf_2|| =_{L_2}= ||H(f_1 - f_2)||_{L_2} = ||f_2 - f_1||_{L_2}.$$

Hence $f_1 = f_2$ and so H is $1-1$.

Now, we like to prove that $(H^*)^* = H$ indeed

$$\begin{aligned}\langle y, (H^*)^* x\rangle &= \langle H^* y, x\rangle\\ &= \overline{\langle x, H^* y\rangle}\\ &= \overline{\langle Hx, y\rangle}\\ &= \langle y, Hx\rangle\,.\end{aligned}$$

Hence $(H^*)^*x = Hx$ for all $x \in L_2$ so

$$(H^*)^* = H. \tag{11}$$

Now, let us consider

$$\begin{aligned}\langle H^*(Hf), f\rangle &= \langle Hf, Hf\rangle\\ &= ||Hf||^2_{L_2} = ||f||^2_{L_2}\\ &= \langle f, f\rangle\\ &= \langle If, f\rangle\,.\end{aligned}$$

Therefore $H^*(H)f = I$, $\forall f \in L_2$ and hence

$$H^*(H) = I. \tag{12}$$

Next, by (11) and (12) we have

$$(H^*)^*(H^*) = I, \quad \text{then} \quad H(H^*) = I. \tag{13}$$

By (11) and (13) we obtain

$$H(H^*) = H^*(H) = I \tag{14}$$

in this way we have shown that H is an unitary operator. By (14)

$$H^*((H^*)^*) = (H^*)^*(H^*) = I. \tag{15}$$

Thus H^* is unitary, and

$$\begin{aligned} ||f||^2_{L_2} = \langle f, f\rangle &= \langle (H^*)H^*f, f\rangle \\ &= \langle H^*f, H^*f\rangle \\ &= ||H^*f||^2_{L_2} \end{aligned}$$

and so $||H^*f||_{L_2} = ||f||_{L_2}$. Which means that H^* is unitary and so H^* is $1-1$, from this fact observe that

$$\begin{aligned} L_2 = \{0\}^{\perp} &= \ker(H^*)^{\perp} \\ &= \overline{\operatorname{rang}(H)} \\ &= \operatorname{rang}(H) \\ &= H(L_2). \end{aligned}$$

Hence $H(L_2) = L_2$, which means that H is onto. Therefore its inverse exists name it $S = H^{-1}$, then $SH = HS = I$, also

$$\begin{aligned} H^* = H^*I &= H^*(HS) \\ &= (H^*H)S \\ &= IS \\ &= S. \end{aligned}$$

Hence $H^* = H^{-1}$. Now,

$$\begin{aligned} H(Hf)\hat{}(\xi) &= -(i \operatorname{sgn} \xi)(Hf)\hat{}(\xi) \\ &= (i \operatorname{sgn} \xi)^2 \widehat{f}(\xi) \\ &= -\widehat{f}(\xi). \end{aligned}$$

Taking the inverse Fourier transform in this way we have proved that

$$H(H) = -I. \tag{16}$$

Next, $H(H) = -I$ implies $H^{-1}(H(H)) = -H^{-1}(I)$ Hence $H = -H^{-1}$. i.e $H^{-1} = -H$. And so

$$H^* = -H. \tag{17}$$

Now that we know the fact that

$$H^{-1} = -H.$$

We will provide an easy but no less interesting proof of Theorem 7.9.

Theorem 8.13. *Let $f \in L_2(\mathbb{R})$ then*

$$f(x) = -\frac{1}{\pi} P.V \int\limits_{-\infty}^{\infty} \frac{Hf(y)}{x-y}\, dy.$$

Proof. Let $f \in L_2(\mathbb{R})$ then $f(x)$ can be written as

$$\begin{aligned} f(x) &= H^{-1}(Hf)(x) \\ &= -H(Hf)(x) \\ &= -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{Hf(y)}{x-y}\, dy. \end{aligned}$$

□

Theorem 8.14. *Let $f, g \in L_2(\mathbb{R})$, then*

$$\int_{-\infty}^{\infty} Hf(x)g(x)\, dx = -\int_{-\infty}^{\infty} f(x)H(g(x))\, dx. \tag{18}$$

Proof. Let $f, g \in L_2(\mathbb{R})$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} Hf(x)g(x)\, dx &= \langle Hf, g\rangle_{L_2} \\ &= \langle Hf, -H(H(g))\rangle_{L_2} \\ &= \langle Hf, H(H(g))\rangle_{L_2} \\ &= -\langle H^*(H(f)), H(g)\rangle_{L_2} \\ &= -\left\langle H^{-1}(H(f)), H(g)\right\rangle_{L_2} \\ &= -\langle f, H(g)\rangle_{L_2} \\ &= -\int_{-\infty}^{\infty} f(x)Hg(x)\, dx. \end{aligned}$$

And hence

$$\int_{-\infty}^{\infty} Hf(x)g(x)\, dx = -\int_{-\infty}^{\infty} f(x)Hg(x)\, dx \quad \text{for} \quad f, g \in L_2(\mathbb{R}).$$

□

Theorem 8.15. *If $f, g \in L_2(\mathbb{C})$, then*

$$\int_{-\infty}^{\infty} Hf(x)\overline{g(x)}\, dx = -\int_{-\infty}^{\infty} f(x)\overline{Hg(x)}\, dx. \tag{19}$$

Proof.

$$\begin{aligned}
\int_{-\infty}^{\infty} Hf(x)\overline{g(x)}\ dx &= \langle Hf(x), g(x)\rangle_{L_2} \\
&= \left\langle \widehat{Hf}(\xi), \widehat{g}(\xi)\right\rangle_{L_2} \\
&= \int_{-\infty}^{\infty} \widehat{Hf}(\xi)\overline{\widehat{g}(\xi)}\ d\xi \\
&= \int_{-\infty}^{\infty} (-i\ \mathrm{sign}(\xi)\widehat{f}(\xi))\overline{\widehat{g}(\xi)}\ d\xi \\
&= \int_{-\infty}^{\infty} \widehat{f}(\xi)\left[-i\ \mathrm{sign}(\xi)\overline{\widehat{g}(\xi)}\right]\ d\xi \\
&= -\int_{-\infty}^{\infty} \widehat{f}(\xi)\left[i\ \mathrm{sign}(\xi)\overline{\widehat{g}(\xi)}\right]\ d\xi \\
&= -\int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{[-i\ \mathrm{sign}(\xi)\widehat{g}(\xi)]}\ d\xi \qquad (i\,\mathrm{sign}\xi = \overline{-i\,\mathrm{sign}\xi}) \\
&= -\int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{Hg}(\xi)}\ d\xi \\
&= -\left\langle \widehat{f}(\xi), \widehat{Hg}(\xi)\right\rangle_{L_2} \\
&= -\langle f(x), Hg(x)\rangle_{L_2} \\
&= -\int_{-\infty}^{\infty} f(x)\overline{Hg(x)}\ dx,
\end{aligned}$$

i.e

$$\int_{-\infty}^{\infty} Hf(x)\overline{g(x)}\ dx = -\int_{-\infty}^{\infty} f(x)\overline{Hg(x)}\ dx.$$

□

Note, in what follows we will exhibit another way to prove results (18) and (19). In order to do that, let us consider

$$\begin{aligned}
\langle Hf, HG\rangle_{L_2} &= \left\langle \widehat{H}f, \widehat{H}G\right\rangle_{L_2} \\
&= \left\langle -i\ \mathrm{sign}(\xi)\widehat{f}(\xi), -i\ \mathrm{sign}(\xi)\widehat{G}(\xi)\right\rangle_{L_2} \\
&= -i\ \mathrm{sign}(\xi)\overline{(-i\ \mathrm{sign}(\xi))}\left\langle \widehat{f}(\xi), \widehat{G}(\xi)\right\rangle_{L_2} \\
&= -i\ \mathrm{sign}(\xi)(i\ \mathrm{sign}(\xi))\left\langle \widehat{f}(\xi), \widehat{G}(\xi)\right\rangle_{L_2} \\
&= -i^2\ (\mathrm{sign}(\xi))^2\left\langle \widehat{f}(\xi), \widehat{G}(\xi)\right\rangle_{L_2}
\end{aligned}$$

$$= \left\langle \widehat{f}(\xi), \widehat{G}(\xi) \right\rangle_{L_2}$$
$$= \langle f, G \rangle_{L_2}.$$

Now, taking $G = Hg$ for f and g reals then

$$\langle Hf, H(Hg) \rangle_{L_2} = \langle f, Hg \rangle_{L_2}$$
$$\langle Hf, -g \rangle_{L_2} = \langle f, Hg \rangle_{L_2}$$
$$-\int_{-\infty}^{\infty} Hf\ g = \int_{-\infty}^{\infty} f\ Hg$$

i.e

$$\int_{-\infty}^{\infty} Hf(x)\ g(x)\ dx = -\int_{-\infty}^{\infty} f(x)\ Hg(x)\ dx.$$

And for f and g complex functions we get

$$\int_{-\infty}^{\infty} Hf(x)\ \overline{g(x)}\ dx = -\int_{-\infty}^{\infty} f(x)\ \overline{Hg(x)}\ dx.$$

9 A theorem due to E. M. Stein and G. Weiss

Let us start by stating and demonstrating the following useful Lemma.

Lemma 9.1. *Let $P(x) = x^n + a_n x^{n+1} + \cdots + a_2 x + a_1$ be polynomial of degree n. Let $r_1, r_2, \cdots, r_n$ be the roots of $P(x) = 0$, then*

$$\sum_{k=1}^{n} r_k = -a_n.$$

Proof. By induction for $n = 2$ the polynomial $x^2 + a_2 x + a_1 = 0$ has two roots r_1 and r_2 such that

$$(x - r_1)(x - r_2) = 0$$

then

$$x^2 - r_1 x - r_2 x + r_1 r_2 = 0$$

and so

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0.$$

Hence $r_1 + r_2 = -a_2$.

Now, for $n = 3$, the polynomial $x^3 + a_3 x^2 + a_2 x + a_1 = 0$ has three roots named r_1, r_2 and r_3 such that

$$(x - r_1)(x - r_2)(x - r_3) = 0$$
$$(x^2 - (r_1 + r_2)x + r_1 r_2)(x - r_3) = 0$$

$$x^3 - (r_+ r_2)x^2 + (r_1 r_2)x - r_3 x^2 + (r_1 + r_2)r_3 x - r_1 r_2 r_3 = 0$$
$$x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)x - r_1 r_2 r_3 = 0$$

from this last equality we have

$$r_1 + r_2 + r_3 = -a_3.$$

Next, suppose that for

$$x^n + a_n x^{n+1} + \cdots + a_2 x + a_1 = 0,$$

$$\sum_{k=1}^{n} r_k = -a_n \qquad \text{holds.}$$

Now, the polynomial $x^{n+1} + a_{n+1}x^n + \cdots + a_2 x + a_1 = 0$, has $n+1$ roots named $r_1, r_2, \cdots, r_{n+1}$ such that

$$(x - r_1)(x - r_2)\cdots(x - r_n)(x - r_{n+1}) = 0$$
$$\left[x^n - \left(\sum_{k=1}^{n} r_k\right) x^{n-1} + \cdots\right](x - r_{n+1}) = 0$$
$$\left[x^{n+1} - \left(\sum_{k=1}^{n} r_k\right) x^n + \cdots\right] + \left[-r_{n+1}x^n + \left(\sum_{k=1}^{n} r_k\right) r_{n+1}x^{n-1} + \cdots\right] = 0$$
$$x^{n+1} - \left(\sum_{k=1}^{n+1} r_k\right) x^n + \cdots = 0.$$

Therefore

$$\sum_{k=1}^{n+1} r_k = -a_{n+1}.$$

□

Lemma 9.2. *Suppose that a_i, b_i $(i = 1, 2, 3, \cdots, n)$ are real numbers satisfying $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ and let g be the rational function*

$$g(x) = \prod_{k=1}^{n} \frac{x - a_k}{x - b_k} \qquad (x \in \mathbb{R}). \tag{20}$$

If $\Delta \neq 1$, then the equation $g(x) = |\Delta|$ has n distinct roots $r_1, r_2, \cdots, r_n$ which satisfy

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} r_k + (1 - \Delta)^{-1} \sum_{k=1}^{n} (b_k - a_k). \tag{21}$$

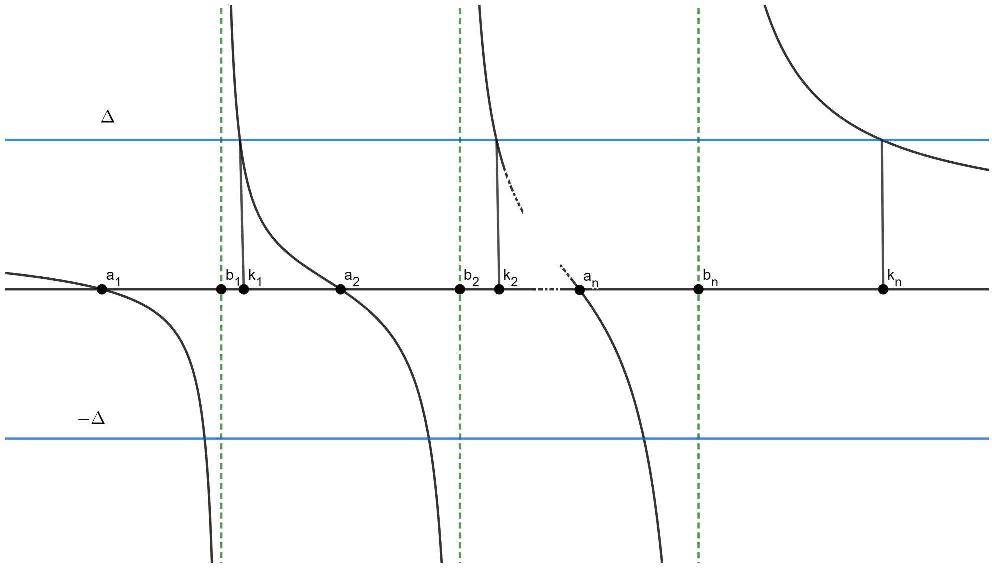

Figure 3: $\prod_{k=1}^{n} \frac{x-a_k}{x-b_k}$

Furthermore if $\Delta > 1$, *then*

$$(\Delta - 1)m(\{g > \Delta\}) = (\Delta + 1)m(\{g < -\Delta\}) = \sum_{k=1}^{n}(b_k - a_k). \tag{22}$$

Proof. Since g has a simple pole at each b_k, $(k = 1, 2, 3, \cdots, n)$ and

$$\lim_{|x|\to\infty} g(x) = \lim_{|x|\to\infty} \prod_{k=1}^{n} \frac{x-a_k}{x-b_k} = \prod_{k=1}^{n} \lim_{|x|\to\infty} \frac{x-a_k}{x-b_k} = 1 \tag{23}$$

there are exactly n distinct solutions, say $r_1, r_2, \cdots, r_n$ to the equation $g(x) = |\Delta|$ $(\Delta \neq 1)$. Then

$$\prod_{k=1}^{n} \frac{x-a_k}{x-b_k} = \Delta \qquad \text{and} \qquad \prod_{k=1}^{n} \frac{x-a_k}{x-b_k} = -\Delta.$$

For $\prod_{k=1}^{n} \frac{x-a_k}{x-b_k} = \Delta$, then

$$\prod_{k=1}^{n}(x-a_k) = \Delta \prod_{k=1}^{n}(x-b_k)$$

and so

$$\prod_{k=1}^{n}(x-a_k) - \Delta \prod_{k=1}^{n}(x-b_k) = 0,$$

where

$$P(x)=\sum_{k=0}^{n}p_kx^k=\prod_{k=1}^{n}(x-a_k)-\Delta\prod_{k=1}^{n}(x-b_k)=0 \tag{24}$$

Then

$$\prod_{k=1}^{n}(x-a_k)-\Delta\prod_{k=1}^{n}(x-b_k)=0$$

implies

$$\left[x^n-\left(\sum_{k=1}^{n}a_k\right)x^{n+1}+\cdots\right]-\Delta\left[x^n-\left(\sum_{k=1}^{n}b_k\right)x^{n+1}+\cdots\right]=0$$

$$(1-\Delta)x^n+\left(-\sum_{k=1}^{n}a_k+\Delta\sum_{k=1}^{n}b_k\right)x^{n-1}+\cdots=0$$

$$x^n+\frac{-\sum_{k=1}^{n}a_k+\Delta\sum_{k=1}^{n}b_k}{(1-\Delta)}x^{n-1}+\cdots=0.$$

since $r_1,r_2,\cdots,r_n$ are the roots of the polynomial $P(x)=0$, then by Lemma 9.1 we have

$$\sum_{k=1}^{n}r_k=-\frac{-\sum_{k=1}^{n}a_k+\Delta\sum_{k=1}^{n}b_k}{(1-\Delta)}.$$

And then

$$\begin{aligned}\sum_{k=1}^{n}r_k&=(1-\Delta)^{-1}\sum_{k=1}^{n}a_k-\frac{\Delta}{(1-\Delta)}\sum_{k=1}^{n}b_k\\&=(1-\Delta)^{-1}\sum_{k=1}^{n}a_k+\left(\frac{1-\Delta-1}{(1-\Delta)}\right)\sum_{k=1}^{n}b_k\\&=(1-\Delta)^{-1}\sum_{k=1}^{n}a_k+\left(1-\frac{1}{(1-\Delta)}\right)\sum_{k=1}^{n}b_k\\&=(1-\Delta)^{-1}\sum_{k=1}^{n}a_k+\left(1-(1-\Delta)^{-1}\right)\sum_{k=1}^{n}b_k\\&=(1-\Delta)^{-1}\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k-(1-\Delta)^{-1}\sum_{k=1}^{n}b_k\end{aligned}$$

Hence

$$\sum_{k=1}^{n}b_k=\sum_{k=1}^{n}r_k+(1-\Delta)^{-1}\sum_{k=1}^{n}b_k-(1-\Delta)^{-1}\sum_{k=1}^{n}a_k$$

$$= \sum_{k=1}^{n} r_k + (1-\Delta)^{-1} \sum_{k=1}^{n} (b_k - a_k).$$

If $\Delta > 1$, then $\{g > \Delta\} = \bigcup_{k=1}^{n} (b_k, r_k)$,(see figure 3) and so

$$\begin{aligned} m(\{g > \Delta\}) &= m\left(\bigcup_{k=1}^{n} (b_k, r_k)\right) \\ &= \sum_{k=1}^{n} (r_k - b_k). \end{aligned}$$

Since

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} r_k + (1-\Delta)^{-1} \sum_{k=1}^{n} (b_k - a_k)$$

then

$$\sum_{k=1}^{n} (b_k - r_k) = (1-\Delta)^{-1} \sum_{k=1}^{n} (b_k - a_k)$$

and so

$$\begin{aligned} -\sum_{k=1}^{n} (r_k - b_k) &= (1-\Delta)^{-1} \sum_{k=1}^{n} (b_k - a_k) \\ -m(\{g > \Delta\}) &= (1-\Delta)^{-1} \sum_{k=1}^{n} (b_k - a_k) \\ (\Delta - 1) m(\{g > \Delta\}) &= \sum_{k=1}^{n} (b_k - a_k). \end{aligned}$$

Moreover, if $-\Delta < -1$, then $\{g < -\Delta\} = \bigcup_{k=1}^{n} (r_k, b_k)$, (see figure 3). Hence

$$\begin{aligned} m(\{g < -\Delta\}) &= m\left(\bigcup_{k=1}^{n} (r_k, b_k)\right) \\ &= \sum_{k=1}^{n} (b_k - r_k). \end{aligned}$$

Now, we have

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} r_k + (\Delta+1)^{-1} \sum_{k=1}^{n} (b_k - a_k).$$

$$\sum_{k=1}^{n}(b_k - r_k) = (\Delta+1)^{-1}\sum_{k=1}^{n}(b_k - a_k)$$
$$m(\{g < -\Delta\}) = (\Delta+1)^{-1}\sum_{k=1}^{n}(b_k - a_k)$$
$$(\Delta+1)m(\{g < -\Delta\}) = \sum_{k=1}^{n}(b_k - a_k).$$

And thus we conclude that

$$(\Delta-1)m(\{g > \Delta\}) = (\Delta+1)m(\{g < -\Delta\}) = \sum_{k=1}^{n}(b_k - a_k). \tag{25}$$

□

In the following result we can observe that the distribution function of $H\chi_E$ depends only on the measure of E and not on the way in which E happens to be distributed over the real line.

Theorem 9.16 (Stein-Weiss)**.** *Let E be the union of finitely many disjoint intervals, each of finite length. Then*

$$D_{H_{\chi_E}}(\lambda) = \frac{2m(E)}{\sinh(\pi\lambda)}; \qquad \lambda > 0. \tag{26}$$

Where $D_{H_{\chi_E}}(\lambda) = m(\{|H\chi_E| > \lambda\})$ which is known as the distribution function of $H\chi_E$.

Proof. We may express E in the form

$$E = \bigcup_{j=1}^{n}(a_j, b_j) \tag{27}$$

such that $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$. We already known that

$$H_{\chi_E}(x) = \frac{1}{\pi}\left[\sum_{i=1}^{n}\int_{a_i}^{b_i}\frac{dy}{x-y}\right]$$
$$= \frac{1}{\pi}\left[\sum_{i=1}^{n}\log\left|\frac{x-a_i}{x-b_i}\right|\right]$$
$$= \frac{1}{\pi}\log\left|\prod_{i=1}^{n}\frac{x-a_i}{x-b_i}\right|.$$

Fix $\lambda > 0$ and $F = \{|H_{\chi_E}| > \lambda\}$, then $m(F) = D_{H_{\chi_E}}(\lambda)$. Since

$$H_{\chi_E}(x) = \frac{1}{\pi}\left[\log\left|\prod_{i=1}^{n}\frac{x-a_i}{x-b_i}\right|\right]$$

we have

$$e^{\pi H_{\chi_E}(x)} = \left|\prod_{i=1}^{n}\frac{x-a_i}{x-b_i}\right|.$$

If we set $g(x) = \prod_{i=1}^{n}\frac{x-a_i}{x-b_i}$. F may be decompose into the disjoint union

$$F = \{|g| > e^{\pi\lambda}\} \cup \{|g| < e^{-\pi\lambda}\} = F_1 \cup F_2. \tag{28}$$

Now, applying Lemma 9.2 to g we obtain

$$\begin{aligned}
m(F_1) &= m(\{|g| > e^{\pi\lambda}\}) \\
&= m(\{g > e^{\pi\lambda}\}) + m(\{g < -e^{\pi\lambda}\}) \\
&= \frac{\sum_{i=1}^{n}(b_i-a_i)}{e^{\pi\lambda}-1} + \frac{\sum_{i=1}^{n}(b_i-a_i)}{e^{\pi\lambda}+1} \\
&= \frac{m(E)}{e^{\pi\lambda}-1} + \frac{m(E)}{e^{\pi\lambda}+1} \\
&= \frac{2e^{\pi\lambda}m(E)}{e^{2\pi\lambda}-1} \\
&= \frac{m(E)}{\sinh(\pi\lambda)}.
\end{aligned}$$

Next, for F_2,

$$\begin{aligned}
m(F_2) &= m(\{|g| < e^{-\pi\lambda}\}) \\
&= m(\{g > -e^{-\pi\lambda}\}) + m(\{g < e^{-\pi\lambda}\}) \\
&= \frac{\sum_{i=1}^{n}(b_i-a_i)}{-e^{-\pi\lambda}-1} + \frac{\sum_{i=1}^{n}(b_i-a_i)}{-e^{-\pi\lambda}+1} \\
&= \frac{m(E)}{-e^{-\pi\lambda}-1} + \frac{m(E)}{-e^{-\pi\lambda}+1} \\
&= \frac{-2e^{\pi\lambda}m(E)}{e^{-2\pi\lambda}-1} \\
&= \frac{-m(E)}{\sinh(-\pi\lambda)} \\
&= \frac{m(E)}{\sinh(\pi\lambda)}.
\end{aligned}$$

Finally

$$m(\{|H_{\chi_E}| > \lambda\}) = m(F_1) + m(F_2) = \frac{2m(E)}{\sinh(\pi\lambda)} \tag{29}$$

i.e

$$m(\{|H_{\chi_E}| > \lambda\}) = \frac{2m(E)}{\sinh(\pi\lambda)}.$$

□

Let us recall the Gamma function and the Riemann zeta function are defined by

$$\Gamma(p) = \int_0^\infty u^{p-1} e^{-u}\, du.$$

and

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{for} \quad p > 1.$$

Theorem 9.17. *Let $E \subset \mathbb{R}$ be the union of finitely many disjoint intervals, each of finite length. Then*

$$\int_{\mathbb{R}} |H\chi_E(x)|^p\, dx = \frac{4pm(E)}{\pi^p}(1 - 2^{-p})\zeta(p)\Gamma(p) \quad \textit{for} \quad p > 1.$$

Proof. For $p > 1$. By the Cavalieri principle (see[9]) and theorem 9.16 we have

$$\begin{aligned}
\int_{\mathbb{R}} |H\chi_E(x)|^p\, dx &= p \int_0^\infty \lambda^{p-1} D_{H\chi_E}(\lambda)\, d\lambda \\
&= p \int_0^\infty \lambda^{p-1} \frac{2m(E)}{\sinh(\pi\lambda)}\, d\lambda \\
&= 2p\; m(E) \int_0^\infty \frac{\lambda^{p-1}}{\sinh(\pi\lambda)}\, d\lambda
\end{aligned}$$

$$\boxed{w = e^{\pi\lambda} \Rightarrow \lambda = \frac{\ln w}{\pi} \Rightarrow d\lambda = \frac{dw}{\pi w}} \quad = 2p\; m(E) \int_0^\infty \frac{\lambda^{p-1} e^{\pi\lambda}}{e^{2\pi\lambda} - 1}\, d\lambda$$

$$= 4p\; m(E) \int_1^\infty \frac{\left(\frac{\ln w}{\pi}\right)^{p-1} w}{(w^2 - 1)\pi w}\, dw$$

$$\boxed{w = \frac{1}{r} \Rightarrow dw = -\frac{dr}{r^2}} \quad = \frac{4p\; m(E)}{\pi^p} \int_1^\infty \frac{(\ln w)^{p-1}}{w^2 - 1}\, dw$$

$$\begin{aligned}
&= \frac{4p\; m(E)}{\pi^p} \left[-\int_1^0 \frac{\left(\ln \frac{1}{r}\right)^{p-1}}{\frac{1}{r^2} - 1} \frac{dr}{r^2} \right] \\
&= \frac{4p\; m(E)}{\pi^p} \int_0^1 \frac{(-\ln r)^{p-1}}{1 - r^2}\, dr
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4p\ m(E)}{\pi^p} \int_0^1 (-\ln r)^{p-1} \sum_{n=0}^{\infty} r^{2n}\ dr \\
\boxed{-\ln r = u \Rightarrow dr = -e^{-u}\ du} \quad &= \frac{4p\ m(E)}{\pi^p} \sum_{n=0}^{\infty} \int_0^1 (-\ln r)^{p-1}\, r^{2n}\ dr \\
&= \frac{4p\ m(E)}{\pi^p} \sum_{n=0}^{\infty} \int_{\infty}^0 -(u)^{p-1} e^{-2nu} e^{-u}\ du \\
&= \frac{4p\ m(E)}{\pi^p} \sum_{n=0}^{\infty} \int_0^{\infty} u^{p-1} e^{-(2n+1)u}\ du \\
&= \frac{4p\ m(E)}{\pi^p} \sum_{n=0}^{\infty} \int_0^{\infty} \left(\frac{s}{2n+1}\right)^{p-1} e^{-s}\ \frac{ds}{2n+1} \\
&= \frac{4p\ m(E)}{\pi^p} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^p} \int_0^{\infty} s^{p-1} e^{-s}\ ds \\
&= \frac{4p\ m(E)}{\pi^p} \Gamma(p) \sum_{n=1}^{\infty} \frac{1}{(2n+1)^p} \\
&= \frac{4p\ m(E)}{\pi^p} \Gamma(p) \left[\sum_{n=1}^{\infty} \frac{1}{n^p} - \sum_{n=1}^{\infty} \frac{1}{(2n)^p}\right] \\
&= \frac{4p\ m(E)}{\pi^p} \Gamma(p)(1-2^{-p}) \sum_{n=1}^{\infty} \frac{1}{n^p} \\
&= \frac{4p\ m(E)}{\pi^p} (1-2^{-p})\zeta(p)\Gamma(p).
\end{aligned}
$$

□

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